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A CLASS OF CONTINUA WHICH ADMITS NO EXPANSIVE HOMEOMORPHISMS

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ABSTRACT. It is proved that any Suslinian, hereditary θ-continuum admits no expansive homeomorphisms.

1. Introduction. A compact, connected metric space is called a *continuum.* A homeomorphism $f : X \twoheadrightarrow X$ of a continuum X is called *expansive* if there exists a constant $c > 0$ (called the *expansive constant*) which satisfies the following condition. For each pair of distinct points x, y of X, there exists an integer n such that $d(f^n(x), f^n(y)) > c$, where d is a metric of X. Expansiveness does not depend on the choice of metrics of X. It is an interesting problem whether a given continuum has an expansive homeomorphism of itself.

To consider this problem, the first author suggested the idea of using monotone decompositions of continua in [**7**]. Using this idea, we show that any Suslinian, hereditary θ-continuum admits no expansive homeomorphisms.

Definition 1. Let X be a continuum. 1) X is called a *θ-continuum* (a *θ_n-continuum*, respectively) if for each subcontinuum Y of X, the number of components of $X - Y$ is finite (at most n, respectively). If each subcontinuum of X is a θ-continuum (θ_n-continuum, respectively), X is called a *hereditary θ-continuum* (a *hereditary θ_n-continuum*, respectively).

2) X is called *Suslinian* if it has no uncountable collection of mutually disjoint nondegenerate subcontinua of X.

3) X is called *decomposable* if $X = A \cup B$ for some proper subcontinua A and B of X. If each subcontinuum of X is decomposable, X is called *hereditarily decomposable.*

It is easy to see that Suslinian continua are hereditarily decomposable.

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Our main theorem is

Theorem 2. *Any Suslinian, hereditary θ-continuum admits no expansive homeomorphisms.*

2. The proof of Theorem 2. First, we prepare some results needed in the proof.

Theorem 3. [**2**, Theorem 1, **8**, Corollary of Theorem 8, **3**, Theorem 3].

1) *Any hereditarily decomposable, θ-continuum is a θ_n-continuum for some n.*

2) *Let X be a hereditarily decomposable θ_n-continuum. Then X admits an upper semi-continuous monotone decomposition $\mathcal{D}$ such that $X/\mathcal{D}$ is a nondegenerate finite graph which is a θ_n-continuum. Furthermore, $\mathcal{D} = \{T^{2n}(x) | x \in X\}$, where T is the aposyndetic set function defined in* [**4, 2**].

Notice that each homeomorphism $f : X \to X$ satisfies $f(T(x)) = T(f(x))$ for each $x \in X$.

Lemma 4. [**6**, Lemma 2.2]. *Let $f : X \twoheadrightarrow X$ be an expansive homeomorphism of a compact metric space X. There exists a $\delta > 0$ such that, for each nondegenerate subcontinuum A of X, there exists an integer $n_0 > 0$ which satisfies one of the following conditions*

($*$) $\operatorname{diam} f^n(A) \geq \delta$ *for each* $n \geq n_0$ *or*

($**$) $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each* $n \geq n_0$.

Let G be a finite connected graph which is not a simple closed curve. The set of all branch points of G is denoted by $B(G)$ and the set of all end points of G is denoted by $E(G)$. The set of all vertices of G, denoted by $V(G)$, is $B(G) \cup E(G)$. A circle C in G is called a *free circle* if $C \cap \operatorname{cl}(G - C)$ is a point. Let $S(G) = \{b \in B(G) |$ there exists a free circle C such that $\{b\} = C \cap \operatorname{cl}(G - C)\}$. Let e be an edge of G whose end points are u and $v \in V(G)$. The open arc $e - \{u, v\}$ is denoted by $\operatorname{int} e$.

Lemma 5. *Let G be a finite connected graph which is not a simple closed curve. There exists an integer $N > 0$ such that each homeomorphism $h : G \twoheadrightarrow G$ satisfies $h^N|V(G) = id_{V(G)}$ and $h^N(e) = e$ for each edge e of G, $h^N(C) = C$ and $h^N|C$ is orientation preserving for each free circle C of G.*

Remark. Any "irrational rotation" of the unit circle has no periodic points. So Lemma 5 does not hold for simple closed curves.

Lemma 6. *Let X be a Suslinian continuum and Y be a continuum. Suppose that $f : X \twoheadrightarrow X$ is an expansive homeomorphism, $p : X \twoheadrightarrow Y$ is a monotone map which is not a homeomorphism, and $h : Y \twoheadrightarrow Y$ is a homeomorphism. If $h \cdot p = p \cdot f$, then h has a periodic point.*

Proof. Suppose, on the contrary, that h does not have a periodic point. Since f is expansive, we can take a $\delta > 0$ as in Lemma 4. For any subset M of Y, we define M^δ as follows.

1) $M^\delta = \{y \in Y|$ there exists a sequence (y_i) of points of M such that $y_i \to y$, $y_i \neq y$, and $\operatorname{diam} p^{-1}(y_i) \geq \delta$ for each $i\}$.

Then we have

2) M^δ is closed in Y and

3) $(M^\delta)^\delta = (M^\delta)' \subset M^\delta$ where $(M^\delta)'$ denotes the derived set of M^δ. For each ordinal number α, we define M_α by $M_1 = Y$, $M_{\alpha+1} = (M_\alpha)^\delta$ and $M_\alpha = \cap_{\beta<\alpha} M_\beta$, where α is a limit ordinal.

We claim that

4) $M_\alpha \neq \varnothing$ for each countable ordinal α.

It is clear that $M_1 \neq \varnothing$. Take a $y_1 \in Y$ such that $p^{-1}(y_1)$ is not a point. Applying Lemma 4, there exists an integer $n_0 > 0$ such that one of the following conditions holds:

$(*)$ $\operatorname{diam} f^n(p^{-1}(y_1)) = \operatorname{diam} p^{-1}(h^n(y_1)) \geq \delta$ for each $n \geq n_0$ or

$(**)$ $\operatorname{diam} f^{-n}(p^{-1}(y_1)) = \operatorname{diam} p^{-1}(h^{-n}(y_1)) \geq \delta$ for each $n \geq n_0$.

Assume that $(*)$ holds. As the point y_1 is not a periodic point of h, $\{h^n(y_1)\}_{n\geq n_0}$ is infinite. So we can take a convergent subsequence $\{h^{n_k}(y_1)\}$ such that $h^{n_k}(y_1) \to y_2$ for some $y_2 \in Y$ as $k \to \infty$ and

$h^{n_k}(y_1) \neq y_2$ for each k. By the definition 1), we have $y_2 \in M_2$. Further, we easily have that $h^i(y_2) \in M_2$ for each integer i. The case $(**)$ is similar.

Take a countable ordinal λ and assume that for each $\alpha < \lambda$, there exists a y_α such that $h^i(y_\alpha) \in M_\alpha$ for each integer i. If $\lambda = \alpha + 1$, we can find a y_λ by the same argument as above. If λ is a limit ordinal, we take an increasing sequence $\alpha_1 < \alpha_2 < \cdots \to \lambda$. We may assume that the y_{α_i}'s converge to a point y_λ. Then y_λ is the desired point. So we have proved 4).

Since Y is separable, there exists a countable ordinal α_0 such that $M_\alpha = M_{\alpha_0}$ for each $\alpha \geq \alpha_0$. In particular, $(M^\delta_{\alpha_0}) = M_{\alpha_0+1} = M_{\alpha_0+2} = (M^\delta_{\alpha_0})^\delta = (M^\delta_{\alpha_0})'$. Hence, M_{α_0+1} is a perfect and compact set, and so is uncountable. But for each $y \in M_{\alpha_0+1}$, $\operatorname{diam} p^{-1}(y) \geq \delta > 0$, which contradicts the assumption that X is Suslinian. This completes the proof. □

Proof of Theorem 2. Let X be a Suslinian hereditary θ-continuum and suppose that $f : X \twoheadrightarrow X$ is an expansive homeomorphism. Take $\delta > 0$ as in Lemma 4.

Step 1. Let $\mathcal{F} = \{K | K$ is a nondegenerate subcontinuum of X such that $f(K) = K\}$.

Take a minimal element M of $\mathcal{F}$. The existence of M is guaranteed by Lemma 4 and Zorn's Lemma. By Theorem 3 and the fact that $f(T(x)) = T(f(x))$, there exists a monotone map $m : M \twoheadrightarrow G$ onto a graph G and a homeomorphism $h : G \twoheadrightarrow G$ such that $m\cdot(f|M) = h\cdot m$. Define an integer N_1 as follows. If G is not a simple closed curve, let N_1 be the integer as in Lemma 5. If G is a simple closed curve, then h has a periodic point $v \in G$ by Lemma 6 (note that a simple closed curve admits no expansive homeomorphisms [**1**], so m is not a homeomorphism). Let N_1 be a period of v such that h^{N_1} is orientation preserving. Clearly, $m \cdot (f/M)^{N_1} = h^{N_1} \cdot m$. We consider two cases.

Case 1.1. For each $t \in \operatorname{Fix}(h^{N_1})$ (= the set of all fixed points of h^{N_1}), $m^{-1}(t)$ is a point.

If G is neither a simple closed curve nor a one point union of simple closed curves, fix an edge e of G. Then by the choice of N_1, $h^{N_1}(e) = e$.

Note that $h^{N_1}|e \neq \mathrm{id}$ (see [**1**, Theorem 4]). We may assume that there exist two distinct points $p, q \in e \cap \mathrm{Fix}\,(h^{N_1})$ such that

1) for each $t \in (p,q)$, $h^{N_1 k}(t) \to p$ as $k \to \infty$ and $h^{N_1 k}(t) \to q$ as $k \to -\infty$.

Suppose that $m^{-1}(t)$ is a point for each $t \in [p,q]$. Then $m^{-1}[p,q]$ is an arc which is invariant under f^{N_1}. This contradicts the assumption that f^{N_1} is expansive [**1**, Theorem 4]. So there exists a $t_0 \in (p,q)$ such that $m^{-1}(t_0)$ is not a point.

Notice that $\mathrm{diam}\, f^{N_1 k}(m^{-1}(t_0)) = \mathrm{diam}\, m^{-1}(h^{N_1 k}(t_0)) \to 0$ as $k \to \pm\infty$. Using this fact and the monotonicity of m, we can take two distinct points $x, y \in m^{-1}(t_0)$ such that $d(f^{N_1 k}(x), f^{N_1 k}(y)) < c$ for each $k \in \mathbf{Z}$, where c is an expansive constant of f^{N_1}. This contradicts the assumption.

Next we assume that G is a simple closed curve. If v is the unique fixed point of h^{N_1}, then $h^{N_1 k}(t) \to v$ as $k \to \pm\infty$, for each $t \in G - v$. If there are fixed points other than v, we can find distinct points $p, q \in \mathrm{Fix}\,(h^{N_1})$ and an open arc (p,q) in G such that $h^{N_1 k}(t) \to p$ ($\to q$, respectively) as $k \to \infty$ ($\to -\infty$, respectively) for each $t \in (p,q)$. In both cases, we have a contradiction by the same argument as above. Also, in the case that G is a one point union of simple closed curves, we have a contradiction.

Case 1.2. There exists a $t_1 \in \mathrm{Fix}\,(h^{N_1})$ such that $m^{-1}(t_1)$ is not a point. By the choice of M, $t_1 \notin \mathrm{Fix}\,(h)$ and $N_1 \geq 2$. So there exists an integer k_1 such that

2) $k_1 \geq 2$ and k_1 divides N_1.

3) $h^i(t_1) \neq h^j(t_1)$ for each $0 \leq i \neq j \leq k_1 - 1$ and $h^{k_1}(t_1) = t_1$. Let $X_i = m^{-1}(h^i(t_1)) = f^i(m^{-1}(t_1))$, $i = 0, \dots, k_1 - 1$. Then $\{X_i |\, 0 \leq i \leq k_1 - 1\}$ is a disjoint collection of nondegenerate subcontinua of X and $f^{N_1}(X_i) = X_i$ for each i. By Lemma 4,

4) $\mathrm{diam}\, X_i \geq \delta$ for each $i = 0, \dots, k_1 - 1$.

Now we proceed to Step 2.

Step 2. Let $f_2 = f^{N_1}$ and $\mathcal{F}_2 = \{K | K$ is a nondegenerate subcontinuum of X_0 such that $f_2(K) = K\}$.

Take a minimal element M_2 of $\mathcal{F}_2$. By Theorem 3 again, there exists a monotone map $m_2 : M_2 \twoheadrightarrow G_2$ onto a finite graph G_2 and a homeomorphism $h_2 : G_2 \twoheadrightarrow G_2$ such that $h_2 \cdot m_2 = m_2 \cdot (f_2|M_2)$. Define N_2 as in Step 1.

Case 2.1. For each $t \in \mathrm{Fix}\,(h_2^{N_2})$, $m_2^{-1}(t)$ is a point.

In this case, we can deduce a contradiction by the same argument as in the Case 1.1.

Case 2.2. There exists a $t_2 \in \mathrm{Fix}\,(h_2^{N_2})$ such that $m_2^{-1}(t_2)$ is nondegenerate continuum. As in the Case 1.2, we can take an integer k_2 such that

5) $k_2 \geq 2$ and k_2 divides N_2.

6) $h_2^u(t_2) \neq h_2^v(t_2)$ for each $0 \leq u \neq v \leq k_2 - 1$, and $h_2^{k_2}(t_2) = t_2$.

Let $X_{iu} = f^i(m_2^{-1}(h_2^u(t_2))) = f^i(f_2^u(m_2^{-1}(t_2)))$, $0 \leq i \leq k_1 - 1$ and $0 \leq u \leq k_2 - 1$. Then $\{X_{iu} \,|\, 0 \leq i \leq k_1 - 1, 0 \leq u \leq k_2 - 1\}$ is a disjoint collection of nondegenerate subcontinua of X and $f_2^{N_2}(X_{iu}) = X_{iu}$ and $X_{iu} \subset X_i$. Again,

7) $\mathrm{diam}\, X_{iu} \geq \delta$ for each $i = 0, \ldots, k_1 - 1$ and $u = 0, \ldots, k_2 - 1$.

Continuing these processes, we obtain an uncountable disjoint collection

$$\{K_{i_1 i_2} \ldots \,|\, 0 \leq i_1 \leq k_1 - 1, 0 \leq i_2 \leq k_2 - 1, \ldots\}$$

defined by $K_{i_1 i_2 i_3 \ldots} = X_{i_1} \cap X_{i_1 i_2} \cap X_{i_1 i_2 i_3} \ldots$. By conditions 4), 7) and so on, each $K_{i_1 i_2 i_3} \ldots$ is a nondegenerate subcontinuum of X. This contradicts the assumption that X is Suslinian and completes the proof. $\square$

It would be interesting if the hypothesis "hereditary θ-continuum" can be replaced by "θ-continuum." In our situation, θ-continuum is a θ_n-continuum for some n. An easy example shows that a Suslinian, θ_n-continuum need not be a hereditary θ-continuum.

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