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EXTENSION OF TOPOLOGICAL INVARIANT MEANS ON A LOCALLY COMPACT AMENABLE GROUP

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1. Introduction. Let G be a locally compact group associated with its left Haar measure. For a Borel set $A \subset G$ we denote by $|A|$ the measure of A. Let $L^\infty(G)$ be the Banach space of all essentially bounded Borel measurable functions on G, and let $CB(G)$ be that of all bounded continuous functions. For a function f in $CB(G)$, we say that f is left uniformly continuous if, given $\varepsilon > 0$, there is a neighborhood U of e, the identity in G, such that

$$|f(x) - f(xy)| < \varepsilon, \quad x \in G,\ y \in U.$$

The space of all left uniformly continuous bounded functions is denoted by $UCB_l(G)$. The right uniform continuity and the space $UCB_r(G)$ are defined symmetrically. The space of all uniformly continuous bounded functions is defined by $UCB(G) = UCB_l(G) \cap UCB_r(G)$. These spaces are all considered as subspaces of $L^\infty(G)$, with the supremum norm $||\cdot||_\infty$.

For each $x \in G$ and $f \in L^\infty(G)$, we define a new function ${}_xf \in L^\infty(G)$ by ${}_xf(t) = f(x^{-1}t)$ for all $t \in G$. For a closed subspace X of $L^\infty(G)$, we say that X is (left) translation invariant if $f \in X$ implies that ${}_xf \in X$ for all $x \in G$. Each of the above spaces is (two sided) translation invariant. For a translation invariant space X containing $UCB(G)$, we define a left invariant mean μ on X to be a positive element in X^* ($\mu(f) \geq 0$ if $f \in X$ is nonnegative) of norm 1 such that $\mu({}_xf) = \mu(f)$ for all f in X and $x \in G$. The existence of an invariant mean on $L^\infty(G)$, or on $UCB(G)$, or on any intermediate space is equivalent. If G admits an invariant mean on any of these spaces, we say G is amenable. Let G_d be the same algebraic group as G with a discrete topological structure. If G_d admits a left invariant

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mean on $l^\infty(G)$, we say that G is amenable as discrete. A group which is amenable as a discrete group is itself amenable. Properties of amenable groups and left invariant means can be found in Greenleaf [**2**] and Pier [**7**].

Hulanicki [**3**] introduced the topological invariance of a mean. Let X be a translation invariant subspace of $L^\infty(G)$ such that $g * f \in X$ for any $f \in X$ and $g \in L^1(G)$ with $g \geq 0$ and $||g||_1 = 1$. Here $g * f$ is defined by

$$(g * f)(s) = \int_G g(t) f(t^{-1}s)\, dt, \qquad s \in G.$$

A mean μ is topological left invariant on X if $\mu(g * f) = \mu(f)$ for any f and g as above. Any topological left invariant mean is left invariant, and any left invariant mean on $UCB(G)$ is topological left invariant. Each left invariant mean on $UCB(G)$ extends uniquely to a topological left invariant mean on $L^\infty(G)$, and any topological left invariant mean on $L^\infty(G)$ or any intermediate space is such an extension [**2, 3**]. Consider the set M of all left invariant means on $UCB(G)$. We see from the above that, equivalently, M is also the set of all topological left invariant means on any intermediate space X. We say that M has unique extension to X if each left invariant mean on X is the unique extension of some element in M; i.e., every left invariant mean on X is topological left invariant.

Granirer [**1**] and Rudin [**10**] first showed that if G is nondiscrete and amenable as discrete, then the extension of M to $L^\infty(G)$ is not unique. Their proofs are based on the fact that if G is amenable as discrete, then a permanently positive set bears a left invariant mean on $L^\infty(G)$. (A set $A \in G$ is permanently positive if $|x_1 A \cap \cdots \cap x_n A| > 0$ for any selection $x_1, \ldots, x_n \in G$.) Liu and von Rooij [**5**] proved that the extension of M to $CB(G)$ is not unique when G is noncompact, nondiscrete, and amenable as discrete. Rosenblatt [**9**] showed that for a nondiscrete σ-compact group G amenable as discrete, there is a continuum of left invariant means which are not topological left invariant. He also proved [**8**] that with the additional condition that G is metric, any element in M has 2^c many different extensions to $L^\infty(G)$.

A group G is unimodular if its left and right Haar measures coincide. G is said to be an [IN]-group if it has a compact neighborhood which is invariant under all inner automorphisms. A group is an [IN]-group if and only if it is the extension of a compact group by an [SIN]-group (a group whose left and right uniformities are coincident). For a discussion of these classes of groups, see Palmer [**6**].

In Section 2 of this paper, we consider the extension of M to the space $UCB_l(G)$. We showed that if G is an [IN]-group then the extension is unique. But for a large class of nonunimodular groups, there is a left uniformly continuous function that has a left invariant mean value one and is topologically null. (A function f is topologically null if $\mu(|f|) = 0$ for any topological left invariant mean μ.) We prove, in this case, that there are exactly $2^{2^{d(G)}}$ mutually singular left invariant means on $UCB_l(G)$ which are not topological left invariant, where $d(G)$ is the smallest cardinality of a compact cover of G.

In Section 3 we discuss the number of left invariant means on $CB(G)$. While it is known that there are at least $2^{2^{d(G)}}$ left invariant means on $CB(G)$ which are not topological left invariant for G noncompact, nondiscrete, and amenable as discrete, we show that the number is at least $2^{2^{d(G)}}\chi(G)$, where $\chi(G)$ is the smallest cardinality of a neighborhood base at e.

2. Extension of M to $UCB_l(G)$. In this section and the next a group G will mean a noncompact, nondiscrete group, unless otherwise specified.

Let G be an amenable group. Let M be the set of all left invariant means on $UCB(G)$, as before. The extension of M to $UCB_r(G)$ is unique as is proved in [**2**]. In this section we deal with the extension of M to the space $UCB_l(G)$. We consider two different classes of groups: [IN]-groups and nonunimodular groups. Also, we have a brief discussion for the gap between them. First we prove a useful lemma.

Let N be a compact normal subgroup of G, with the normalized left Haar measure. Let $f \in UCB_l(G)$ and g be any L^1-function on N. We define a function $g * f \in UCB_l(G)$ by

$$g * f(x) = \int_N g(y) f(y^{-1}x)\, dy, \qquad x \in G.$$

Lemma 2.1. *Let μ be a left invariant mean on $UCB_l(G)$. Then for any $f \in UCB_l(G)$ and $g \in L^1(N)$,*

$$\mu(g * f) = \mu(f) \int_N g(y)\, dy.$$

Proof. For fixed $f \in UCB_l(G)$, $g \to \mu(g*f)$ defines a linear functional on $L^1(N)$. Also, $\mu({}_y g * f) = \mu({}_y(g * f)) = \mu(g * f)$ for any $y \in N$. Thus, $\mu(g * f) = c_f \int_K g(y)\, dy$ for some constant c_f depending only on f. Let $\{U_\alpha\}$ be a neighborhood base at e in N, and let $g_\alpha = |U_\alpha|^{-1} 1_{U\alpha}$. Since the restrictions of f on the cosets of N form an equicontinuous family, we have $||f - g_\alpha * f||_\infty \to 0$. Therefore, $c_f = \mu(f)$. □

Proposition 2.2. *If G is an amenable* [IN]*-group, then every left invariant mean on $UCB_l(G)$ is topological left invariant.*

Proof. There is a compact normal subgroup N of G such that G/N is an [SIN]-group [**6**]. Let μ be a left invariant mean on $UCB_l(G)$ and $f \in UCB_l(G)$. The function $1_{N^*} f$ is constant on every coset of N. So it is uniformly continuous (both left and right) on G, since G/N is an [SIN]-group. Let μ_1 be the unique topological left invariant mean on $UCB_l(G)$ that coincides with μ on $UCB(G)$. Then by Lemma 2.1, $\mu_1(f) = \mu_1(1_{N^*} f) = \mu(1_{N^*} f) = \mu(f)$. Therefore, μ_1 and μ are identical. □

We conjecture that if G is not an [IN]-group, then the extension of M to $UCB_l(G)$ is not unique. But we can only prove a weaker result (Proposition 2.3).

For any subset A of G, we define the cardinal $d(A)$ to be the smallest cardinality of a cover of A by compact sets in G. (See [**4, 11**] for applications of $d(G)$ on problems concerning the number of topological invariant means on G.) Consider the following condition on G:

(A) G is nonunimodular and there exists a compact neighborhood E of e and a real number L such that

$$d\{x \in G : |ExE| < L\} = d(G).$$

Proposition 2.3. *Suppose G satisfies Condition* (A) *and is amenable as discrete. Then there is a family $\{f_\beta\}_{\beta<d(G)}$ of left uniformly continuous functions on G with disjoint supports, such that for each β, $0 \le f_\beta \le 1$, the set $\{x : f_\beta(x) = 1\}$ is permanently positive, and f_β is topologically null.*

Proof. Let E be a symmetric compact neighborhood of e in G that satisfies Condition (A) for some $L > 0$. Then, for every positive integer n, there is an open neighborhood U_n of e such that

$$d\{x \in G : |ExU_n| < 1/n\} = d(G).$$

For let Δ be the modular function on G. Choose $z \in G$ such that $\Delta(z) < 1/nL$. Let U_n be a neighborhood of e in G such that $zU_nz^{-1} \subset E$. Then for any $x \in G$ such that $|ExE| < L$, we have

$$|ExzU_n| = |ExzU_nz^{-1}z| \le |ExEz| = |ExE|\Delta(z) < 1/n.$$

We may suppose that the sets U_n are all symmetric and such that $U_n \supset U_{n+1}$ for all n. For each fixed U_n, define $\mathcal{C}_n$ to be a cover of G by sets in the form yU_n, $y \in G$, and such that the cardinality of $\mathcal{C}_n$ is $d(G)$. Let Λ be the set of all finite sequences $\{y_1U_{n_1}, y_2U_{n_2}, \dots, y_kU_{n_k}\}$, $y_iU_{n_i} \in \mathcal{C}_{n_i}$, such that

$$\sum_{i=1}^{k} \frac{\Delta(y_i^{-1})}{\min_{1\le i\le k}(n_i)} < 1.$$

Write Λ as $\{\lambda_\alpha : \alpha < d(G)\}$. Let j be a 1–1 onto mapping from $d(G)$ to $d(G) \times d(G)$. We are going to define a family $\{f_{\beta,\gamma} : \beta, \gamma < d(G)\}$ of left equicontinuous functions with compact supports. Let H be a σ-compact open subgroup of G with $|H| = \infty$, and take a sequence $\{H_n\}$ of symmetric compact neighborhoods of e such that $|H_n| \to \infty$ and $H_n \subset H$. If G is not σ-compact, define $H_\alpha = H$ for each infinite ordinal $\alpha < d(G)$. Now we can define subsets $S_{j(\alpha)}$, $\alpha < d(G)$, of G satisfying the following properties by induction on α:

(1) $S_{\beta,\gamma} = \cup_{i=1}^{k} y_iU_{n_i}x_{\beta,\gamma}E$, where $x_{\beta,\gamma} \in G$ and $\lambda_\gamma = \{y_1U_{n_1}, \dots, y_kU_{n_k}\}$;

(2) $|S_{\beta,\gamma}^{-1}| < 1$;

(3) $S_{j(\alpha)} \cap (\cup_{\beta<\alpha} S_{j(\beta)}) H_\alpha = \varnothing$.

This is possible because we can always choose $x_{j(\alpha)} \notin \cup_{i=1}^{k} \cup_{\delta<\alpha} U_{n_i} y_i^{-1} S_{j(\delta)} H_\alpha E$, where $j(\alpha) = (\beta,\gamma)$ and $\lambda_\gamma = \{y_1 U_{n_1}, \dots, y_k U_{n_k}\}$, such that $|E x_\alpha^{-1} U_{n_i}| < 1/n_i$, for every $1 \le i \le k$. So (2) is satisfied.

Now, on each $S_{\beta,\gamma}$ we define a function $f_{\beta,\gamma}$ as follows. Choose an open symmetric neighborhood E_1 of e such that $E_1^2 \subset E$. Then let $f_{\beta,\gamma} = |E_1|^{-1} 1_{K_{\beta,\gamma}E_1} * 1_{E_1}$, where $K_{\beta,\gamma} = \cup_{i=1}^{k} y_i U_{n_i} x_{\beta,\gamma}$ ($S_{\beta,\gamma} = K_{\beta,\gamma} E$ as in (1)). Then $f_{\beta,\gamma}$ is supported on $S_{\beta,\gamma}$ and is one on $K_{\beta,\gamma}$.

For every $\beta < d(G)$, let $f_\beta = \sum_{\gamma<d(G)} f_{\beta,\gamma}$. Since the functions $f_{\beta,\gamma}$ are disjointly supported, the sum is well defined and each f_β is left uniformly continuous, and the family $\{f_\beta\}_{\beta<d(G)}$ is disjointly supported. The set $\{x \in G : f_\beta(x) = 1\}$, for each β, is permanently positive. For let $a_1, \dots, a_k$ be arbitrary elements in G. Take $z \in G$ such that $\Delta(z) > k\Delta(a_i) \cdot \sup\{\Delta(a) : a \in U_1\}$ for every i. Then for each i, there is an integer n_i and $y_i \in G$ such that $a_i^{-1} z \in y_i U_{n_i}$, and we see that $\Delta(y_i^{-1}) < \Delta(a_i)\Delta(z^{-1}) \inf\{\Delta(a) : a \in U_{n_i}\} < 1/k$. So $\{y_1 U_{n_1}, \dots, y_k U_{n_k}\} \in \Lambda$, and there is a $\gamma < d(G)$ such that $K_{\beta,\gamma} = \cup_{i=1}^{k} y_i U_{n_i} x_{\beta,\gamma}$. Since $z \in \cap_{i=1}^{k} a_i K_{\beta,\gamma}$ and each $a_i K_{\beta,\gamma}$ is an open set, the set $\{x : f_\beta(x) = 1\}$ is permanently positive.

Finally, we prove that the sum $f = \sum_\beta f_\beta$, a left uniformly continuous function on G, is topologically null. Let n be any integer. Then for any $x \in G$, $|1_{H_n} * f_{\beta,\gamma}(x)| \le |H_n \cap x S_{\beta,\gamma}^{-1}| \le 1$ if $(\beta,\gamma) = j(\alpha)$ for some $\alpha > n$, by (2). From condition (3) we see $f = \sum_{\beta,\gamma} f_{\beta,\gamma}$, $|1_{H_n} * f(x)| \le 1$ for all x except on a compact set, and therefore $|\mu(f)| = |H_n|^{-1} |\mu(1_{H_n} * f)| < |H_n|^{-1}$ for all topological invariant means. Since $|H_n| \to \infty$, f is topologically null. □

From the family $\{f_\beta\}_{\beta<d(G)}$ we can construct many mutually singular left invariant means that are not topological left invariant. For each $\beta < d(G)$, let μ_β be a left invariant mean on $UCB_l(G)$ such that $\mu_\beta(f_\beta) = 1$. As in [**11**], we define a mapping $\pi : L^\infty(G) \to l^\infty(d(G))$ by $\pi(f)(\beta) = \mu_\beta(f)$ for $f \in L^\infty(G)$ and $\beta < d(G)$. The mapping is linear, positive, and $||\pi|| = 1$. Also, we have that π^* is a linear isometry from $l^\infty(d(G))^*$ to $L^\infty(G)^*$. Furthermore, π^* maps the set $\beta\mathbf{d(G)}$ of all ultrafilters on $d(G)$ into the set of all left invariant means on $UCB_l(G)$. It is easy to see that if μ is in the image of $\beta\mathbf{d(G)}$ then

$\mu(f) = 1$, where $f = \sum_\beta f_\beta$. Also, if μ_1 and μ_2 are two different image points, then there is a subset A of $d(G)$ such that $\mu_1(\sum_{\beta\in A} f_\beta) = 1$ and $\mu_2(\sum_{\beta\in d(G)\setminus A} f_\beta) = 1$. Thus, we can prove the following.

Proposition 2.4. *Suppose that G satisfies Condition* (A) *and is amenable as discrete. Then there are exactly $2^{2^{d(G)}}$ many mutually singular left invariant means on $UCB_l(G)$, each of which is singular to all topological left invariant means.*

Proof. From the above argument we see that we have at least $2^{2^{d(G)}}$ such left invariant means on $UCB_l(G)$. Take a compact normal subgroup N of G such that G/N is metric. Then by Lemma 2.1 there is a 1–1 correspondence between the left invariant means on $UCB_l(G)$ and $UCB_l(G/N)$. Since the dimension of the space $CB(G/N)$ is $2^{d(G/N)} = 2^{d(G)}$, the cardinality of $CB(G/N)^*$ is at most $2^{2^{d(G)}}$. Thus the set of all left invariant means on $UCB_l(G)$ has a cardinality $2^{2^{d(G)}}$. □

For an infinite cardinal κ, we say that κ has a cofinality $> \aleph_0$ if κ is not the sum of countably many smaller cardinals. Examples of such cardinals include $\aleph_1, \aleph_2, \ldots$, and the cardinality of any power set.

Corollary 2.5. *Let G be a nonunimodular group amenable as discrete. If $d(G)$ has a cofinality $> \aleph_0$, then there are exactly $2^{2^{d(G)}}$ many mutually singular left invariant means on $UCB_l(G)$ which are not topological left invariant.*

Proof. We need only to show that G satisfies Condition (A). Let E be any compact neighborhood of e. Then $G = \cup A_n$, where $A_n = \{x \in G : |ExE| < n\}$. Since $d(G) = \sum d(A_n)$ and $d(G)$ has a cofinality $> \aleph_0$, some $d(A_n)$ must equal $d(G)$. □

Corollary 2.6. *Let G be a nonunimodular group amenable as discrete. Then G can be embedded into a group G_1 such that $d(G) = d(G_1)$ and there are $2^{2^{d(G)}}$ mutually singular left invariant means on $UCB_l(G_1)$ which are not topological left invariant.*

Proof. Let H be any abelian group with $d(H) = d(G)$, and G_1 the direct product of G and H. Then G_1 is not unimodular and for any compact neighborhood E and any $a \in H$, $|EaE| = |E^2|$. Thus Condition (A) is satisfied since $d(G_1) = d(H)$. □

Generally, Condition (A) is not true for all nonunimodular groups. The group G defined by

$$\begin{bmatrix} 1 & a & b \\ 0 & c & 0 \\ 0 & 0 & c^{-2} \end{bmatrix}, \qquad a, b, c \in \mathbf{R}, \quad c > 0,$$

is a σ-compact, nonunimodular group, amenable as discrete (solvable in fact). Yet for every choice of a compact neighborhood E and a real number L, the set $\{x : |ExE| < L\}$ is precompact. But even for this group it can be shown that there are left invariant means on $UCB_l(G)$ that are not topological left invariant. In fact, it can be proved in a manner similar to Proposition 2.3 that if a group G satisfies the following condition

(B) There exists a compact neighborhood E of e such that for any $\varepsilon > 0$, there is a neighborhood U of e such that

$$d\{x \in G : |yE \cap ExU| < \varepsilon \text{ for any } y \in G\} = d(G);$$

and such that G is amenable as discrete, then there exist left invariant means on $UCB_l(G)$ which are not topological left invariant. We conjecture that Condition (B) is true for any non-[IN]-group, but we cannot provide a proof.

3. Extension of M to $CB(G)$. It is well known that when the group G is noncompact, nondiscrete, and amenable as discrete, then the extension of M to $CB(G)$ is not unique. By the technique we employed in Section 2, we can prove that there are $2^{2^{d(G)}}$ mutually singular left invariant means on $CB(G)$ that are not topological left invariant; or equivalently, there are as many *singular* left invariant means as topological left invariant ones. In this section we give a structure theorem about the set of all left invariant means on $CB(G)$. It shows that it is possible that there are more left invariant means in number than topological left invariant means. From now on, we assume that the group G is noncompact, nondiscrete, and amenable as discrete.

Let $\chi(G)$ be the smallest cardinality of a neighborhood base at e. Without loss of generality, we assume that G is σ-compact and $\chi(G)$ is uncountable. (See comments after Proposition 3.2.) Then there is a compact normal subgroup N such that G/N is metric. Let N be associated with its normalized Haar measure and let μ_0 be a left invariant mean on $CB(G/N)$. Then the convolution μ of μ_0 with the Haar measure on N, defined by $\mu(f) = \mu_0(1_{N^*}f)$, $f \in CB(G)$, is a left invariant mean on $CB(G)$. Denote the set of all left invariant means on $CB(G)$ obtained this way by M_N. Lemma 2.1 shows that topological left invariant means lie in the intersection of the sets M_N for all such N. On the other hand, starting from the union of these sets, we can obtain the set of all left invariant means on $CB(G)$ by taking the w^*-closure.

Proposition 3.1. *The union of all M_N is w^*-dense in the set of all left invariant means on $CB(G)$.*

Proof. First we prove that the union is a convex set. Each M_N, being isomorphic to the set of all left invariant means on $CB(G/N)$, is convex. Also, for any two compact normal subgroups N_1 and N_2, $M_{N_1} \cap M_{N_2} \subset M_{N_1 \cap N_2}$. Thus, the whole union is also convex.

Let μ be a left invariant mean on $CB(G)$, and let $f \in CB(G)$. Now it is enough to show that there exists a compact normal subgroup N of G and a left invariant mean $\mu_0 \in M_N$ such that $\mu_0(f) = \mu(f)$.

Fix an integer n. For each $x \in G$, we choose an open neighborhood $E_{(x,n)}$ of e such that the oscillation of f on the set $xE^2_{(x,n)}$ is smaller than $1/n$. Let $x_k E_{(x_k,n)}$ be a cover of G. There is a compact normal subgroup N such that $N \subset \cap_{k,n} E_{(x_k,n)}$ and G/N is metric. Let xN be a coset of N in G. Then for any integer n, x is contained in some $x_k E_{(x_k,n)}$. Thus, xN is contained in $x_k E^2_{(x_k,n)}$ and hence the oscillation of f on xN is less than $1/n$. Since n is arbitrary, we have that f is constant on each coset of N. Let μ_0 be the restriction of μ on $CB(G/N)$, considered as the subset of $CB(G)$ of all functions that are constant on each coset of N. Then μ_0 is a left invariant mean on $CB(G/N)$ and $\mu_0(f) = \mu(f)$. $\square$

Now we calculate the cardinality of the union $M = \cup_N M_N$. It gives an estimate of the number of left invariant means on $CB(G)$.

Proposition 3.2. *The union M of all M_N has a cardinality $\geq \chi(G)2^c$.*

Proof. Since each M_N has a cardinality 2^c, we need only to show that the set M has at least $\chi(G)$ many elements.

If this is not so, then M is the union of less than $\chi(G)$ many M_N's. This means that there is a compact normal subgroup N_0 (the intersection of all such N) of G such that $\chi(G/N_0) < \chi(G)$ and every element in M is a mean on $CB(G/N_0)$. Therefore, every left invariant mean on $CB(G)$ can be considered as on $CB(G/N_0)$, by Proposition 3.1. We show that this is not possible.

Let λ be the normalized Haar measure on N_0. For any integer $n > 0$, there is a neighborhood V_n of e in N_0 such that $\lambda(V_n) < 1/2n$. Let U_n be a symmetric compact neighborhood of e in G such that $U_n^4 \cap N_0 \subset V_n$. The set U_n has the property that $\lambda(xU_n^2 \cap N_0) < 1/2n$ for any $x \in G$. For if $xU_n^2 \cap N_0 \neq \varnothing$, then there is an $a \in N_0$ such that $ax \in U_n^2$. And so $|xU_n^2 \cap N_0| < |a^{-1}U_n^4 \cap N_0| < 1/2n$. For each n, let $y_{n,1}U_n, y_{n,2}U_n, \dots,$ be a covering of G by translates of U_n, and let Λ_n be the set of all n-element subsets of this covering. Then the union $\Lambda = \cup_n \Lambda_n$ is a countable set, and we denote its elements by λ_m, $m = 1, 2, 3, \dots$. By induction on m, we can define a sequence $S_1, S_2, S_3, \dots,$ of subsets of G with the properties that each $S_m = \cup_{i=1}^n y_{n,m_i} U_n x_m$, where $\lambda_m = \{y_{n,m_1}U_n, \dots, y_{n,m_n}U_n\}$ and that the sets $\pi(\cup_{i=1}^n y_{n,m_i} U_n^2 x_m)$ are mutually disjoint, where π is the projection from G to G/N_0.

We can now define a continuous function f on G such that f is supported on $\cup_{m=1}^\infty \cup_{i=1}^n y_{n,m_i} U_n^2 x_m$ and is one on each S_m. It can be proved as in the proof of Proposition 2.3 that the union of all S_m is permanently positive. Therefore, f supports a left invariant mean on $CB(G)$. But $||1_{N_0} * f||_\infty \leq 1/2$. So we obtain the contradiction as promised. □

Remark 1. The above argument works only for σ-compact groups. But with the modification that the sets M_N are defined for all compact normal subgroups N of G such that $\chi(G/N) \leq d(G)$, then Propositions

3.1 and 3.2 remain true for any noncompact group. The proofs are similar.

Remark 2. We see from Proposition 3.2 and Remark 1 that for a noncompact group amenable as discrete, there are at least $2^{2^{d(G)}}\chi(G)$ many left invariant means on $CB(G)$. It is not difficult to see that for each compact subset K of G the space $CB(K)$ has a dimension at most $\chi(G)$. Thus the dimension (as well as the cardinality) of $CB(G)$ is $\kappa(G) = \chi(G)^{d(G)}$. this shows that the number of left invariant means on $CB(G)$ is at most $2^{\kappa(G)}$. This cardinal is generally larger than $2^{2^{d(G)}}\chi(G)$ when $\chi(G) > 2^{d(G)}$. This leads to the open problem of how many left invariant means there are on $CB(G)$, or more general, on any translation invariant space between $CB(G)$ and $L^\infty(G)$ inclusive.

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