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BACKWARD EXTENSIONS AND STRONG HAMBURGER MOMENT SEQUENCES

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ABSTRACT. Strong Hamburger moment sequences are studied. Necessary and sufficient conditions for the strong Hamburger moment problem to have a unique solution are obtained. These results are based on certain of the solutions to the classical Hamburger moment problem. The nested disks and orthogonal polynomials associated with the corresponding Jacobi type continued fraction are used.

1. Introduction. A double sequence of real numbers $\{c_n : n = 0, \pm 1, \pm 2, \dots\}$ is called a strong Hamburger sequence (SHMS) if there is a bounded nondecreasing function ϕ on the interval $-\infty < t < \infty$ such that

$$c_n = \int_{-\infty}^{\infty} t^n \, d\phi(t) \tag{1}$$

for all integers n. A real sequence $\{c_n : n = 0, 1, 2, \dots\}$ is a (classical) Hamburger moment sequence (HMS) if there is a bounded nondecreasing ϕ such that (1) holds for all nonnegative integers n. In each case, the function ϕ is called a distribution function for the given sequence.

Extensive work has been done in recent years on the strong moment problems. In particular, we mention the work of Jones, Thron, and Waadeland [**7**] on a strong Stieltjes moment sequence (a Hamburger moment sequence in which the distribution function ϕ is constant on $(-\infty, 0)$). In their study, they emphasized continued fraction methods, specifically T-fractions. More recently, Jones, Njástad, and Thron [**4,5**] and Njástad and Thron [**9**] have investigated strong Hamburger moment sequences. Their studies use several methods, including continued fractions, orthogonal polynomials, and nested disks. Good historical sketches of moment problems can be found in the works of Jones and Thron [**6**] and [**9**].

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An HMS is said to have an m-fold backward extension if there exist real numbers $c_{-2m}, c_{-2m+1}, \dots, c_{-1}$ such that $\{c_{n-2m} : n = 0, 1, 2, \dots\}$ is itself an HMS. The concept of backward extension can be traced to Hamburger [**2**], and later the theory was developed by Wall [**13**]. In this paper, we make use of the fact that a double sequence $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ is an SHMS if and only if the subsequence $C_0 = \{c_n : n = 0, 1, 2, \dots\}$ has an m-fold backward extension to $C_m = \{c_n : n = -2m, -2m+1, \dots\}$ for each positive integer m [**6**, pp. 20-21]. This enables us to parallel classical results on Hamburger moment sequences to obtain new necessary and sufficient conditions for the distribution function of an SHMS to be substantially unique. Two approaches are presented. The first is in terms of nested circles and the second is in terms of orthogonal polynomials that are denominators of certain J-fractions.

2. Characterization of an SHMS. If ϕ is a bounded nondecreasing function on $-\infty < t < \infty$, then the complex valued function

$$I(z, \phi) = \int_{-\infty}^{\infty} \frac{d\phi(t)}{z - t} \tag{2}$$

is analytic for z in the upper half-plane $\operatorname{Im} z > 0$ (and in the lower half-plane). For the class D of bounded nondecreasing functions ϕ on $(-\infty, \infty)$ that are normalized to be right continuous and such that $\phi(-\infty) = \lim_{t \to -\infty} \phi(t) = 0$, the linear transformation (2) is one-to-one into the class of analytic functions in the upper half-plane [**12**], [**14**, p. 247ff].

The fact that distinct bounded nondecreasing functions correspond by (2) to distinct functions I is used throughout this paper. In particular, this enables us to obtain a characterization of an SHMS that ties strong moment problems to the classical Hamburger moment problem.

Theorem 1. *The sequence* $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *of real numbers is an* SHMS *if and only if, for each integer* m, *the sequence* $C_m = \{c_{n-2m} : n = 0, 1, 2, \dots\}$ *is an* HMS.

Proof. If C is an SHMS, there is a ϕ such that (1) holds for all integers n. For each integer m, set

$$\phi_m(t) = \int_{-\infty}^{t} u^{-2m}\, d\phi(u).$$

By the hypothesis, ϕ_m is a distribution function such that

$$\int_{-\infty}^{\infty} t^n\, d\phi_m(t) = \int_{-\infty}^{\infty} t^{n-2m}\, d\phi(t) = c_{n-2m}$$

for all integers n. In particular, C_m is an HMS with distribution function ϕ_m.

Conversely, for each integer m, there is a ϕ_m in D such that

$$c_{n-2m} = \int_{-\infty}^{\infty} t^n\, d\phi_m(t), \qquad n = 0, 1, 2, \ldots.$$

For integers k, m, $0 \le k < m$, set

$$\psi_{m,k}(t) = \int_{-\infty}^{t} u^{2m-2k}\, d\phi_m(u). \tag{3}$$

Then $\psi_{m,k}$ is a distribution function and $\int_{-\infty}^{\infty} d\psi_{m,k}(t) = c_{-2k}$. Thus for fixed k, the sequence $\{\psi_{m,k} : m = k+1, k+2, \ldots\}$ is uniformly bounded. Furthermore, by (2) and (3) we have for $k \ge 1$,

$$\begin{aligned} I(z, \psi_{m,k}) &= \int_{-\infty}^{\infty} \left[\sum_{j=0}^{2k-1} \left(\frac{t^j}{z^{j+1}} \right) + \frac{(t/z)^{2k}}{z-t} \right] t^{-2k}\, d\psi_{m,0}(t) \\ &= \sum_{j=0}^{2k-1} \frac{c_{j-2k}}{z^{j+1}} + z^{-2k} I(z, \psi_{m,0}). \end{aligned} \tag{4}$$

We apply a result of Grommer [**1**] (see also [**10**, p. 207ff]) to obtain a subsequence $\{m_i\}$ of indices and distribution functions ψ_k, ψ_0 in D such that $\psi_{m_i,k} \to \psi_k$, $\psi_{m_i,0} \to \psi_0$, $I(z, \psi_{m_i,k}) \to I(z, \psi_k)$, and $I(z, \psi_{m_i,0}) \to I(z, \psi_0)$. It follows that (4) holds when $\psi_{m,k}$ and $\psi_{m,0}$ respectively are replaced by ψ_k and ψ_0. As in (4) we also have

$$I(z, \psi_k) = \sum_{j=0}^{2k-1} \frac{c_{j-2k}}{z^{j+1}} + z^{-2k} \int_{-\infty}^{\infty} \frac{t^{2k}}{z-t}\, d\psi_k(t).$$

It follows from the uniqueness of the functions (2) over D that $\psi_0(t) = \int_{-\infty}^t u^{2k}\, d\psi_k(u)$. In particular,

$$c_n = \int_{-\infty}^{\infty} t^{n+2k}\, d\psi_k(t) = \int_{-\infty}^{\infty} t^n\, d\psi_0(t), \qquad n = -2k, -2k+1, \dots .$$

Since k was arbitrary, we conclude ψ_0 is a distribution function for the sequence C. Thus, C is an SHMS and the proof is complete. (We note that this result appears to have been observed by a number of individuals including W.B. Gragg (see [**6**, p. 20]) and the present authors.) □

For the (double) sequence C of real numbers, the Hankel determinants are $H_0^{(n)} = H_0^{(n)}(C) = 1$ and

$$H_k^{(n)} = H_k^{(n)}(C) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ \vdots & \vdots & & \vdots \\ c_{n+k-1} & c_{n+k} & & c_{n+2k-2} \end{vmatrix},$$
$$n = 0, \pm 1, \pm 2, \dots;\ k = 1, 2, \dots .$$

A necessary condition for a sequence $\{c_n : n = 0, 1, 2, \dots\}$ of real numbers to be an HMS is that $H_k^{(0)} \geq 0$, $k = 0, 1, 2, \dots$. The distribution function for such a sequence has infinitely many points of increase if and only if $H_k^{(0)} > 0$, $k = 0, 1, 2, \dots$, [**11**; Theorem 1.2, p. 5]. By Theorem 1, therefore, a double sequence C is an SHMS with a distribution function that has infinitely many points of increase if and only if $H_k^{(-2m)} > 0$, $m, k = 0, 1, 2, \dots$. There is, however, a redundancy in these conditions as indicated in the following result.

Corollary 1. *The real sequence C is an* SHMS *with a distribution function that has infinitely many points of increase if and only if, for some integer q, one of the following conditions holds*:

(a) $H_{2m}^{(2q-2m)} > 0$, $H_{2m+1}^{(2q-2m)} > 0$, $m = 0, 1, 2, \dots$,

or

(b) $H_{2m-1}^{(2q-2m)} > 0$, $H_{2m}^{(2q-2m)} > 0$, $m = 1, 2, \dots$.

Proof. The Jacobi identities [**3**, p. 595]

$$H_k^{(n-1)}H_k^{(n+1)} - H_{k-1}^{(n+1)}H_{k+1}^{(n-1)} = [H_k^{(n)}]^2,$$
$$n = 0, \pm 1, \pm 2, \dots, \ k = 1, 2, \dots,$$

imply $H_k^{(n-1)}H_k^{(n+1)} \geq H_{k-1}^{(n+1)}H_{k+1}^{(n-1)}$. Therefore, we obtain by induction from either condition (a) or (b) that $H_k^{(-2m)} > 0$, $k = 0, 1, 2, \dots$, for all integers m. □

Condition (a) of Corollary 1 is, when $q = 0$, known [**4**]. Owing to its relationship with quadratic forms, we call a double sequence C positive definite whenever either condition (a) or (b) holds. A sequence C can be an SHMS and not be positive definite.

Corollary 2. *The double sequence C is an* SHMS *with a distribution function that has exactly a finite number p of points of increase if and only if there is an integer q such that $H_k^{(2q)} > 0$ for $0 \leq k \leq p$, $H_k^{(2q)} = 0$ for $k > p$, and $H_p^{(2q+2)} > 0$.*

Proof. It is known [**11**, p. 5] that for a fixed integer q the sequence $\{c_{2q+n} : n = 0, 1, 2, \dots\}$ is an HMS with a distribution function ϕ that has exactly p points of increase if and only if $H_k^{(2q)} > 0$ for $0 \leq k \leq p$ and $H_k^{(2q)} = 0$ for $k > p$. For a ϕ with p points of increase at $a_1, a_2, \dots, a_p$, we have

$$c_{2q+n} = \int_{-\infty}^{\infty} t^n \, d\phi(t) = \sum_{j=1}^{p} \lambda_j a_j^n, \qquad n = 0, 1, 2, \dots,$$

where $\lambda_j > 0$, $j = 1, 2, \dots, p$. If $a_j \neq 0$, $j = 1, 2, \dots, p$, there is a unique backward extension to an SHMS, namely, $\{\sum_{j=1}^{p} \lambda_j a_j^n : n = 0, \pm 1, \pm 2, \dots\}$. On the other hand, if $a_j = 0$ for some j, $1 \leq j \leq p$, then $\int_{-\infty}^{\infty} t^{-n} \, d\phi(t)$ fails to exist for $n > 0$. The HMS cannot be extended backwards and, therefore, cannot be part of an SHMS. Thus, an HMS with a distribution function that has only a finite number of points of increase has a unique backward extension to an SHMS if and only if the origin is not a point of increase of the distribution function. In terms of Hankel determinants it is known [**8**, Lemma 2] that the origin is not a point of increase for a distribution function with p points of increase if and only if $H_p^{(2q+2)} > 0$. (This fact can also be established directly using elementary properties of determinants.) The corollary now follows from Theorem 1. □

Corollary 3. *The sequence* $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *is an* SHMS *if and only if* $\hat{C} = \{c_{-n} : n = 0, \pm 1, \pm 2, \dots\}$ *is an* SHMS.

Proof. It is easily proved that the Hankel determinants for C and $\hat{C}$ are related by the identity

$$H_{k+1}^{(n-k)}(\hat{C}) = H_{k+1}^{(-n-k)}(C)$$

for $k = 0, 1, 2, \dots$ and all integers n. The stated result follows from this identity and the previous corollaries. □

By (1) the index choice in a double sequence C appears to play a role in determining when the sequence is an SHMS. We conclude from Theorem 1 that a shift of indices by an even integer transforms one SHMS into another.

Corollary 4. *The sequence* $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *is an* SHMS *if and only if for some integer* m *the sequence* $C_m^\infty = \{c_{n-2m} : n = 0, \pm 1, \pm 2, \dots\}$ *is an* SHMS.

Since the odd indexed entries of an SHMS can be negative, there is no analogue for a shift of indices from even to odd integers.

3. Determinate and indeterminate sequences. A strong or classical moment sequence is said to be *determinate* if the distribution function of the sequence, when normalized to be in the class D, is unique. Otherwise the moment sequence is called *indeterminate*.

If the distribution function of a moment sequence has only a finite number of points of increase, then it is uniquely determined in D. This observation, combined with a result of Wall [**13**; Theorem 9, p. 527] for the case of a distribution function with infinitely many points of increase, establishes the following theorem.

Theorem 2. *Let* $C_0 = \{c_n : n = 0, 1, 2, \dots\}$ *be a determinate* HMS *with distribution function* ϕ. *This sequence admits a unique backward extension to an* SHMS $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *if and only if* $\int_{-\infty}^{\infty} t^{-n}\, d\phi(t)$ *exists and equals* c_{-n} *for all integers* $n > 0$. *The* SHMS C *is determinate in this case.*

An immediate consequence of Theorem 1 and Theorem 2 is the following sufficient condition for an SHMS to be determinate.

Corollary 5. *The* SHMS C *is determinate if there is an integer* m *such that* C_m *is a determinate* HMS.

The converse of the implication in Corollary 5 is false. That is, each subsequence C_m of a positive definite double sequence C can be an indeterminate HMS while C is a determinate SHMS. In fact we characterize in the next section a determinate (and an indeterminate) SHMS C based on classical properties applied to the subsequences C_m.

When C_0 is an indeterminate HMS, Wall [**13**; Theorem 10, p. 530] (see also [**8**]) proved there are infinitely many backward extensions to strong Hamburger moment sequences. More precisely, we paraphrase his result as our next theorem.

Theorem 3. *Let* $C_0 = \{c_n : n = 0, 1, 2, \dots\}$ *be an indeterminate* HMS *and let* m *be a given positive integer. Then there exists an* SHMS $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *such that* $C_{m+s} = \{c_n : n = -2m - 2s, -2m - 2s + 1, \dots\}$ *is an indeterminate* HMS *when* $-m \leq s < 0$ *and a determinate* HMS *when* $s \geq 0$. *In addition, there is an* SHMS C' *such that* C'_s *is an indeterminate* HMS *for all indices* s *and* $C'_0 = C_0$.

4. Nested disks. Let $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ be a given positive definite double sequence, that is, an SHMS with a distribution function that has infinitely many points of increase. For each integer m, the sequence $C_m = \{c_{n-2m} : n = 0, 1, 2, \dots\}$ is an HMS by Theorem 1. Let ϕ_m be any distribution function of the sequence C_m. It is known [**11**, pp. 46-47] that the function $I(z, \phi_m)$ of (2) is asymptotically represented by the formal power series $\sum_{j=0}^{\infty} c_{j-2m}/z^{j+1}$ in each wedge $\varepsilon < \arg z < \pi - \varepsilon$, where $0 < \varepsilon < \pi/2$, and the equivalent function of the positive definite real J-fraction

$$(5) \qquad \frac{\beta_{0,m}}{z-\alpha_{1,m}} - \frac{\beta_{1,m}}{z-\alpha_{2,m}} - \cdots - \frac{\beta_{n,m}}{z-\alpha_{n+1,m}} - \cdots,$$

where

$$(6) \quad \beta_{0,m} = c_{-2m}, \qquad \beta_{n,m} = H_{n+1}^{(-2m)} H_{n-1}^{(-2m)} / [H_n^{(-2m)}]^2, \qquad n \geq 1.$$

(The constants $\alpha_{n,m}$ of (5) are, of course, also determined by the moment sequence (see [**10**, p. 165]) but their formulas are not needed in the sequel.) Let the nth approximant of (5) be $P_n^{(-2m)}(z)/Q_n^{(-2m)}(z)$, where the polynomial $Q_n^{(-2m)}(z)$ is of degree n with leading coefficient 1. For a fixed z in the upper half-plane, the linear fractional transformation

$$(7) \qquad w = \frac{P_{n+1}^{(-2m)}(z) - \sigma P_n^{(-2m)}(z)}{Q_{n+1}^{(-2m)}(z) - \sigma Q_n^{(-2m)}(z)}$$

maps $\operatorname{Im}\sigma > 0$ onto the interior of a closed disk $\Gamma_{n+1}^{(-2m)}(z)$ in the lower half plane $\operatorname{Im} w \leq 0$. Each boundary point of this disk is a value of the linear functional $I(z, \phi_m^{(n)})$ of (2), where $\phi_m^{(n)}$ is a uniquely determined distribution function in D with a finite number of points of increase such that for $\phi = \phi_m^{(n)}$,

$$(8) \qquad \int_{-\infty}^{\infty} t^k \, d\phi(t) = c_{k-2m}, \qquad k = 0, 1, 2, \ldots, 2n.$$

The linear functional $I(z, \phi)$ is in the closed disk $\Gamma_{n+1}^{(-2m)}(z)$ whenever ϕ is a distribution function such that (8) holds and it is an interior point of the disk if ϕ has infinitely many points of increase [**11**, p. 48]. By (6) and the determinant formula for continued fractions, the radius of the disk $\Gamma_{n+1}^{(-2m)}(z)$ is found to be

$$(9) \qquad r_{n+1}^{(-2m)} = r_{n+1}^{(-2m)}(z) = \frac{H_{n+1}^{(-2m)}/H_n^{(-2m)}}{2|\operatorname{Im}[Q_{n+1}^{(-2m)}(z) Q_n^{(-2m)}(\bar{z})]|}.$$

As n increases, the disks $\Gamma_{n+1}^{(-2m)}(z)$ are nested [**11**; Theorem 2.8, p. 48]. They converge as $n \to \infty$ to a disk $\Gamma^{(-2m)}(z)$ with a positive radius if and only if C_m is an indeterminate HMS. The boundary of the disk $\Gamma^{(-2m)}(z)$ is the circle

$$W = \frac{A^{(-2m)}(z) - \sigma C^{(-2m)}(z)}{B^{(-2m)}(z) - \sigma D^{(-2m)}(z)}, \qquad -\infty \le \sigma \le \infty,$$

where $A^{(-2m)}, B^{(-2m)}, C^{(-2m)}$, and $D^{(-2m)}$ are certain entire functions of z when C_m is indeterminate [**11**; Theorem 2.12, p. 57]. Furthermore,

$$\Gamma^{(-2m)}(z) = \{I(z, \phi_m) : \phi_m \text{ is a distribution function of } C_m\}. \tag{10}$$

These classical results play a vital role in our characterization of indeterminate strong Hamburger moment sequences. Indeed, the disks $\Gamma^{(-2m)}(z)$ can be mapped onto closed disks containing $I(z, \phi)$ whenever ϕ is a distribution function of the SHMS C and C is such that C_m is indeterminate for each integer m. We then prove the SHMS C is itself determinate if and only if the intersection of the images under the mapping of $\Gamma^{(-2m)}(z)$ is a unique point. The details are provided by a sequence of lemmas.

Lemma 1. *For a given positive integer n, let $\eta_{n+1}^{(0)} = \Gamma_{n+1}^{(0)}(z)$, $\eta_{n+1}^{(-2m)} = \{z^{2m}w - \sum_{j=1}^{2m} c_{-j} z^{j-1} : w \in \Gamma_{n+1}^{(-2m)}(z)\}$ for $m = 1, 2, \ldots$. Then, for $m \ge 1$,*

$$\eta_{n+1}^{(-2m)} = \{I(z, \psi_m^{(n)}) : \psi_m^{(n)} \in D \text{ and } \int_{-\infty}^{\infty} t^k \, d\psi_m^{(n)}(t) = c_k \text{ for } k = -2m, -2m+1, \ldots, -2m+2n\}. \tag{11}$$

Proof. For a fixed integer $m \ge 1$ there is a one-to-one correspondence between the distribution functions ϕ of D such that (8) holds and the ψ of D such that $\int_{-\infty}^{\infty} t^k \, d\psi(t) = c_k$ for $k = -2m, -2m+1, \ldots, -2m+2n$. This correspondence is defined by

$$\psi(t) = \int_{-\infty}^{t} u^{2m} \, d\phi(u). \tag{12}$$

As in (4) we have

$$I(z,\phi)=\sum_{j=0}^{2m-1}\frac{c_{j-2m}}{z^{j+1}}+z^{-2m}I(z,\psi). \tag{13}$$

Since

$$\Gamma_{n+1}^{(-2m)}(z)=\{I(z,\phi):\phi\in D \text{ and (8) holds}\},$$

the result (11) follows from (13). □

Lemma 2. *For $n\geq m>1$, the disks (11) are nested in the sense that*

$$\eta_{n+1}^{(-2m)}\subset\eta_{n}^{(-2m+2)}\subset\cdots\subset\eta_{n-m+1}^{(0)}=\Gamma_{n-m+1}^{(0)}(z).$$

Proof. If $\rho\in\eta_{n+1}^{(-2m)}$, there is a distribution function ψ in D such that $I(z,\psi)=\rho$ and

$$\int_{-\infty}^{\infty}t^k\,d\psi(t)=c_k,\qquad k=-2m,-2m+1,\dots,-2m+2n.$$

The last equality holds for the parameter k in the more restricted range $-2m+2\leq k\leq -2m+2n$ which implies $\rho\in\eta_n^{(-2m+2)}$ by (11).

In addition, we have the nesting $\eta_{n+1}^{(-2m)}\subset\eta_n^{(-2m)}$ for all integers $n>0$ which, by Lemma 1, is equivalent to the nesting of $\Gamma_{n+1}^{(-2m)}$ as n increases. □

Define $\eta^{(-2m)}=\eta^{(-2m)}(z)$ by

$$\eta^{(-2m)}=\bigcap_n\eta_{n+1}^{(-2m)}.$$

Then by (10), (12), and (13), we have

$$\eta^{(-2m)}=\{I(z,\psi_m):\psi_m\in D \text{ and } \int_{-\infty}^{\infty}t^k\,d\psi_m(t)=c_k \text{ for } k=-2m,-2m+1,\dots\}.$$

Thus, the distribution functions associated with $\eta^{(-2m)}$ are those for C_0 that have an m-fold backward extension to C_m. Let

$$\eta = \eta(z) = \bigcap_m \eta^{(-2m)}. \tag{14}$$

Then η is a proper disk, called the limit circle case, or a single point, called the limit point case. The points of $\eta(z)$ are the values of $I(z, \psi)$ where $\psi \in D$ and ψ is a distribution function of the SHMS C.

The analysis so far was confined to a fixed z in the upper half-plane. Without significant change, we could consider a fixed z where $\operatorname{Im} z < 0$. In particular, if the limit point case holds for a nonreal z, then it holds also for $\bar{z}$ since for all $m \geq 0$, we have $\eta^{(-2m)}(\bar{z}) = \overline{\eta^{(-2m)}(z)}$. There is, however, a stronger "invariability theorem" connected with the strong moment problem.

Lemma 3. *If the limit point case holds for some nonreal z_0, then it holds for all nonreal z.*

Proof. If for some integer m the subsequence C_m of C is a determinate HMS, then $\eta^{(-2m)}(z)$ is a point for each z in $\operatorname{Im} z > 0$ (or $\operatorname{Im} z < 0$). The limit point case, therefore, holds for all nonreal z. Assume that C_m is an indeterminate HMS for all integers $m \geq 0$ and that the limit point case holds at z_1 while the limit circle case holds at z_2. Replacing either z_1 or z_2 by its conjugate, if necessary, we can assume $\operatorname{Im} z_1 > 0$ and $\operatorname{Im} z_2 > 0$. There is a one-to-one correspondence, independent of the choice of z, between points on the circumference of $\Gamma^{(-2m)}(z)$ and a subset of the distribution functions for the indeterminate HMS C_m [**11**; Theorem 2.12, p. 57] for each $m \geq 0$. By Lemma 1, therefore, there is a one-to-one correspondence, independent of z, betweeen points on the circumference of $\eta^{(-2m)}(z)$ and a particular subset of distribution functions of C_0 that have a backward extension to C_m. Now let ψ be a distribution function of C such that $I(z_2, \psi)$ is an interior point of $\eta(z_2)$. Since the limit point case holds at z_1, there is a sequence of distribution functions $\{\psi_m\}$ such that, for each integer $m \geq 0$, $I(z_1, \psi_m)$ is on the boundary of $\eta^{(-2m)}(z_1)$ and $\psi_m \to \psi$ as $m \to \infty$ pointwise on $(-\infty, \infty)$. By a theorem of Grommer [**1**] we conclude $I(z_2, \psi_m) \to I(z_2, \psi)$ as $m \to \infty$. However, each $I(z_2, \psi_m)$ is on the boundary of $\eta^{(-2m)}(z_2)$

whereas $I(z_2, \psi)$ is an interior point of the intersection of these disks. This contradiction proves the lemma. $\square$

If the limit point case holds, then $I(z, \phi) = I(z, \psi)$ for all z, $\operatorname{Im} z \neq 0$, whenever ϕ and ψ are distribution functions of C. Thus $\phi = \psi$ if each distribution function is normalized to be in D.

Theorem 4. *Let $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ be a positive definite double sequence. Then C is a determinate* SHMS *if and only if the limit point case holds for some z_0,* $\operatorname{Im} z_0 \neq 0$.

In the limit circle case, the radius of the disk $\Gamma^{(-2m)}(z)$ is $[2\mathrm{Im} \cdot \{B^{(-2m)}(z)D^{(-2m)}(\bar{z})\}]^{-1}$, since the entire functions $A^{(-2m)}, B^{(-2m)}$, C^{-2m}, and $D^{(-2m)}$ are real for real z and related by the identity $A^{(-2m)}D^{(-2m)} - B^{(-2m)}C^{(-2m)} \equiv 1$ for all z [**11**, p. 52]. The radius of $\eta^{(-2m)}(z)$ is $|z|^{2m}$ times the radius of $\Gamma^{(-2m)}$. This observation and Theorem 4 imply the following result.

Corollary 5. *Let C be a positive definite double sequence such that C_m is an indeterminate* HMS *for all integers $m \geq 0$. Then C is a determinate* SHMS *if and only if for some nonreal z*

$$\lim_{m \to \infty} \frac{|\mathrm{Im}\,[B^{(-2m)}(z)D^{(-2m)}(\bar{z})|}{|z|^{2m}} = \infty.$$

The nesting of the disks $\eta^{(-2m)}(z)$ assures the existence of the limit in the wide sense.

Rather than letting $n \to \infty$ and then $m \to \infty$ independently, we can obtain the limit point or limit circle $\eta(z)$ by relating n to m. One such choice is the subject of the next lemma.

Lemma 4. *The limit point case holds if and only if for some nonreal z*

$$\lim_{m \to \infty} |z|^{2m} r_{2m+2}^{(-2m)} = 0, \tag{15}$$

where $r_{2m+2}^{(-2m)}$ is given by (9).

Proof. By Lemma 2 and the nesting of $\eta_{n+1}^{(-2m)}$ with incresing n, we have

$$\eta_{2m+2}^{(-2m)} \subset \eta_{2m+1}^{(-2m)} \subset \eta_{2m}^{(-2(m-1))}.$$

Since $|z|^{2m} r_{2m+2}^{(-2m)}$ is the radius of $\eta_{2m+2}^{(-2m)}$, the limit on the left of (15) exists and it is zero if and only if

$$\hat{\eta}(z) = \bigcap_m \eta_{2m+2}^{(-2m)}$$

is a unique point. In view of (11) with $n = 2m+1$, we have

$$\hat{\eta}(z) = \{I(z,\psi) : \psi \in D \text{ and } \psi \text{ is a distribution function of } C\},$$

that is, $\hat{\eta}(z) = \eta(z)$, where $\eta(z)$ is given by (14). Therefore, (15) holds if and only if $\eta(z)$ is a unique point. □

Corollary 6. *Let C be a positive definite double sequence. Then C is a determinate* SHMS *if and only if* (15) *holds for some nonreal z.*

5. Characterization using orthogonal polynomials. Let $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ be a positive definite double sequence with distribution function ϕ. For each integer m, define, on the space P of all polynomials in z, the bilinear functional

$$\langle f, g\rangle_m = \int_{-\infty}^{\infty} f(t)\overline{g(t)} t^{-2m}\, d\phi(t) \tag{16}$$

for all polynomials f, g in P. In fact, this bilinear functional is an inner product since $C_m = \{c_{n-2m} : n = 0, 1, 2, \dots\}$ is positive definite and $\phi_m(t) = \int_{-\infty}^{t} u^{-2m}\, d\phi(u)$ is a distribution function of C_m. The denominators of the approximants of the J-fraction (5) form, relative to this inner product, an orthogonal sequence $\{Q_n^{(-2m)}\}_{n=0}^{\infty}$ that spans the space P. Thus, the sequence

$$\omega_n^{(-2m)}(z) = \left[\frac{H_n^{(-2m)}}{H_{n+1}^{(-2m)}}\right]^{1/2} Q_n^{(-2m)}(z), \qquad n = 0, 1, 2, \dots \tag{17}$$

is an orthonormal basis of P [**11**, p. 46ff].

The sequence $\hat{C} = \{c_{-n} : n = 0, \pm 1, \pm 2, \dots\}$ is, by Corollary 3, also positive definite. Let ψ be a distribution function of $\hat{C}$. For each integer m, we have that the bilinear functional

$$(18) \qquad {}_m\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}t^{-2m}\, d\psi(t),$$

defined for all polynomials f, g in P, is also an inner product on P. An orthonormal basis $\{\hat{\omega}_n^{(-2m)}\}_{n=0}^{\infty}$ of P relative to this inner product can be obtained from the J-fraction associated with $\hat{C}$.

Let L be the space of complex polynomials in z and $1/z$. Then for f, g in L, the bilinear functional

$$(19) \qquad \langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, d\phi(t)$$

is an inner product on L since C is a positive definite double sequence. By a diagonalization type process applied to the orthonormal bases of P for the sequences C_m and $\hat{C}_m$, we obtain an orthonormal basis of L.

Lemma 5. *Let $W_0 = \omega_0^{(0)}$ and, for $h = 0, 1, 2, \dots$, let*

$$(20) \qquad W_{4h+1}(z) = z^{-2h}\omega_{4h+1}^{(-4h)}(z), \qquad W_{4h+2}(z) = z^{-2h}\omega_{4h+2}^{(-4h)}(z),$$

(21)
$$W_{4h+3}(z) = z^{2h+2}\hat{\omega}_{4h+3}^{(4h+4)}\left(\frac{1}{z}\right), \qquad W_{4h+4}(z) = z^{2h+2}\hat{\omega}_{4h+4}^{(4h+4)}\left(\frac{1}{z}\right).$$

Then $\{W_n\}_{n=0}^{\infty}$ is an orthonormal basis of L relative to the inner product (19).

Proof. Since $W_0 = \omega_0^{(0)} = c_0^{-1/2}$, we have by (16) that $\langle W_0, W_0\rangle = 1$. For $k = 0, 1, 2, \dots$, we obtain from (16) and (19),

$$\langle z^k, \omega_{4h+1}^{(-4h)}\rangle_{2h} = \int_{-\infty}^{\infty} t^{k-4h}\omega_{4h+1}^{(-4h)}(t)\, d\phi(t) = \langle z^{k-2h}, W_{4h+1}\rangle.$$

Since $\omega_{4h+1}^{(-4h)}$ is a polynomial in P of degree $4h+1$, the orthogonality of $\omega_{4h+1}^{(-4h)}$ implies W_{4h+1} is orthogonal to all polynomials in $L(-2h, 2h)$, where $L(-p,q)$ denotes the polynomials in L of degrees at most p in $1/z$ and q in z. Moreover, by (16), (17), (19), (20), and properties of inner products, we have

$$\begin{aligned}\langle W_{4h+1}, W_{4h+1}\rangle &= a_{4h+1}\left\langle z^{2h+1}, W_{4h+1}\right\rangle = a_{4h+1}\left\langle z^{4h+1}, \omega_{4h+1}^{(-4h)}\right\rangle_{2h}\\ &= \left\langle \omega_{4h+1}^{(-4h)}, \omega_{4h+1}^{(-4h)}\right\rangle_{2h} = 1,\end{aligned}$$

where $a_{4h+1} = [H_{4h+1}^{(-4h)}/H_{4h+2}^{(-4h)}]^{1/2}$. By a similar argument, W_{4h+2} is a unit vector in L orthogonal to all polynomials in $L(-2h, 2h+1)$.

From (18), (19), and (21), we obtain for $k = 0,1,2,\dots,$

$$\begin{aligned}{}_{2h+2}\left\langle z^k, \hat{\omega}_{4h+3}^{(4h+4)}\right\rangle &= \int_{-\infty}^{\infty} t^{k-4h-4}\hat{\omega}_{4h+3}^{(4h+4)}(t)\,d\psi(t)\\ &= \int_{-\infty}^{\infty} t^{4h+4-k}\hat{\omega}_{4h+3}^{(4h+4)}(1/t)\,d\phi(t)\\ &= \left\langle t^{2h+2-k}, W_{4h+3}\right\rangle.\end{aligned}$$

We conclude that W_{4h+3} is orthogonal to all polynomials in $L(-2h, 2h+2)$ since the left hand inner product is zero for $k = 0,1,2,\dots,4h+2$. Since $\hat{\omega}_{4h+3}^{(4h+4)}$ is a polynomial of exact degree $4h+3$, we can prove W_{4h+3} is a unit vector in L by an argument similar to that for W_{4h+1}. In the same manner, we can also prove W_{4h+4} is a unit vector orthogonal to all polynomials in $L(-2h-1, 2h+2)$.

This proves $\{W_n\}_{n=0}^{\infty}$ is an orthonormal sequence in L. Since, for each integer n, there is a nonnegative integer m such that $\langle z^n, W_m\rangle \neq 0$, this orthonormal sequence spans the space L and the proof of the lemma is complete. □

The orthonormal basis of Lemma 5 can also be obtained by applying the Gram-Schmidt process to the sequence $\{1, z, z^2, z^{-1}, z^{-2}, z^3, \dots\}$. The order of the sequence gives us the connection with denominators of J-fractions and hence ties the discussion to the classical moment problem. On the other hand, Njåstad and Thron [**9**] considered the sequence $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$ and thereby made an association with T-fractions.

Our next lemma is the analogue in the L space of the reproducing property of the kernel polynomial for ordinary orthogonal polynomials. The proof follows the standard pattern for orthogonal polynomials and it is therefore omitted.

Lemma 6. *Let* $K_n(x,y) = \sum_{k=0}^{n} W_k(x)W_k(y)$, *and let* p *be a polynomial in the subspace* L_n *of* L *spanned by* $\{W_k\}_{k=0}^{n}$. *Then for any nonzero real* y

$$\langle p(x), K_n(x,y)\rangle = p(y). \tag{22}$$

Conversely, if $K(x,y)$ *is in* L_n *whenever either* x *or* y *is fixed and if (22) with* K_n *replaced by* K *holds for all* p *in* L_n *and all nonzero real* y, *then* $K(x,y) = K_n(x,y)$ *for all nonzero real* x *and* y.

A Christoffel-Darboux type identity also holds for the orthonormal basis in Lemma 5. For the applications to strong moment sequences, we require only the special cases of the following lemma.

Lemma 7. *If* $x \neq y$ *and* h *is a nonnegative integer, then*

$$\frac{a_{4h+1}}{a_{4h+2}} \frac{W_{4h+2}(x)W_{4h+1}(y) - W_{4h+2}(y)W_{4h+1}(x)}{x-y} = \sum_{k=0}^{4h+1} W_k(x)W_k(y), \tag{23}$$

where a_{4h+1} *and* a_{4h+2} *are respectively the leading coefficients of* W_{4h+1} *and* W_{4h+2}.

Proof. Let $K(x,y)$ denote the expression on the left-hand side of the equality (23). Since W_{4h+1} and W_{4h+2} are in $L(-2h, 2h+2)$, it follows that for fixed nonzero y, the function $K(x,y)$ is in $L(-2h, 2h+1)$ when K is defined by continuity for $x = y$.

For any positive integer n and for p in L, the algebraic identity

$$\begin{aligned} p(x)&\frac{W_{n+1}(x)W_n(y)-W_{n+1}(y)W_n(x)}{x-y} \\ &= \Big[W_{n+1}(x)W_n(y)-W_{n+1}(y)W_n(x)\Big]q(x,y) \\ &\quad + p(y)\Big[W_n(x)\frac{W_{n+1}(x)-W_{n+1}(y)}{x-y} \\ &\qquad - W_{n+1}(x)\frac{W_n(x)-W_n(y)}{x-y}\Big], \end{aligned}$$

where

$$q(x,y)=\frac{p(x)-p(y)}{x-y}$$

holds for all distinct nonzero x and y. Let $n=4h+1$ and let y be a fixed nonzero real number. If the above identity is integrated with respect to a distribution function ϕ of the positive definite sequence C, we obtain from (19) that

$$\begin{aligned} \frac{a_{n+1}}{a_n}\langle p(x),K(x,y)\rangle &= W_n(y)\langle W_{n+1}(x),q(x,y)\rangle \\ &\quad - W_{n+1}(y)\langle W_n(x),q(x,y)\rangle \\ &\quad + p(y)\left\langle W_n(x),\frac{W_{n+1}(x)-W_{n+1}(y)}{x-y}\right\rangle \\ &\quad - p(y)\left\langle W_{n+1}(x),\frac{W_n(x)-W_n(y)}{x-y}\right\rangle. \end{aligned}$$

By the orthogonality of W_n and W_{n+1} the terms on the right hand side of the equality are zero when $p\in L(-2h,2h+1)$ except for the third. This term is $p(y)$ multiplied by

$$\left\langle W_n(x),\frac{W_{n+1}(x)-W_{n+1}(y)}{x-y}\right\rangle = a_{n+1}\langle W_n(x),x^n\rangle = a_{n+1}/a_n.$$

It follows that

$$\langle p(x),K(x,y)\rangle = p(y)$$

for all polynomials p in L_n. By Lemma 6, therefore, (23) is proved for all nonzero real x and y, $x\neq y$. Since the function of (23) is rational, the identity holds as well for all complex nonzero x and y. □

We apply these lemmas to obtain a characterization of indeterminate strong Hamburger moment sequences. This is analogous to one of the series characterizations of indeterminate Hamburger moment sequences [**11**, p. 50].

Theorem 5. *Let C be a positive definite double sequence, and let $\{W_j\}_{j=0}^{\infty}$ be the corresponding orthonormal polynomials of Lemma 5. A necessary and sufficient condition for C to be an indeterminate* SHMS *is that the series $\sum_{j=0}^{\infty} |W_j(z)|^2$ converges for some nonreal z. In this case the series converges for all z,* $\operatorname{Im} z \neq 0$.

Proof. By Corollary 6 and (9) C is an indeterminate SHMS if and only if for a nonreal z the sequence

$$R_{4h+2}^{(-4h)} = |z|^{4h} \frac{H_{4h+2}^{(-4h)}/H_{4h+1}^{(-4h)}}{2|\operatorname{Im}[Q_{4h+2}^{(-4h)}(z)Q_{4h+1}^{(-4h)}(\bar{z})]|}$$

converges to a nonzero limit as $h \to \infty$. By Lemma 5 and (17), we have

$$\begin{aligned} R_{4h+2}^{(-4h)} &= \frac{(H_{4h+2}^{(-4h)}/H_{4h+1}^{(-4h)})(H_{4h+1}^{(-4h)}/H_{4h+2}^{(-4h)})^{1/2}(H_{4h+2}^{(-4h)}/H_{4h+3}^{(-4h)})^{1/2}}{|z|^{-4h}|\omega_{4h+2}^{(-4h)}(z)\omega_{4h+1}^{(-4h)}(\bar{z}) - \omega_{4h+2}^{(-4h)}(\bar{z})\omega_{4h+1}^{(-4h)}(z)|} \\ &= \left\{ \frac{a_{4h+1}}{a_{4h+2}} |W_{4h+2}(z)W_{4h+1}(\bar{z}) - W_{4h+2}(\bar{z})W_{4h+1}(z)| \right\}^{-1}, \end{aligned}$$

where

$$a_{4h+1} = \left[H_{4h+1}^{(-4h)}/H_{4h+2}^{(-4h)}\right]^{1/2}, \qquad a_{4h+2} = \left[H_{4h+2}^{(-4h)}/H_{4h+3}^{(-4h)}\right]^{1/2}.$$

By (17) and (20) a_{4h+1} and a_{4h+2} are, respectively, the leading coefficients of W_{4h+1} and W_{4h+2}. In view of Lemma 7, we conclude

$$R_{4h+2}^{(-4h)} = \{2\operatorname{Im} z \sum_{j=0}^{4h+1} W_j(z)W_j(\bar{z})\}^{-1}.$$

Thus, $R_{4h+2}^{(-4h)}$ tends to a nonzero limit as $h \to \infty$ if and only if $\sum_{j=0}^{\infty} |W_j(z)|^2$ converges. The last statement in the theorem is a consequence of the invariability result Lemma 3. □

A sufficient condition in terms of Hankel determinants for a positive definite double sequence to be determinate is obtained from this theorem.

Corollary 7. *A positive definite double sequence* $C = \{c_n : n = 0, \pm 1, \pm 2, \dots\}$ *is a determinate* SHMS *if*

$$\liminf_{h\to\infty} \{c_{-4h}^2 H_{4h+1}^{(-4h)} H_{4h+3}^{(-4h)} / [H_{4h+2}^{(-4h)}]^2\}^{1/(2h)} = 0.$$

Proof. As in the classical case [**11**, p. 59], we have for nonreal z that

$$|\mathrm{Im}\, z||z|^{-4h} c_{-4h}^{-1} \beta_{4h+2,2h}^{-1/2} \le |W_{4h+1}(z)|^2 + |W_{4h+2}(z)|^2.$$

The condition stated in this corollary implies the series $\sum_{h=0}^{\infty} c_{-4h}^{-1} \cdot \beta_{4h+2,2h}^{-1/2} \rho^h$ is divergent for all $\rho \neq 0$. By Theorem 5, therefore, C is a determinate SHMS. □

6. Open problems. A determinate HMS need not have a backward extension to an SHMS by Theorem 2. When such an extension exists, the backward extension is a unique determinate SHMS. The behavior of an indeterminate HMS is quite different. Such a sequence always admits infinitely many backward extensions to strong Hamburger moment sequences. It would be interesting to obtain necessary and sufficient conditions on an indeterminate HMS for it to have a backward extension to an indeterminate SHMS. Perhaps every indeterminate HMS has such a backward extension.

It is known that an HMS $C_0 = \{c_n : n = 0, 1, 2, \dots\}$ is determinate if and only if $\lim_{n\to\infty} (H_{n+1}^{(0)} / H_{n-1}^{(4)}) = 0$. The analogue of this result for an SHMS to be determinate has not been found by the present authors.

In recent years various moment problems have been studied from a geometric point of view (see [**8**]). No such study has been attempted for strong moment problems. A geometric discussion of an SHMS may provide additional insights and new criteria for the strong Hamburger moment problem.

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