ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

A BASIC CONSTRUCTION IN DUALS OF SEPARABLE BANACH SPACES

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ABSTRACT. A basic construction of the Cantor set Δ in the dual of a separable Banach space X is presented. If X^* is nonseparable, a modification of this construction yields bounded ε-trees in X^* (Stegall). A continuous linear surjection from X to $C(\Delta)$ is obtained if ℓ^1 embeds in X (Pelczynski) by a further modification of this construction. Through it the delicate nature of the difference between the cases (i) X^* is nonseparable and (ii) ℓ^1 embeds in X is highlighted.

A. Introduction. Let Δ^0 denote the usual Cantor set with dyadic partitions $(C^0_{ni} : i = 1, \dots, 2^n)_{n=0}^\infty$ and Haar measure λ^0 (where $\lambda^0(C^0_{ni}) = 2^{-n}$ for all i and n). Let $\lambda^0_{ni}(\cdot) = 2^n\lambda^0((\cdot) \cap C^0_{ni})$.

Now let Δ denote the natural copy of Δ^0 in $C(\Delta^0)^*$, the points of Δ corresponding to point-masses on $C(\Delta^0)$. Let λ_{ni} denote λ^0_{ni} as a measure on Δ. We think of λ^0_{ni} in $C(\Delta^0)^*$ as the barycenter of the measure λ_{ni} on Δ. Note that the λ^0_{ni}'s form a bounded ε-tree, with $\varepsilon = 2$, as $\lambda^0_{ni} = (1/2)\ (\lambda^0_{n+12i-1} + \lambda^0_{n+1,2i})$ and $||\lambda^0_{n+1,2i-1} - \lambda^0_{n+1,2i}|| = 2$.

Now suppose X is a separable Banach space and X^* is nonseparable. Then it is easy (see Corollary 2 below) to construct a topological copy of Δ in (B^*, weak*) which is norm discrete (and conversely the existence of such a set obviously implies X^* is nonseparable). C. Stegall [**7**] showed how to construct such a Δ and corresponding dyadic partitions (C_{ni}), with Haar measure λ, so that the barycenters x^*_{ni} of the measures $\lambda_{ni}(\cdot) = 2^n\lambda((\cdot) \cap C_{ni})$ on Δ form a bounded ε-tree in X^*.

On the other hand, the Pelczynski–Hagler theorem states that ℓ^1 embeds in a separable Banach space X if and only if there exists a continuous linear surjection from X to $C(\Delta^0)$ [**3, 4**]. In this paper a basic construction is presented which obtains these two results and highlights the delicate differences between them.

This work forms part of the author's Ph.D. thesis at the Pennsylvania State University.
Received by the editors on March 27, 1986.

Basically our notation follows that of [**2**]. Throughout, X is a separable Banach space and $X^*, X^{**}, \dots$ the successive duals of X. Let $B, B^*, \dots$ be the closed unit ball of $X, X^*, \dots$. We say that a Banach space Y embeds in a Banach space X (or equivalently X contains a copy of Y) if there exists an isomorphism from Y into X.

A sequence $(x_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ in X is called a *tree* if $x_{ni} = (1/2)(x_{n+1,2i-1} + x_{n+1,2i})$ for all n, i. If we also have that $||x_{n+1,2i-1} - x_{n+1,2i}|| > \varepsilon$ for some positive ε and for all n, i, then $(x_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ is called an *ε-tree*. An *ε-Rademacher tree* (Riddle, Uhl [**5**]) is a tree such that $|| \sum_{i=1}^{2^n} (-1)^i x_{ni} || \geq 2^n \varepsilon$ for every n.

B. The basic construction. The following is a standard result from topology. Its proof is the core of the constructions in this paper.

Lemma 1. *Let A be an uncountable subset of a compact metric space M. Then $\bar{A}$, the closure of A in M, contains a subset Δ homeomorphic to Δ^0.*

Proof. Let ρ be the metric on M and $B(x, \alpha) = \{y \in M : \rho(x, y) < \alpha\}$. As M is second countable, all but countably many points of any uncountable subset are condensation points. We build by induction on n, a sequence $(A_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$, of subsets of A with the following properties for $n = 1, 2, \dots,\ i = 1, \dots, 2^n$:

(i) $A_{n+1,2i-1} \cup A_{n+1,2i} \subseteq A_{n,i}$.

(ii) For fixed n, $\overline{A_{ni}} \cap \overline{A_{nj}} = \phi$ if $i \neq j$.

(iii) The diameter of $\overline{A_{ni}} < 2^{-n}$.

(iv) Each A_{ni} is uncountable.

Then having done so it is clear that $\Delta = \cap_{n=0}^{\infty} \cup_{i=1}^{2^n} \overline{A_{ni}}$ is homeomorphic to Δ^0 with $C_{ni} = \Delta \cap \overline{A_{ni}}$ homeomorphic to the dyadic intervals.

Let x_{01} be a condensation point of A and let $A_{01} = A \cap B(x_{01}, 1)$.

Now suppose the construction has been made for all n, i for $n = 0, \dots, m$. Choose $x_{m+1,2i-1}, x_{m+1,2i}$ to be weak* condensation points of A_{mi}. As M is a metric space we can easily find neighborhoods $U_{m+1,2i-1}$ of $x_{m+1,2i-1}$ and $U_{m+1,2i}$ of $x_{m+1,2i}$ so that $\overline{U_{m+1,2i-1}} \cap$

$\overline{U_{m+1,2i}} = \phi$. Then define

$$A_{m+1,2i-1} = A_{mi} \cap B(x_{m+1,2i-1}, 2^{-m+1}) \cap U_{m+1,2i-1}$$
$$A_{m+1,2i} = A_{mi} \cap B(x_{m+1,2i}, 2^{-m+1}) \cap U_{m+1,2i}.$$

It is clear that these subsets indeed satisfy (i) through (iv) above. □

An immediate consequence of the above lemma is that B^* contains a subset weak* homeomorphic to the Cantor set whenever X is a separable Banach space. In fact, the main results of this chapter are obtained by judiciously selecting an uncountable set A in B^* and more carefully constructing copies of Δ^0 in $\bar{A}$ as in Lemma 1. For instance, one can easily show the following. (We prove a stronger result in the next section.)

Corollary 2. *If X is a separable Banach space such that X^* is nonseparable, then there is a norm discrete subset Δ of B^* that is weak* homeomorphic to Δ^0.*

Notation. Let $W(x^*; x, \varepsilon) = \{y^* \in X^* : |x^*(x) - y^*(x)| < \varepsilon\}$. If A is a subset of X^*, let $\bar{A}$ denote the weak* closure of A. Clearly, if $A \subset W(x^*; x, \varepsilon)$ and $z^* \in \bar{A}$, then $|z^*(x) - x^*(x)| \leq \varepsilon$.

C. The case when X^* is nonseparable. In this section we considerably simplify the published proofs of Stegall's theorem [**7, 2**]. We still need the following lemma from [**7**].

Lemma 3. *Let Y be a nonseparable Banach space and let ω_1 be the first uncountable ordinal number. Then for every $\varepsilon > 0$, there exist sets $\{y_\alpha : \alpha < \omega_1\}$ in Y and $\{y^*_\alpha : \alpha < \omega_1\}$ in Y^* such that for all $\alpha, \beta < \omega_1$, $||y_\alpha|| = 1$, $||y^*_\alpha|| < 1 + \varepsilon$ and*

$$y^*_\beta(y_\alpha) = \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Proof. Choose $y_1 \in Y$ and $y^*_1 \in Y^*$ such that $||y_1|| = ||y^*_1|| = y^*_1(y_1) = 1$. Let $\beta < \omega_1$. Assume that we have made the construction for all $\alpha < \beta$. Since $\{y_\alpha : \alpha < \beta\}$ spans a separable subspace of the nonseparable space Y, there exists a $y^*_\beta \in Y^*$ such that $y^*_\beta(y_\alpha) = 0$ for all $\alpha < \beta$ and $||y^*_\beta|| = 1 + \varepsilon/2$. Then choose $y_\beta \in Y$ such that $||y_\beta|| = 1 = y^*_\beta(y_\beta)$. □

Theorem 4. (Stegall [**7**]). *Let X be a separable Banach space such that X^* is nonseparable. Then for every $\varepsilon > 0$, there exists a subset Δ of B^* which is weak* homeomorphic to the Cantor set, along with subsets $(C_{ni})_{n=0}^{\infty}\,{}_{i=1}^{2^n}$ of Δ weak* homeomorphic to the dyadic intervals, and a sequence $\{x_{ni}\}_{n=0}^{\infty}\,{}_{i=1}^{2^n}$ in X such that $||x_{ni}|| < 1+\varepsilon$ for all n, i and*

$$|x^*(x_{ni}) - \chi_{C_{ni}}(x^*)| \le \varepsilon 2^{-n} \quad \textit{for all } x^* \in \Delta.$$

Proof. Let $\varepsilon > 0$ be given. Use Lemma 3 to find sets $A = \{x^*_\alpha : \alpha < \omega_1\}$ in X^* and $\{x^{**}_\alpha : \alpha < \omega_1\}$ in X^{**} such that $||x^*_\alpha|| = 1$, $||x^{**}_\alpha|| < 1+\varepsilon$ and

$$x^{**}_\beta(x^*_\alpha) = \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Claim. If $A_1, \dots, A_n$ are uncountable disjoint subsets of A and $1 \le j \le n$, then for every $\eta > 0$ there exist uncountable sets $A'_i \subset A_i$ $(i = 1, \dots, n)$ and a point x_j in X with $||x_j|| < 1 + \varepsilon$ and such that

$$|x^*(x_j) - \chi_{A'_j}(x^*)| < \eta \quad \text{for all } x^* \in \bigcup_{i=1}^{n} A'_i.$$

To prove the claim, fix j and for $i = 1, \dots, n$, let $x^*_{\beta_i}$ be a weak* condensation point of A_i, such that $\beta_j > \beta_i$ if $i \ne j$. Then $||x^{**}_{\beta_j}|| < 1+\varepsilon$ and $x^{**}_{\beta_j}(x^*_{\beta_i}) = \delta_{ij}$. By the weak* density of B in B^{**}, choose x_j in X with $||x_j|| < 1 + \varepsilon$ and $|x_j(x^*_{\beta_i}) - x^{**}_{\beta_j}(x^*_{\beta_i})| < \eta/2$ $(i = 1, \dots, n)$. Let

$$A'_i = A_i \cap W(x^*_{\beta_i}; x_j, \eta/2).$$

This proves the claim.

To prove the theorem, apply the construction of Lemma 1 to A with the following change. At each stage, where $A_{n1}, \dots, A_{n2^n}$ have been chosen, inductively apply the above claim 2^n times to choose x_{ni} and uncountable $B_{ni} \subset A_{ni}$ $(i = 1, \ldots, 2^n)$ with $||x_{ni}|| < 1 + \varepsilon$ and such that for each i

$$|x^*(x_{ni}) - \chi_{B_{ni}}(x^*)| < \varepsilon 2^{-n} \quad \text{for all } x^* \text{ in } \bigcup_{j=1}^{2^n} B_{nj}.$$

Then replace each A_{ni} by B_{ni} and continue the construction. □

Let Δ and $(C_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ be as constructed in the above theorem (for a given ε, $0 < \varepsilon < 1/4$). Since Δ is weak* homeomorphic to Δ^0, the natural evaluation map $T : X \to C(\Delta^0)$ given by $T(x)(x^*) = x^*(x)$ is a continuous linear operator. Also, having a sequence $(x_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ in X which approximates $(\chi_{C_{ni}})_{n=0}^{\infty}{}_{i=1}^{2^n}$, it is easy to see that T maps X onto a dense subspace of $C(\Delta^0)$. In general, however, T cannot map X onto all of $C(\Delta^0)$, for this would imply X contains ℓ^1 as the remarks at the beginning of this chapter indicated. We would like to be able to say that this evaluation mapping has some property that characterizes separable spaces X with nonseparable duals. Of course, the first thing that comes to mind is that T maps X onto a dense subspace of $C(\Delta^0)$, However, this does not characterize separable spaces with nonseparable duals as the following example shows.

Example 5. Define $T : \ell^2 \to C(\Delta^0)$ by $T((\alpha_n)) = \sum_{n=1}(1/n)\alpha_n t^n$. As the range of T clearly contains the polynomials, it is dense in $C(\Delta^0)$. Clearly though, $(\ell^2)^* = \ell^2$ is separable.

Now consider the adjoint of T, $T^* : C(\Delta^0)^* \to X^*$. Let λ_{ni}^0 be defined as before ($\lambda_{ni}^0(\cdot) = 2^n\lambda^0((\cdot) \cap C_{ni})$ where λ^0 is then Haar measure on Δ^0). Let $x_{ni}^* = T^*(\lambda_{ni}^0)$. Clearly, (x_{ni}^*) forms a bounded tree in X^*. In fact, since $0 < \varepsilon < 1/4$, (x_{ni}^*) forms a bounded 2/5-tree in X^*. We have

$$\begin{aligned} ||x^*_{n+1,2i-1} - x^*_{n+1,2i}|| &= \sup_{x \in B} |x^*_{n+1,2i-1}(x) - x^*_{n+1,2i}(x)| \\ &\geq \frac{4}{5}|x^*_{n+1,2i-1}(x_{n+1,2i-1}) - x^*_{n+1,2i}(x_{n+1,2i-1})| \end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{5}|T^*(\lambda^0_{n+1,2i-1} - \lambda^0_{n+1,2i})(x_{n+1,2i-1})| \\
&= \frac{4}{5}\bigg| \int_\Delta x^*(x_{n+1,2i-1})\, d\lambda_{n+1,2i-1} \\
&\qquad - \int_\Delta x^*(x_{n+1,2i-1})\, d\lambda_{n+1,2i} \bigg| \\
&\ge \frac{4}{5}\left|\frac{3}{4} - \frac{1}{4}\right| = \frac{2}{5}.
\end{aligned}
$$

Let us summarize as follows.

Theorem 6. *Let X be a separable Banach space. Then the following are equivalent*:

(i) *X^* is nonseparable.*

(ii) *For every $\varepsilon > 0$, there exists a subset Δ of B^* that is weak* homeomorphic to Δ^0 and a sequence $(x_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ in X with $||x_{ni}|| < 1 + \varepsilon$ such that*

$$|x^*(x_{ni}) - \chi_{C_{ni}}(x^*)| < \varepsilon 2^{-n} \quad \textit{for all } x^* \in \Delta,$$

where the C_{ni}'s are homeomorphic to the dyadic intervals. Hence, there exists a $\delta > 0$ and a continuous linear operator $T : X \to C(\Delta^0)$ such that $T^*(C(\Delta^0)^*)$ contains a bounded δ-tree.

(iii) *There exists for every $\varepsilon > 0$ a subset Δ of B^*, weak* homeomorphic to Δ^0 such that for every $x^* \in \Delta^0$, there is an x^{**} in X^{**} with $||x^{**}|| < 1 + \varepsilon$ such that $x^{**}(x^*) = 1$ and $x^{**}(y^*) = 0$ for all y^* in Δ, $y^* \neq x^*$.*

(iv) *There exists a subset Δ of B^* that is weak* homeomorphic to Δ^0, but is discrete in the weak topology.*

Proof. (i) $\Rightarrow$ (ii). See Theorem 4 and the remarks following its proof.

(ii) $\Rightarrow$ (iii). Let $\varepsilon > 0$ and let Δ be the copy of Δ^0 satisfying the conditions of (ii). Let x^* be in Δ, and let (i_n) be the unique sequence such that $x^* \in A_{ni_n}$. Let x^{**} be any weak* cluster point in X^{**} of the sequence $\{x_{ni_n}\}$. Then $||x^{**}|| < 1 + \varepsilon$ and, since $|x^*(x_{ni_n}) - 1| < \varepsilon 2^{-n}$, the sequence $x^*(x_{ni_n})$ converges to 1 but clusters at $x^{**}(x^*)$. Consequently, $x^{**}(x^*) = 1$. Now if $y^* \in \Delta$ but $y^* \neq x^*$, then, for some N, if $n \geq N$ then y^* is not in A_{ni_n}. Therefore, $|y^*(x_{ni_n})| < \varepsilon 2^{-n}$ for $n \geq N$ and clearly $x^{**}(y^*) = 0$.

(iii) $\Rightarrow$ (iv). For any fixed $x^* \in \Delta$, let x^{**} be as in (iii). Then $\{x^*\} = \Delta \cap \{y^* : x^{**}(y^*) > 0\}$, so $\{x^*\}$ is weak open in Δ.

(iv) $\Rightarrow$ (i). As Δ is uncountable and weak discrete it is also norm discrete and, consequently, X^* is nonseparable. □

D. The case when ℓ^1 embeds in X. The Pelczynski–Hagler theorem states that a separable Banach space X contains a copy of ℓ^1 if and only if there is a continuous linear surjection from X to $C(\Delta^0)$. The standard proof of this uses the following fact [**4**].

Theorem (Pelczynski). *If a separable Banach space Z contains a subspace Z_1 isomorphic to $C(\Delta)$, then there is a subspace $Z_2 \subset Z_1$ such that Z_2 is isomorphic to $C(\Delta)$ and complemented in Z.*

Here we obtain the Pelczynski–Hagler theorem directly by modifying the construction of Lemma 1.

For each n and dyadic partition $C_{n1}, \dots , C_{n2^n}$ of Δ^0, there are 2^{2^n} different continuous functions $(\varphi_{nj})_{j=1}^{2^{2^n}}$, $\varphi_{nj} : \Delta^0 \to \{-1, 1\}$ that are constant on C_{ni}, $i = 1, \dots , 2^n$. Let $(\sigma^{(j)})_{j=1}^{2^{2^n}}$ be an enumeration of all possible choices of $\sigma = (\sigma_1, \dots , \sigma_{2^n})$, where $\sigma_i = \pm 1$. We can identify each φ_{nj} with a $\sigma^{(j)}$ as follows:

$$\varphi_{nj} = \sum_{i=1}^{2^n} \sigma_i^{(j)} \chi_{C_{ni}}.$$

These functions are called *Rademacher-type functions*. Recall that Theorem 4 loosely says that if X^* is nonseparable, then we can construct a copy of Δ^0 in (B^*, weak^*), and a bounded sequence $(x_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ in X such that for all n, i, x_{ni} as a function on Δ^0 approximates $\chi_{C_{ni}}$. Consequently, the point $w_{nj} = \sum_{i=1}^{2^n} \sigma_i^{(j)} x_{ni}$ approximates φ_{nj} on Δ^0. However, we have no control over the norm of w_{nj}. The idea in the following is that we can approximate the φ_{nj}'s by a bounded sequence in X whenever ℓ^1 embeds in X.

Lemma 7 [**4**]. *If X is a separable Banach space containing ℓ^1, then $\ell^1(\mathbf{R})$ embeds in X^*.*

Proof. We first show that $\ell^1(\mathbf{R})$ embeds in ℓ^∞ isometrically. Let D denote the collection of all $((I_i, \varepsilon_i) : i \in F)$, where F is finite, $(I_i)_{i\in F}$ are disjoint intervals in $\mathbf{R}$ with rational endpoints and $\varepsilon_i = \pm 1$. Hence, D is countable and $\ell^\infty = \ell^\infty(D)$. Define $T : \ell^1(\mathbf{R}) \to \ell^\infty(D)$ by

$$T_x((I_i, \varepsilon_i) : i \in F) = \sum_{i\in F} \varepsilon_i \sum_{\alpha \in I_i} x(\alpha).$$

$|T_x((I_i, \varepsilon_i)) : i \in F| \leq ||x||_1$, and, if x has finite support, $||T_x||_\infty = ||x||_1$. Hence, T is an isometry.

Let ε_λ be the unit basis vector at λ in $\ell^1(\mathbf{R})$, and let $z_\lambda \in \ell^\infty$ be such that $T(e_\lambda) = z_\lambda$. Let $S : \ell^1 \to X$ be an isomorphic embedding such that $||S|| \leq 1$. Then $S^* : X^* \to \ell^\infty$ is onto and hence open. Thus, there exists a constant M and $x^*_\lambda \in X^*$ such that $S^*(x^*_\lambda) = z_\lambda$ and $||x^*_\lambda|| \leq M$. We will show that $(x^*_\lambda)_{\lambda \in \mathbf{R}}$ is isomorphic to the unit vector basis of $\ell^1(\mathbf{R})$.

Let $x^*_{\lambda 1}, \dots, x^*_{\lambda n}$, and scalars $\alpha_{\lambda 1}, \dots, \alpha_{\lambda n}$ be given. Let δ_p denote the pth unit vector basis of ℓ^1. Then

$$\begin{aligned} M \sum_{i=1}^n |\alpha_{\lambda i}| &\geq || \sum_{i=1}^n \alpha_{\lambda i} x^*_{\lambda i} || \geq \sup_p \left| \sum_{i=1}^n \alpha_\lambda \left\langle x^*_{\lambda i}, S(\delta_p) \right\rangle \right| \\ &= \sup_p \left| \sum_{i=1}^n \alpha_{\lambda i} \left\langle z_{\lambda i}, \delta_p \right\rangle \right| = \left\| \sum_{i=1}^n \alpha_{\lambda i} z_{\lambda i} \right\|_\infty \\ &= \left\| T\left(\sum_{i=1}^n \alpha_{\lambda i} e_{\lambda i} \right) \right\| = \sum_{i=1}^n |\alpha_{\lambda i}|. \qquad \square \end{aligned}$$

Theorem 8. *Let X be a separable Banach space such that ℓ^1 embeds in X. Then, for every $\varepsilon > 0$, there exists a subset Δ of B^* that is weak* homeomorphic to the Cantor set along with subsets $(C_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ of Δ that are weak* homeomorphic to the dyadic intervals and a bounded sequence $\{(w_{nj}) : j = 1, \dots, 2^{2^n}\}_{n=0}^\infty$ in X such that*

$$|x^*(w_{nj}) - \varphi_{nj}(x^*)| < \varepsilon 2^{-n} \quad \textit{for all } x^* \in \Delta.$$

Proof. Use Lemma 7 to find a norm 1 isomorphism $T : \ell^1(\mathbf{R}) \to X^*$. Hence, $T^* : X^{**} \to \ell^\infty(\mathbf{R})$ is onto and open. So there exists $M < \infty$ such that $T^*(MB^{**})$ covers the unit ball of $\ell^\infty(\mathbf{R})$. Let $(e_\lambda)_{\lambda \in \mathbf{R}}$ be the usual basis for $\ell^1(\mathbf{R})$, $x^*_\lambda = T(e_\lambda)$ and $A = (x^*_\lambda)_{\lambda \in \mathbf{R}}$.

Claim. If $A_1, \dots, A_n$ are uncountable disjoint subsets of A and if $\sigma = (\sigma_i)_{i=1}^n$ is such that $\sigma_i = \pm 1$ for each i, then for every $\eta > 0$ there exist uncountable sets $A'_i \subset A_i$ $(i = 1, \dots, n)$ and a point w_σ in X such that $||w_\sigma|| \le M$ and

$$\left| x^*(W_\sigma) - \sum_{i=1}^n \sigma_i \chi_{A'_i}(x^*) \right| < \eta \quad \text{for all } x^* \in \bigcup_{i=1}^n A'_i.$$

To prove the claim, fix $\sigma = (\sigma_i)_{i=1}^n$ and define $z \in \ell^\infty(\mathbf{R})$ by

$$z_\lambda = \begin{cases} \sigma_i & \text{if } x^*_\lambda \in A_i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $||z|| = 1$ and so there exists x^{**} in X^{**} with $||x^{**}|| \le M$ such that $T^*(x^{**}) = z$. Hence, if $x^*_\lambda \in A_i$,

$$x^{**}(x^*_\lambda) = x^{**}(T(e_\lambda)) = T^*(x^{**})(e_\lambda) = z(e_\lambda) = \sigma_i.$$

Choose x^*_i to be weak* condensation points of A_i, and by weak* density of B in B^{**}, choose w_σ in X such that $||w_\sigma|| \le M$ and $|x^*_i(w_\sigma) - \sigma_i| = |x^*_i(w_\sigma) - x^{**}(x^*_i)| < \eta/2$ $(i = 1, \dots, n)$. Let

$$A'_i = A_i \cup W(x^*_i, w_\sigma, \eta/2).$$

This proves the claim.

To prove the theorem, apply the construction of Lemma 1 to A with the following change. At each stage, when $A_{n1}, \dots, A_{n2^n}$ have been chosen, inductively apply the above claim 2^{2^n} times to choose w_{nj} and uncountable $B_{ni} \subset A_{ni}$ $(i = 1, \dots, 2^n; j = 1, \dots, 2^{2^n})$ with $||w_{nj}|| \le M$ and such that for each of the 2^{2^n} choices of $\sigma^{(j)}$,

$$\left| x^*(w_{nj}) - \sum_{i=1}^{2^n} \sigma_i^{(j)} \chi_{Bni}(x^*) \right| < \varepsilon 2^{-n} \quad \text{for all } x^* \in \bigcup_{i=1}^{2^n} B_{ni}.$$

Then replace each A_{ni} by B_{ni} and continue the construction. □

Let $\mu \in C(\Delta^0)^*$. It is clear that $||\mu|| = \sup_{n,\sigma} |\mu(\varphi_{n\sigma})|$. Let Δ be the copy of the Cantor set constructed in Theorem 8, and let $T : X \to C(\Delta^0)$ be the evaluation map given by $T(x) = x^*(x)$ for all $x^* \in \Delta$. Again, as Δ is weak* homeomorphic to the Cantor set, T is a continuous linear operator. Let $T^* : C(\Delta^0)^* \to X^*$ be the adjoint of T. If $\mu \in C(\Delta^0)^*$, then

$$\begin{aligned} ||T^*(\mu)|| &= \sup_{x\in B} |T^*(\mu)(x)| \geq \frac{1}{m} \sup_{n,j} |T^*(\mu)(w_{nj})| \\ &= \frac{1}{m} \sup_{n,j} \left| \int_\Delta x^*(w_{nj})\, d\mu \right| \\ &\geq \frac{1}{m} \sup_{n,j} \left[\left| \int_\Delta \varphi_{nj}(x^*)\, d\mu \right| - \varepsilon 2^{-n} \right] \\ &= \frac{1}{m} ||\mu||. \end{aligned}$$

Hence, T^* is an isomorphism of $C(\Delta^0)^*$ into X^* and, consequently, T is an onto map. This yields the following.

Corollary 9. *If X is a separable Banach space such that ℓ^1 embeds in X, then there exists a continuous linear surjection from X to $C(\Delta^0)$.*

We conclude this paper with

Theorem 10. *Let X be a separable Banach space. Then the following are equivalent*:

(i) *ℓ^1 embeds in X.*

(ii) *$\ell^1(\Gamma)$ embeds in X^*, where Γ is some uncountable set.*

(iii) *For every $\varepsilon > 0$, there exists a subset Δ of B^*, weak* homeomorphic to the Cantor set and a bounded sequence $\{(w_{nj}) : j = 1, \dots, 2^{2^n}\}_{n=0}^\infty$ in X such that for every Rademacher-type function φ_{nj}, $|x^*(w_{nj}) - \varphi_{nj}(x^*)| < \varepsilon 2^{-n}$ for all x^* in Δ.*

(iv) *There exists a continuous linear surjection from X to $C(\Delta^0)$.*

(v) *There exists an isomorphism from* $C(\Delta^0)^*$ *into* X^*. *Consequently,* X^* *contains a bounded* ε*-Rademacher tree.*

Proof. (i) $\Rightarrow$ (ii). See Lemma 7.

(ii) $\Rightarrow$ (iii). The proof of Theorem 8 will clearly work if $A = (T(e_\lambda))_{\lambda\in\Gamma}$, where $(e_\lambda)_\lambda$ is the usual basis for $\ell^1(\Gamma)$ and T is a norm 1 isomorphism of $\ell^1(\Gamma)$ into X^*.

(iii) $\Rightarrow$ (i). Let Δ, (w_{nj}), (φ_{nj}) be as in statement (iii). A natural subsequence of the Rademacher type functions (φ_{nj}) is the sequence of Rademacher functions $(r_n)_{n=1}^\infty$ defined by $r_n = \sum_{i=1}^{2^n}(-1)^{i+1}\chi_{C_{ni}}$ for all n. Let $(z_n)_{n=1}^\infty$ be the subsequence of (w_{nj}) that approximates the r_n's, i.e., such that $|x^*(z_n) - r_n(x^*)| < \varepsilon 2^{-n}$ for all $x^* \in \Delta$. It suffices to show that $(z_n)_{n=1}^\infty$ is isomorphic to the usual ℓ^1-basis.

Let M be such that $||z_n|| \le M < \infty$, and let $(\alpha_i)_{i=1}^k$ be a finite sequence of scalars. Then

$$M\sum_{i=1}^k |\alpha_i| \ge \left\|\sum_{i=1}^k \alpha_i z_i\right\| = \sup_{x^*\in B^*}\left|\sum_{i=1}^k \alpha_i x^*(z_i)\right| \ge \sup_{x^*\in\Delta}\left|\sum_{i=1}^k \alpha_i x^*(z_i)\right|.$$

Choose x^* in the appropriate C_{kj} to ensure $x^*(z_i)\alpha_i > |\alpha_i|/2$ for all i. Hence, $M\sum_{i=1}^k |\alpha_i| \ge ||\sum_{i=1}^k \alpha_i z_i|| > (1/2)\sum_{i=1}^k |\alpha_i|$.

(i) $\Rightarrow$ (iv). See Corollary 9.

(iv) $\Rightarrow$ (v). It is easy to see that $(\lambda_{ni})_{n=0}^{\infty}{}_{i=1}^{2^n}$ defined at the beginning of this paper forms a bounded 1-Rademacher tree in $C(\Delta^0)^*$. If T is a surjection from X to $C(\Delta^0)$, then T^* is an isomorphism from $C(\Delta^0)^*$ into X^*. Clearly, isomorphic images of ε-Rademacher trees are ε'-Rademacher trees and, hence, X^* contains a bounded ε'-Rademacher tree.

(v) $\Rightarrow$ (ii). Since $\ell^1(\Delta)$ embeds in $C(\Delta^0)^*$, it must also embed in X^*. $\square$

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