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A SKOROHOD REPRESENTATION AND AN INVARIANCE PRINCIPLE FOR SUMS OF WEIGHTED i.i.d. RANDOM VARIABLES

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ABSTRACT. A Skorohod representation is obtained for sums of weighted i.i.d. random variables, extending the i.i.d. case. This leads to a functional law of the iterated logarithm and other invariance results. In this setting, the results are not included as special cases of previous martingale results.

1. Introduction. Let $\{X_k : k = 1, 2, \dots\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = 1$. Let $\{a_k : k = 1, 2, \dots\}$ be a sequence of real numbers. We refer to these as "weights." Define the sum, S_n, of weighted i.i.d. random variables as $S_n = \sum_{k=1}^n a_k X_k$.

In Section 2 of this paper a Skorohod representation is obtained for the sums S_n. This is the content of Theorem 2.1 and Theorem 2.2. These results extend the original representation by Skorohod [**11**] for sums of i.i.d. random variables.

Section 3 consists of applications of the Skorohod representation derived in Section 2. In particular, we obtain a functional law of the iterated logarithm (Theorem 3.2) and an almost sure invariance principle for sums of weighted i.i.d. random variables (Theorem 3.3). These are analogous to the results obtained by Strassen [**13**] in the i.i.d. case. A central limit theorem (Theorem 3.1) and a classical law of the iterated logarithm (Corollary 3.4) are also obtained. We remark that all the results derived here are extensions of the i.i.d. case, i.e., where $a_k \equiv 1$.

Since Skorohod and Strassen proved their results for i.i.d. random variables, analogous results have been obtained for martingales, notably by Strassen [**14**], Jain, Jogdeo and Stout [**7**], Heyde and Scott [**6**], and Hall and Heyde [**5**]. However, the Skorohod representation derived in

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this paper differs from the martingale results. In the specific setting of weighted i.i.d. random variables, an explicit representation for the "embedding times" $\{T_n\}$ is obtained (see Theorem 2.2). This is crucial in obtaining results for weighted i.i.d. random variables that are more easily accessible and of wider applicability than the martingale results as applied in this setting. We close the paper with a simple example illustrating this.

2. A Skorohod representation.

We use the construction derived by Billingsley [**1**] in his proof of the Skorohod embedding for i.i.d. random variables as the basis for the construction used here for weighted i.i.d. random variables.

Let X be a random variable defined on $(\Omega, \mathcal{F}, P)$ with $EX = 0$ and $EX^2 = 1$. In [**1**] it is shown that, on some probability space, a standard Brownian motion $\{B(t) : t \geq 0\}$ and a stopping time τ exist so that $B(\tau) = X$ in dist. (distribution) and $E\tau = 1$.

Theorem 2.1. *Let $a \in \mathbf{R}$. There exists a stopping time σ such that $B(\sigma) = aX$ in dist. and $\sigma = a^2\tau$ in dist.*

Let $\{X_k\}$, $\{a_k\}$ and $\{S_n\}$ be as described in Section 1. Theorem 2.2, the Skorohod representation for the sums S_n, will follow in the standard manner.

We note that all Brownian motions described in this paper are standard Brownian motions.

Theorem 2.2. *There exists a probability space with a Brownian motion $\{B(t) : t \geq 0\}$ and nonnegative random variables $\{T_n\}$ defined on it such that $(S_1, S_2, \dots) = (B(T_1), B(T_2), \dots)$ in dist. and $T_n = \sum_{k=1}^{n} a_k^2 \tau_k$ with $\{\tau_k\}$ i.i.d. satisfying $\tau_k \geq 0$ and $E\tau_k = 1$.*

Proof of Theorem 2.1. The proof is divided into three parts. We use as the foundation in the proof a construction used by Billingsley [**1**, Theorem 37.6]. There, a martingale $\{(X_k, G_k) : k = 1, 2, \dots\}$ is defined with $X_k \to X$ a.s. Let $\{B(t) : t \geq 0\}$ be a Brownian motion on a possibly different probability space. A sequence of stopping times

$\tau_1 \le \tau_2 \le \dots$ is obtained such that

$$(B(\tau_1), B(\tau_2), \dots) = (X_1, X_2, \dots) \quad \text{in dist.} \tag{1}$$

Further, it is shown that

$$\tau = \lim \tau_k \quad \text{as } k \to \infty \quad \text{exists a.s.} \tag{2}$$

such that

$$B(\tau) = X \quad \text{in dist.} \quad \text{and} \quad E\tau = 1. \tag{3}$$

In part (i) of the proof we describe the basic construction used by Billingsley. Refer to [**1**] for more details.

In part (ii) we show that under this construction the distribution of τ is uniquely determined independent of the Brownian motion and the space on which it is defined.

In part (iii) we obtain a stopping time σ as described in the statement of Theorem 2.1.

Part (i). We may assume that X is nondegenerate. Define $X_k = E[X|G_k]$ where G_k is constructed inductively. Define G_1 to be the σ-field generated by the partition $\{X \in (-\infty, 0], X \in (0, \infty)\}$. Denote G_1 as $\sigma\{X \in I_{1,j} : j = 1, 2\}$ where $I_{1,1}$ and $I_{1,2}$ are the intervals $(-\infty, 0]$ and $(0, \infty)$.

Denote by μ the measure on the Borel sets of $\mathbf{R}$ induced by X, i.e., $\mu = PX^{-1}$. For H, an interval in $\mathbf{R}$ with $\mu(H) > 0$ define $M(H)$ as

$$M(H) = (1/\mu(H)) \int_H x \, d\mu(x).$$

Denote the interior of H as H^0.

In general, suppose G_n has been defined as

$$G_n = \sigma\{X \in I_{n,k} : k = 1, 2, \dots, k_n\}$$

for some set of intervals $\{I_{n,k}\}$ partitioning $\mathbf{R}$. The σ-field G_{n+1} is defined as

$$G_{n+1} = \sigma\{X \in I_{n+1,k} : k = 1, 2, \dots, k_{n+1}\}$$

where $\{I_{n+1,k}\}$ is a set of intervals further partitioning $\mathbf{R}$. In particular, the set of intervals $\{I_{n+1,k}\}$ is obtained from the set of intervals $\{I_{n,k}\}$ in the following way: If $\mu(I^0_{n,k}) > 0$, subdivide $I_{n,k}$ with $M(I_{n,k})$ into two subintervals $I_{n+1,l}$ and $I_{n+1,m}$. If $\mu(I^0_{n,k}) = 0$, leave $I_{n,k}$ intact.

If $\mu(I_{n+1,k}) > 0$, let $X_{n+1}(w) = M(I_{n+1,k})$ for $w \in [X \in I_{n+1,k}]$. If $\mu(I_{n+1,k}) = 0$, we can arbitrarily assign $X_{n+1}(w) = X_n(w)$ for $w \in [X \in I_{n+1,k}]$.

For a discrete random variable Z, define $R(Z)$ to be the set of points where its distribution is concentrated.

Let $\{B(t) : t \geq 0\}$ be a Brownian motion on some probability space. We define a sequence of stopping times $\{\tau_n : n = 1, 2, \dots\}$ for $\{B(t)\}$ inductively. Let $\tau_0 = 0$. Then, defining τ_n as

$$\tau_n = \inf\{t \geq \tau_{n-1} : B(t) \in R(X_n)\} \tag{4}$$

Billingsley [**1**] obtains (1)–(3).

Part (ii). Let $\tau_1 = \tau^{(1)}$ and

$$\tau_n = \tau^{(1)} + \tau^{(2)} + \tau^{(3)} + \cdots + \tau^{(n)} \tag{5}$$

where $\tau^{(k)}$ is defined by

$$\tau_k = \tau_{k-1} + \tau^{(k)}. \tag{6}$$

Suppose $x \in R(X_k)$. Conditional on the set $[X_k = x]$, the distribution of X_{k+1} is concentrated at the points we denote u_x and v_x. By (1) and (4) it follows that the conditional distribution of $B(\tau_{k+1})$ on the set $[B(\tau_k) = x]$ is also concentrated at the points u_x and v_x.

Define the Brownian motion $\{B^{(k+1)}(t) : t \geq 0\}$ for $k = 1, 2, \dots, n-1$ as

$$B^{(k+1)}(t) = B(\tau_k + t) - B(\tau_k).$$

Let $x \in R(X_k)$. For $k = 1, 2, \dots, n-1$, define the stopping time $\tau^{(k+1)}[x]$ for $\{B^{(k+1)}(t) : t > 0\}$ by

$$\tau^{(k+1)}[x] = \inf\,[t \geq 0 : B^{(k+1)}(t) \in \{(u_x - x), (v_x - x)\}].$$

Let $r_i \geq 0$ for $i = 1, 2, \ldots, n$. We obtain

$$\begin{aligned} &P[\tau^{(k)} \leq r_k : k = 1, 2, \ldots, n] \\ &\quad = \sum_{x_1} \cdots \sum_{x_n} P[\tau^{(k)} \leq r_k, B(\tau_k) = x_k : k = 1, 2, \ldots, n] \\ &\quad = \sum_{x_1} \cdots \sum_{x_n} P[\tau_1 \leq r_1, B(\tau_1) = x_1, \tau^{(k+1)}[x_k] \leq r_k, \\ &\qquad\qquad B^{(k+1)}(\tau^{(k+1)}_{[x_k]}) = (x_{k+1} - x_k) : k = 1, \ldots, n-1] \end{aligned} \tag{7}$$

where x_i ranges over the set $R(X_i)$. Here we are using the fact that on the set $[B(\tau_k) = x_k]$ the equality $\tau^{(k+1)}[x_k] = \tau^{(k+1)}$ holds.

Using the independence of the random vectors

$$\{(\tau_1, B(\tau_1)), (\tau^{(k+1)}[x_k], B^{(k+1)}(\tau^{(k+1)}[x_k])) : k = 1, \ldots, n-1\}$$

(see [**1**, pp. 461–462]) for $x_k \in R(X_k)$, we obtain the equivalence of (7) to

$$\begin{aligned} \sum_{x_1} \cdots \sum_{x_n} P[\tau_1 \leq r_1, B(\tau_1) = x_1] \prod_{k=1}^{n-1} P[\tau^{(k+1)}[x_k] \\ \leq r_{k+1}, B^{(k+1)}(\tau^{(k+1)}[x_k]) = a(x_{k+1} - x_k)]. \end{aligned} \tag{8}$$

The probabilities within this summation are independent of the Brownian motion [**9**, 62] and hence the distribution of τ_n is uniquely determined.

Define $F(z) = P[Z \leq z]$ for a random variable Z. Clearly, $\tau_k \uparrow \tau$ a.s. Hence, for $t \in \mathbf{R}$, we may write $(\tau \leq t) = \cap_n (\tau_n \leq t)$ and therefore obtain

$$F_\tau(t) = \lim_n F_{\tau_n}(t).$$

Hence, the distribution of τ is uniquely determined, independent of the Brownian motion and the space on which it is defined.

Part (iii). Define the Brownian motion $(B^*(t), t \geq 0)$ by $B^*(t) = (1/a)B(a^2 t)$. Let τ^* be the stopping time derived using the construction described in parts (i) and (ii). Then $B^*(\tau^*) = X$ in dist. and $\tau^* = \tau$ in dist.

It follows that $B(a^2\tau^*) = aX$ in dist. Let σ be the stopping time for $\{B(t), t \geq 0\}$ defined by $\sigma = a^2\tau^*$. This stopping time satisfies the conditions described by the statement of Theorem 2.1. $\square$

Proof of Theorem 2.2. A modification of the proof of Theorem 37.7 in [**1**] is sufficient to prove the result. Let $\{B(t) : t \geq 0\}$ be a Brownian motion on some probability space. Define $\{B^{(1)}(t) : t \geq 0\}$ by $B^{(1)}(t) = B(t)$. Using the construction described in Theorem 2.1, a stopping time δ_1 exists such that

$$B^{(1)}(\delta_1) = a_1 X \quad \text{in dist.}$$

and

$$\delta_1 = a_1^2 \tau \quad \text{in dist.}$$

where τ is a stopping time with $E\tau = 1$.

Proceed inductively, using the construction of Theorem 2.1, to obtain stopping times $\{\delta_k : k = 1, 2, \dots\}$ for Brownian motions $\{B^{(k)}(t) : t \geq 0\}$, where

$$B^{(k+1)}(t) = B^{(k)}(\delta_k + t) - B^{(k)}(\delta_k) : k = 1, 2, \dots$$

such that

$$B^{(k)}(\delta_k) = a_k X_k \quad \text{in dist.} \tag{9}$$

Furthermore, it follows from the remark following the proof of Theorem 2.1 that

$$\delta_k = a_k^2 \tau \quad \text{in dist.}$$

The random vectors $\{(\delta_k, B^{(k)}(\delta_k)) : k = 1, 2, \dots\}$ are independent [**1**, 461–462]. Therefore, defining T_n as

$$T_n = \sum_{k=1}^{n} \delta_k \quad \text{for } n = 1, 2, \dots$$

it follows that

$$T_n = \sum_{k=1}^{n} a_k^2 \tau_k$$

for $\{\tau_k : k = 1, 2, \dots\}$ i.i.d. satisfying $\tau_k \geq 0$ and $E\tau_k = 1$. We observe that $B(T_n) = \sum_{k=1}^{n} B^{(k)}(\delta_k)$. Hence, by (9), the result follows. $\square$

3. Applications.

Let $\{X_k : k = 1, 2, \dots\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = 1$. Let the sequence of real numbers $\{a_n : n = 1, 2, \dots\}$ be given. Define A_n by

$$A_n^2 = \sum_{k=1}^{n} a_k^2 \tag{10}$$

and define the sum, S_n, of weighted i.i.d. random variables as $S_n = \sum_{k=1}^{n} a_k X_k$.

Consider the form of the random variables $\{T_n : n = 1, 2, \dots\}$ as described in Theorem 2.2. In particular, we can write T_n as

$$T_n = \sum_{k=1}^{n} a_k^2 \tau_k$$

where $\{\tau_k : k = 1, 2, \dots\}$ is a sequence of i.i.d. random variables with $\tau_k \geq 0$ and $E\tau_k = 1$.

We say that the "strong law holds for $\{T_n\}$" if

$$T_n / A_n^2 \to 1 \text{ a.s.} \quad \text{as } n \to \infty. \tag{11}$$

Lemma 3.1. *Sufficient conditions for the strong law,* (11), *to hold are that*

$$A_n^2 \to \infty$$

and

$$a_n^2 / A_n^2 = 0\left(\frac{1}{n}\right). \tag{A}$$

Proof. This follows as a special case in [**4**] and [**8**]. $\square$

In a paper by Chow and Teicher [**2**] that is referred to following Corollary 3.4, it is noted that condition (A) and $A_n^2 \to \infty$ include the cases

$$a_n = \pm n^{\beta}, \qquad -(1/2) \le \beta < \infty$$

and

$$a_n = \pm n^{\beta}(\log n)^{\alpha}, \qquad \beta > -(1/2) \quad \text{or} \quad \beta = -(1/2) \le \alpha < \infty,$$

and exclude exponential (geometric) growth.

Note. It is to be pointed out that the conclusions in Theorems 3.1–3.3 and Corollary 3.4 continue to hold if condition (A) in the hypotheses is replaced by any condition(s) ensuring that the strong law, (11), holds.

Also, we note that the results in this section are extensions of results for i.i.d. random variables, i.e., for $a_k \equiv 1$.

We first obtain a central limit theorem for weighted i.i.d. random variables.

Theorem 3.1. *If $A_n^2 \uparrow \infty$ and condition* (A) *holds, then*

$$\frac{S_n}{A_n} \to Z \text{ weakly} \quad \text{as } n \to \infty$$

where Z is a standard normal random variable.

Proof. Using Theorem 2.2 and Lemma 3.1, the proof follows exactly that in [**1**, p. 462]. □

Theorem 3.1 also follows as a special case of the Lindeberg Central Limit Theorem (see Chow and Teicher [**3**]).

Let $C[0,1]$ be the space of continuous functions on the closed interval $[0,1]$ with the uniform metric $\rho(x,y) = \sup_{0\le t\le 1} |x(t) - y(t)|$. Let K be the set of absolutely continuous $x \in C[0,1]$ with $x(0) = 0$ and $\int_0^1 \dot{x}^2(t)\, dt \le 1$.

Define the random function $S(r) : r \ge 0$ by linearly interpolating S_n on $[A_n^2, A_{n+1}^2]$ so that

$$S(r) = S_n + (r - A_n^2)(A_{n+1}^2 - A_n^2)^{-1} a_{n+1} X_{n+1} \quad \text{for } A_n^2 \le r \le A_{n+1}^2. \tag{12}$$

Now, define $U_n(t) \in C[0,1]$ for $n = 1, 2, \ldots$ as

$$U_n(t) = (2A_n^2 \log_2 A_n^2)^{-1/2} S(A_n^2 t).$$

We now prove a functional LIL for weighted i.i.d. random variables.

Theorem 3.2. *If $A_n^2 \uparrow \infty$ and condition* (A) *holds, then with probability one the sequence $\{U_n : n = 1, 2, \ldots\}$ is relatively compact and the set of its a.s. limit points coincides with K.*

Proof. The theorem may be proved, with some modifications, in the manner of Strassen [**13**] or Stout [**12**, pp. 291–293] in the i.i.d. case.

However, a generalization of the argument has been developed by Hall and Heyde [**5**, Theorem B, p. 119]. We will show that their result can be applied here.

Construct the sequence $\{U_n\}$ on a possibly different probability space using the Skorohod representation of Theorem 2.2. In particular, redefine the sequence $\{S_n : n = 1, 2, \ldots\}$ using $\{S_n^* : n = 1, 2, \ldots\}$ in its place where $S_n^* = B(T_n)$.

To apply Theorem B in Hall and Heyde [**5**] it is sufficient to show that the conditions

$$T_n \to \infty \text{ a.s.,} \tag{13}$$

$$T_n^{-1} A_n^2 \to 1 \text{ a.s.} \tag{14}$$

and

$$T_{n+1}^{-1} T_n \to 1 \text{ a.s.} \tag{15}$$

hold. It follows immediately that these conditions hold using condition (A) and its implication that (11) holds and that $A_{n+1}^2 (A_n^2)^{-1} \to 1$. □

Note. In the verification that (13)–(15) hold, condition (A) is used to imply (11) and that $A_{n+1}^2 (A_n^2)^{-1} \to 1$. That $A_{n+1}^2 (A_n^2)^{-1} \to 1$ also follows from $A_n^2 \uparrow \infty$ and (11) (see Jamison et al. [**8**, p. 40] for a standard argument of this).

An integral part of the proof of Theorem B in [**5**, p. 120] is the establishment of an almost sure invariance principle that translates here to one for weighted i.i.d. random variables.

Theorem 3.3. *If* $A_n^2 \uparrow \infty$ *and condition* (A) *holds, then on some probability space, one can define a Brownian motion* $\{B(r) : r \geq 0\}$ *and redefine* $\{S(r) : r \geq 0\}$, (12), *without changing its distribution so that*

$$\lim_{r\to\infty} \frac{S(r) - B(r)}{(r \log_2 r)^{1/2}} = 0 \ a.s.$$

The following LIL follows as a corollary of Theorem 3.2.

Corollary 3.4. *If* $A_n^2 \uparrow \infty$ *and condition* (A) *holds, then*

$$\limsup_{n\to\infty} \frac{S_n}{(2A_n^2 \log_2 A_n^2)^{1/2}} = 1 \ a.s.$$

Proof. See [**5**, Theorem 4.8] for a standard proof that the functional LIL implies the classical LIL. □

The LIL of Corollary 3.4 has also been obtained through a classical proof by Chow and Teicher [**2**, Theorem 1] and can be obtained as a special case of a classical LIL for martingales proved by Tomkins [**16**].

As a note of interest, Teicher has shown that the LIL fails for $\{a_n\}$ of geometric growth (see [**15**, Theorem 5]). We refer the interested reader to a paper by Rosalsky [**10**] where $\{a_n\}$ of this type is considered.

We conclude with a simple example that satisfies the hypotheses of the results here for weighted i.i.d. random variables but fails to satisfy those for martingales, as discussed in Section 1.

Example. Let $P[X = \pm\sqrt{n}] = C/(n^2 \log n (\log_2 n)^2)$, for $n = 3, 4, \dots$. With $a_k = 1$ for $k = 1, 2, \dots$ and $\{X_i : i = 1, 2, \dots\}$ i.i.d. distributed as X, condition (25) of Theorem 3.1 in [**7**] fails as does condition (138) of Theorem 4.4 in [**14**].

With X as defined above and with $a_k = k^{1/2}$ for $k = 1, 2, \ldots,$ one can verify that condition (1) of [**6**, Theorem 1] does not hold.

However, the hypotheses of the theorems in this paper clearly hold for these examples.

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