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A UNIFORM GEOMETRIC PROPERTY OF BANACH SPACES

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ABSTRACT. Property (β) of Banach spaces was introduced by S. Roiewicz as an intermediate property between uniform convexity and nearly uniform convexity. It is proved in this note that there are Banach spaces with property (β) which cannot be renormed in a uniform convex manner. This answers a question in [**8**].

1. Notations. We follow standard terminology which can be found in [**3**] or [**6**]. Let $(X, ||\cdot||)$ be a Banach space. B_X denotes its closed unit ball, $\operatorname{conv}(A)$ ($\overline{\operatorname{conv}}(A)$) is the convex hull (the closed convex hull) of a subset A of X. If (x_n) is a sequence in X, let $\operatorname{sep}(x_n) = \inf\{||x_n - x_m|| : n \neq m\}$. $\mathbf{K}$ denotes the field of the real or complex numbers.

2. Introduction. Several classes of Banach spaces have been introduced in the past according to the fulfillment of certain uniform properties. We can mention:

(UC): Uniform convexity (Clarkson [**1**]): $\forall \varepsilon > 0 \, \exists \delta > 0$ such that if $x, y \varepsilon B_X$ and $||x - y|| > \varepsilon$, then

$$||\frac{x+y}{2}|| < 1 - \delta.$$

(NUC): *Nearly Uniform convexity* (Huff [**5**]): $\forall \varepsilon > 0 \, \exists \delta > 0$ such that, if (x_n) is a sequence in B_X with $\operatorname{sep}(x_n) > \varepsilon$, then $\operatorname{conv}(x_n) \cap (1-\delta) B_X \neq \phi$.

(β): Property (β) (Rolewicz [**8**]): Given an element $x_0 \in X \sim B_X$ define the associated *drop* $D(x_0, B_X)$ as the set $\operatorname{conv}(B_X \cup \{x_0\})$,

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and let $R(x_0, B_X)$ be $D(x_0, B_X) \sim B_X$. The *Kuratowski index* of noncompactness $\alpha(A)$ of a subset A of X is defined as the infimum of $\delta > 0$ such that A is covered by a finite family of sets, each of them with diameter $< \delta$. Then a Banach space $(X, ||\cdot||)$ is said to have *property* (β) if $\forall\varepsilon > 0\, \exists\delta > 0$ such that $\alpha(R(x, B_X)) < \varepsilon\, \forall x \in X$, $1 < ||x|| < 1 + \delta$.

In [**8**] it is proved that every (UC) Banach space has property (β) (thanks to a characterization of (UC) Banach spaces using drops given in [**7**]). Moreover, it is proved that property (β) implies a uniform property called Δ-*uniform convexity*:

(ΔUC) (Goebel and Sekowski [**4**]): a Banach space $(X, ||\cdot||)$ is Δ-*uniform convex* if $\forall\varepsilon > 0\, \exists\delta > 0$ such that for every convex set A contained in B_X such that $\alpha(A) > \varepsilon$ we have $A \cap (1-\delta)B_X \neq \phi$.

It is easy to prove that Δ-uniform convexity coincides with nearly uniform convexity. It follows that every Banach space with property (β) is an (NUC)-space. Moreover, [**5**], (NUC)-Banach spaces coincide with the reflexive Banach spaces with *uniform Kadec-Klee* property (UKK): A Banach space $(X, ||\cdot||)$ has (UKK) property if $\forall\varepsilon > 0\, \exists\delta > 0$ such that for every sequence (x_n) in B_X weakly convergent to x and such that $\operatorname{sep}(x_n) > \varepsilon$ then $x \in (1-\delta)B_X$.

We have finally

$$(UC) \Rightarrow (\beta) \Rightarrow (\text{NUC}) = (\Delta\text{UC}) = (\text{reflexive} + (\text{UKK})).$$

Neither of the first two implications can be reversed [**5, 8**].

3. Property (β). It was asked in [**8**] if every Banach space with property (β) can be renormed to be uniformly convex. We prove here that this is not the case. We shall do it using the next result, close in spirit to a theorem given by R. Huff in [**5**] concerning (NUC) Banach spaces.

Theorem 3.1. *Let $(Y, ||\cdot||)$ be a Banach space with basis $(e_i : i \in I)$ (unconditional if I is noncountable) and such that, for every finite subset J of I,*

$$0 \leq |\alpha_j| \leq \beta_j, \qquad \forall j \in J \Rightarrow \left\| \sum_{j\in J} \alpha_j e_j \right\| \leq \left\| \sum_{j\in J} \beta_j e_j \right\|.$$

Let $(X_i, i \in I)$ be a family of finite dimensional Banach spaces. Let

$$Z := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} ||x_i|| e_i \in Y \right\},$$

equipped with the norm $||(x_i)_{i\in I}|| = ||\sum_{i\in I} ||x_i|| e_i||$. Then, if $(Y, ||\cdot||)$ has property (β), $(Z, ||\cdot||)$ has property (β), too.

Corollary 3.2. *There are Banach spaces with property (β) which cannot be renormed in a uniform convex manner.*

Proof of the Corollary. In the former theorem let $(Y, ||\cdot||)$ be the space $(l_2, ||\cdot||_2)$ and $(X_i, ||\cdot||)$ the space $(\mathbf{R}^i, ||\cdot||_\infty)$, $i = 1, 2, \ldots$. Then $(Z, ||\cdot||)$ has property (β) but, according to a result of M.M. Day [**2**] it cannot be renormed to be uniformly convex. $\square$

In order to prove Theorem 3.1 we need the following

Lemma 3.3. *Given Y and Z as in the statement of Theorem* 3.1 *and defining ϕ, a mapping from Z into Y, as $\phi((x_i)_{i\in I}) = \sum_{i\in I} ||x_i|| e_i$, $\forall (x_i)_{i\in I} \in Z$, then ϕ has the following properties*:

(i) $\forall z \in Z$, $||\phi(z)|| = ||z||$.

(ii) $\forall z \in Z$, $\forall \lambda \in \mathbf{K}$, $\phi(\lambda z) = |\lambda| \phi(z)$.

(iii) $\forall z_1, z_2 \in Z$, $||\phi(z_1) - \phi(z_2)|| \leq ||z_1 - z_2||$.

(iv) $\phi(R(z, B_Z)) \subseteq R(\phi(z), B_Y)$, $\forall z \in Z$, $||z|| > 1$.

Proof. (i) is just the definition of the norm in Z while (ii) is obvious. (iii) is easy: Let $z_1 = (x_i^1)_{i\in I}$, $z_2 = (x_i^2)_{i\in I}$ be elements in Z. Then

$$\begin{aligned} ||\phi(z_1) - \phi(z_2)|| &= \left\| \sum_{i\in I} ||x_i^1|| e_i - \sum_{i\in I} ||x_i^2|| e_i \right\| = \left\| \sum_{i\in I} (||x_i^1|| - ||x_i^2||) e_i \right\| \\ &\leq \left\| \sum_{i\in I} ||x_i^1 - x_i^2|| e_i \right\| = ||\phi(z_1 - z_2)|| = ||z_1 - z_2||, \end{aligned}$$

in view of $|\,||x_i^1|| - ||x_i^2||\,| \leq ||x_i^1 - x_i^2||$, $\forall i \in I$.

To prove (iv), let $z \in R(z_0, B_Z)$. Then $z = \lambda z_0 + (1-\lambda)b$, where $0 < \lambda < 1$ (we can assume $z \neq z_0$) and $b \in B_Z$. Moreover, $||z|| > 1$.

$$\left\| \frac{\phi(z) - \lambda\phi(z_0)}{1-\lambda} \right\| = \frac{||\phi(z) - \lambda\phi(z_0)||}{1-\lambda} \leq \frac{||z - \lambda z_0||}{1-\lambda}$$
$$= \left\| \frac{z - \lambda z_0}{1-\lambda} \right\| = ||b|| \leq 1.$$

Therefore,

$$c := \frac{\phi(z) - \lambda\phi(z_0)}{1-\lambda} \in B_Y.$$

It follows that $\phi(z) = \lambda\phi(z_0) + (1-\lambda)c$, where $0 < \lambda < 1$, hence $\phi(z) \in R(\phi(z_0), B_Y)$. □

Proof of Theorem 3.1. The proof combines the result of Lemma 3.3 with the intermediate results contained in the proof provided by R. Huff [**5**, Theorem 2]. We give the details for the sake of completeness.

First of all, we can suppose $(e_i : i \in I)$ to be normalized. If E is a subset of I, let $P_E : Y \to Y$ be defined by

$$P_E\left(\sum_{i\in I} \alpha_i e_i\right) = \sum_{i\in E} \alpha_i e_i$$

(in case I is countable, (say $I = \mathbf{N}$), and (e_i) is not unconditional, we consider only those E's of the form $\{1, 2, \dots, n\}$ and $\{n, n+1, n+2, \dots\}$). Choose $K > 0$ such that $||P_E|| \leq K$ for all E.

Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that if $1 < ||y|| < 1+\delta$ then $\alpha(R(y, B_Y)) < \varepsilon/5K$. Let $z \in Z$ be given such that $1 < ||z|| < 1+\delta$. We shall prove that $\alpha(R(z, B_Z)) \leq 2\varepsilon$.

Let us suppose instead that $\alpha(R(z, B_Z)) > 2\varepsilon$. Then it is possible to choose a sequence $(z^{(n)})$ in $R(z, B_Z)$ such that $\operatorname{sep}(z^{(n)}) \geq \varepsilon$. We claim that there exists a subsequence $(z^{(n_k)})$ of $(z^{(n)})$ such that $\operatorname{sep}(\phi(z^{(n_k)})) \geq \varepsilon/5K$. But $1 < ||\phi(z)|| = ||z|| < 1+\delta$, and Lemma 3.3 shows that $\phi(z^{(n_k)}) \in R(\phi(z), B_Y)$. This gives a contradiction.

To prove the claim, it is sufficient to show that for any finite set $\{z^{(n_1)}, \dots, z^{(n_k)}\}$ there exists $z^{(n)}$ such that $||\phi(z^{(n_j)}) - \phi(z^{(n)})|| \geq \varepsilon/5K$, $\forall j = 1, 2, \dots, k$. Suppose this is not the case for some finite set

$\{z^{(n_1)}, \dots, z^{(n_k)}\}$. Since (e_i) is a basis for Y, there exists some finite set $E \subset I$ such that

$$||P_{I\setminus E}(\phi(z^{(n_j)}))|| < \frac{\varepsilon}{5}, \qquad \forall j = 1, 2, \dots, k.$$

Hence, for all n, there exists some j so that

$$\begin{aligned} ||P_{I\setminus E}(\phi(z^{(n)}))|| &\leq ||P_{I\setminus E}(\phi(z^{(n_j)}))|| + ||P_{I\setminus E}[\phi(z^{(n_j)}) - \phi(z^{(n)})]|| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{2\varepsilon}{5}. \end{aligned}$$

Therefore, for all m and n,

$$\begin{aligned} ||P_{I\setminus E}[\phi(z^{(n)} - z^{(m)})]|| &= \left\| \sum_{i\in I\setminus E} ||x_i^{(n)} - x_i^{(m)}|| e_i \right\| \\ &\leq \left\| \sum_{i\in I\setminus E} (||x_i^{(n)}|| + ||x_i^{(m)}||) e_i \right\| \\ &= ||P_{I\setminus E}(\phi(z^{(n)}))|| + ||P_{I\setminus E}(\phi(z^{(m)}))|| < \frac{4\varepsilon}{5}, \end{aligned}$$

where $z^{(n)} = (x_i^{(n)})_{i\in I}$, $n = 1, 2, 3, \dots$.

Next, for each $i \in E$, $(x_i^{(n)})_{n=1}^{\infty}$ is a bounded sequence in X_i and hence has a Cauchy subsequence. By passing to successive subsequences, we may assume that $(x_i^{(n)})_{n=1}^{\infty}$ is Cauchy for every $i \in E$. Then

$$\lim_{m,n\to\infty} ||P_E[\phi(z^{(n)} - z^{(m)})]|| = \lim_{m,n\to\infty} \left\| \sum_{i\in E} ||x_i^{(n)} - x_i^{(m)}|| e_i \right\| = 0.$$

Choose m and n different and sufficiently large so that

$$||P_E[\phi(z^{(n)} - z^{(m)})]|| < \frac{\varepsilon}{5}.$$

Then we have $m \neq n$ and

$$||z^{(n)} - z^{(m)}|| = ||\phi(z^{(n)} - z^{(m)})||$$
$$\leq ||P_E[\phi(z^{(n)} - z^{(m)})]|| + ||P_{I\setminus E}[\phi(z^{(n)} - z^{(m)})]|| < \frac{\varepsilon}{5} + \frac{4\varepsilon}{5} = \varepsilon,$$

a contradiction. This contradiction proves the claim. □

As far as we know, the following problem is still open:

Problem. Does a Banach space X exist such that X is (NUC) but it cannot be renormed to have property (β)?

The following simple result goes the other way around:

Proposition 3.4. *Let* $(X, ||\cdot||)$ *be an infinite-dimensional* (NUC)-*Banach space. Then, given an arbitrary* $\varepsilon > 0$ *there exists an equivalent norm* $|||\cdot|||$ *on* X *such that* $1/(1+\varepsilon)||\cdot|| \leq |||\cdot||| \leq ||\cdot||$, $(X, |||\cdot|||)$ *is still* (NUC) *but it does not have Property* (β).

Proof. This time denote by $B_{||\cdot||}$ the closed unit ball of $(X, ||\cdot||)$. Let x_0 be an element in X such that $1 < ||x_0|| < 1 + \varepsilon$. Define $|||\cdot|||$ the norm on X with closed unit ball

$$B_{|||\cdot|||} := \mathrm{conv}\,(B_{||\cdot||}, \{x_0\}, \{-x_0\}).$$

We shall prove that $|||\cdot|||$ has the required properties:

Obviously, $1/(1+\varepsilon)||\cdot|| \leq |||\cdot||| \leq ||\cdot||$.

Next, it is easy to prove that $R(x_0, B_{||\cdot||})$ contains a ball, more precisely the ball $B_{||\cdot||}(x_1, \delta)$, where $x_1 = (1 + ||x_0||/2||x_0||)x_0$, and δ is small enough to have $\delta < (||x_0|| - 1)/3$, and $2\delta||x_0||/(||x_0|| - 1) < 1$, hence it has a positive Kuratowski index of noncompactness. Now the sequence $((1 + (1/n))x_0)_{n=1}^{\infty}$ violates the (β)-property for $|||\cdot|||$ (just observe that $\mathrm{conv}\,(R((1 + (1/n))x_0, B_{|||\cdot|||}))$ contains $R(x_0, B_{||\cdot||})$, and consider that the Kuratowski index of noncompactness of a set and of its convex hull coincide).

To prove that $(X, |||\cdot|||)$ is a (NUC)-Banach space, we can use the result mentioned at the introduction: (NUC) is equivalent to (UKK) plus reflexivity. It is then enough to prove that $(X, |||\cdot|||)$ satisfies the (UKK)-property:

We shall find $\delta_1 = \delta_1(\varepsilon) > 0$ such that, for any sequence (x_n) in $B_{|||\cdot|||}$ which satisfies $(x_n) \stackrel{w}{\to} x$ and such that $|||\cdot|||-\text{sep}\,(x_n) > \varepsilon$, then $|||x||| < 1 - \delta_1$. Obviously, $B_{|||\cdot|||} = \text{conv}\,(B_{||\cdot||}, \{x_0\}) \cup \text{conv}\,(B_{||\cdot||}, \{-x_0\})$. We may assume $(x_n) \subseteq \text{conv}\,(B_{||\cdot||}, \{x_0\})$. Therefore, $x_n = \lambda_n b_n + (1 - \lambda_n)x_0$, where $0 \leq \lambda_n \leq 1$, $b_n \in B_{||\cdot||}$, $n = 1, 2, \ldots$. We may also assume $\lambda_n \neq 0$, $n = 1, 2, \ldots$. The fact that $(X, ||\cdot||)$ is a reflexive space allows us to select a subsequence of (x_n) (which still will be denoted by (x_n)) such that $(b_n) \stackrel{w}{\to} b \in B_{||\cdot||}$, and $(\lambda_n) \to \lambda \in [0, 1]$, hence $x = \lambda b + (1 - \lambda)x_0$.

Observe that

$$\begin{aligned} |||x_n - x_m||| &\leq \lambda_n \cdot |||b_n - b_m||| + |\lambda_n - \lambda_m| \cdot |||b_m||| + |\lambda_n - \lambda_m| \cdot |||x_0||| \\ &\leq 2\lambda_n + 2|\lambda_n - \lambda_m|; \end{aligned}$$

hence, there exists a subsequence of (λ_n) (still denoted by (λ_n)) such that $|\lambda_n| > \varepsilon/4$. We shall select the corresponding subsequence of (x_n), denoted also by (x_n).

We have

$$\begin{aligned} |||b_n - b_m||| &\geq \left|\left|\left| \frac{x_n}{\lambda_n} - \frac{x_m}{\lambda_m} \right|\right|\right| - \left| \frac{1-\lambda_m}{\lambda_m} - \frac{1-\lambda_n}{\lambda_n} \right| \cdot |||x_0||| \\ &\geq \frac{1}{\lambda_n}|||x_n - x_m||| - \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_m} \right| \cdot |||x_m||| - \left| \frac{1-\lambda_m}{\lambda_m} - \frac{1-\lambda_n}{\lambda_n} \right| \cdot |||x_0|||. \end{aligned}$$

It follows that there exists $n_0 \in \mathbf{N}$ such that $||b_n - b_m|| \geq |||b_n - b_m||| > \varepsilon/2$ for $m \geq n_0$, $n \neq m$. Property (UKK) of $(X, ||\cdot||)$ gives a $\delta = \delta(\varepsilon/2)$ (we can even assume $\delta\varepsilon/4 < 1$) such that $||b|| < 1 - \delta$.

Finally, we shall construct $\mu = \mu(\varepsilon) > 1$ such that $|||\mu x||| < 1$. Taking $\delta_1(\varepsilon) = 1/\mu(\varepsilon)$, we shall get the conclusion. It is enough to choose $1 < \mu < 1/(1 - (\varepsilon\delta/4))$. A simple calculation gives $\mu x \in \text{conv}\,(B_{||\cdot||}, x_0) \subseteq B_{|||\cdot|||}$. This completes the proof. □

Remark . Theorem 3.1 has been obtained independently by D. Kutzarova (personal communication).

Note added in proof. The Problem stated on page 6 has a positive answer (D.N. Kutzarova, personal communication).

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