ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

A QUASILINEAR SYSTEM MODELING THE SPREAD OF INFECTIOUS DISEASE

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ABSTRACT. Recent results for quasilinear systems are applied to a quasilinear reaction diffusion system modeling the spread of an infectious disease within a system. The long-time behavior of the system is investigated and asymptotic convergence results are obtained.

1. Introduction. We shall be concerned with a system of quasilinear reaction diffusion equations which model the spread within a population of infectious disease. The population is assumed to be subdivided into three classes: the susceptible class S consisting of individuals capable of becoming infected, the infective class I consisting of individuals capable of transmitting the disease, and the removed class R which consists of individuals who have died or recovered from the disease and have become immune.

We make the following assumptions regarding the kinetics of our model: individuals of the susceptible class enter the infective class at a rate proportional to the product of the size of the susceptible class and the infective class with constant of proportionality $\mu > 0$, and individuals who do not survive as infectives enter the removed class at a rate proportional to the size of the class I with a constant of proportionality $\lambda \geq 0$. We assume that the epidemic occurs in a bounded spatial region which is a bounded Lipschitz domain in $\mathbf{R}^n$ with $C^{2+\varepsilon}$ boundary $0 < \varepsilon < 1$ and that the population is constrained to remain for all time. Finally, we assume that diffusion takes place in all classes and that the diffusion mechanism is represented by a quasilinear diagonal diffusion operator which can be written in divergence form. These assumptions lead to the following quasilinear parabolic system

Received by the editors on June 10, 1989, and in revised form on February 14, 1990.
AMS (MOS) *Subject classifications.* 35K57, 34B40, 34G20.
The work of the first author supported in part by NSF grant DMS 8803151 and Texas Advanced Research grant 1100.
The work of the second author supported in part by NSF grant DMS 8813071. The support of these agencies is gratefully acknowledged.

of equations:

$$\partial_t S(x,t) = \nabla \cdot (\phi(S(x,t))\nabla S(x,t)) - \mu S(x,t)I(x,t) \qquad t>0, x\in\Omega, \tag{1.1a}$$

$$\partial_t I(x,t) = \nabla \cdot (\psi(I(x,t))\nabla I(x,t)) + \mu S(x,t)I(x,t) - \lambda I(x,t) \qquad t>0, x\in\Omega, \tag{1.1b}$$

$$\partial_t R(x,t) = \nabla \cdot (\theta(R(x,t))\nabla R(x,t)) + \lambda I(x,t) \qquad t>0, x\in\Omega \tag{1.1c}$$

which are subject to boundary conditions of the form

$$\frac{\partial S(x,t)}{\partial n} = \frac{\partial R(x,t)}{\partial n} = \frac{\partial I(x,t)}{\partial n} = 0 \qquad t>0, x\in\partial\Omega. \tag{1.1d}$$

Finally, we have the initial conditions

$$S(x,0) = S_0(x) \qquad I(x,0) = I_0(x) \qquad R(x,0) = R_0(x) \qquad x\in\Omega. \tag{1.1e}$$

The diffusivities, $\phi(\quad), \psi(\quad), \theta(\quad)$, are assumed to be smooth, non-degenerate, strictly positive, and uniformly bounded. Namely, there are positive constants $\underline{a}$ and $\overline{a}$ so that

$$\underline{a} \le \min\{\phi(u),\psi(u),\theta(u)\} \le \max\{\phi(u),\psi(u),\theta(u)\} \le \overline{a} \quad \text{for all } u\in\mathbf{R}. \tag{1.2}$$

We assume the following about our initial data:

$$S_0(\quad), I_0(\quad), R_0(\quad) \in H^1(\Omega). \tag{1.3}$$

We shall have occasion to impose several different norms on functions defined on Ω. The symbol, $|| \quad ||_{p,\Omega}$, shall denote the standard $L_p(\Omega)$ norm. The norm in $H^1(\Omega)$ will be denoted by $|| \quad ||_{2,\Omega}^{(1)}$.

Webb [**13**] studies this system for constant linear diffusion in each component in the case of one space dimension and is able to analyze the behavior as $t\to\infty$. These results have been extended to the case of arbitrary dimension and distinct diffusion constants in [**1**].

2. Existence and a priori bounds. This section contains a discussion of existence and provides a priori bounds requisite for analyzing the asymptotic convergence of solutions to (1.1a–e). We specify our notion of solutions to (1.1a–e).

Definition 2.1. A triple $(S(\ ,\ ), I(\ ,\ ), R(\ ,\ ))$ is said to be a classical solution to (1.1a–e) on $\overline{\Omega} \times [0, T]$ if the following are satisfied:

(i) For all $t \in (0, T]$, $(S(\ , t), I(\ , t), R(\ , t)) \in C^2(\Omega, \mathbf{R}^3) \cap C^1(\overline{\Omega}, \mathbf{R}^3)$; for $x \in \Omega$, $(S(x,\ ), I(x,\ ), R(x,\ )) \in C^1((0, T], R^3)$ and the partial differential equation is satisfied for $x \in \Omega$ and $t \in (0, T]$.

(ii) $\lim_{t \to 0^+} [||S(\ , t) - S_0(\ )||_{2,\Omega}^{(1)} + ||I(\ , t) - I_0(\ )||_{2,\Omega}^{(1)} + ||R(\ , t) - R_0(\ )||_{2,\Omega}^{(1)}] = 0$.

Our basic existence result follows.

Theorem 2.2. *If* (1.3) *holds, the initial data is nonnegative, and* $T > 0$*, then there exists a unique classical solution of* (1.1a–e) *on* $\overline{\Omega} \times [0, T]$*. Moreover,* $S(x, t), I(x, t), R(x, t) \geq 0$ *for* $x \in \overline{\Omega}$*,* $t \geq 0$*.*

Indication of Proof. Local existence, uniqueness, and continuous dependence results follow from standard arguments of abstract parabolic theory, and standard continuation results produce existence on a maximal interval $[0, T_{\max})$. These results are discussed by Waggoner [**12**]. The positivity follows from the fact that the positive orthant is an invariant rectangle, cf. Smoller [**11**]. If $H(S, I, R) = S + I + R$, we compute a priori $L_1(\Omega)$ bounds on $H(S(\ ,\ ), I(\ ,\ ), R(\ ,\ ))$. Summing the components (1.1a–c) we obtain

$$\partial_t(S + I + R) = \nabla \cdot (\psi(S)\nabla S + \psi(I)\nabla I + \theta(R)\nabla R). \tag{2.3}$$

We integrate to observe that

$$||H(S(\ , t), I(\ , t), R(\ , t))||_{1,\Omega} \leq ||H(S_0(\ ), I_0(\ ), R_0(\ ))||_{1,\Omega} \quad \text{for } t \in (0, T_{\max}).$$

Clearly, there exists a constant $C_1 > 0$ so that for all $0 \leq \tau < t < T_{\max}$

$$\int_\tau^t \int_\Omega H(S, I, R)\, dx\, ds \leq C_1(t - \tau). \tag{2.4}$$

Results appearing in [**12**] may be readily adapted to conclude that the assumption $T_{\max} < \infty$ leads to a contradiction. The crux of the proof consists of showing that Morgan's intermediate sum condition, [**8,9**], is satisfied with intermediate sum matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and constants $K_1 = K_2 = 0$. We may conclude that solutions exist for any interval $[0, T]$ for any $T > 0$. □

It is trivial to observe that the first two components of (1.1a–e) decouple from the third and that determination of $S(\ ,\ )$ and $I(\ ,\ )$ completely determine $R(\ ,\ )$. We shall therefore subsequently confine our attention to the system:

(2.5a) $$\partial_t S(x,t) = \nabla \cdot (\phi(S(x,t))\nabla S(x,t)) - \mu S(x,t)I(x,t) \qquad x \in \Omega, t > 0$$

(2.5b)
$$\partial_t I(x,t) = \nabla \cdot (\psi(I(x,t))\nabla I(x,t)) + \mu S(x,t)I(x,t) - \lambda I(x,t) \qquad x \in \Omega, t > 0$$

subject to boundary conditions,

(2.5c) $$\frac{\partial S(x,t)}{\partial n} = \frac{\partial I(x,t)}{\partial n} = 0 \qquad x \in \partial\Omega$$

with initial data

(2.5d) $$S(x,0) = S_0(x), I(x,0) = I_0(x) \qquad x \in \Omega.$$

We may adopt an argument due to Haraux and Youkana [**4**] to produce a priori $L_p(\Omega)$ cylinder bounds for solutions to (2.5a–d).

Proposition 2.6. *If* $(S(\ ,\ ), I(\ ,\ ))$ *is the solution to* (2.5a–d) *guaranteed by Theorem* 2.3 *on* $\overline{\Omega} \times [0, \infty)$ *and* $p \in [1, \infty)$, *then there exists a constant* $K_p > 0$ *so that for all* $0 < \tau < T < \infty$ *we have*

(2.7) $$||I(\cdot, \cdot)||_{p, \Omega \times (\tau, T)} \leq K_p(T - \tau).$$

Proof. The regularity of parabolic systems will ensure that we shall lose no generality if we assume that S_0 and I_0 are C^1 on $\overline{\Omega}$. The fact that the first component of our kinetic term is nonpositive implies that for all $p \in [1, \infty]$

$$||S(\ ,t)||_{p,\Omega} \le ||S_0(\ )||_{p,\Omega} \qquad \text{for } t \ge 0. \tag{2.8}$$

It will simplify subsequent calculation if we set $g_1(S, I) = \mu SI$ and $g_2(S, I) = \mu SI - \lambda I$. Clearly, for $S, I \ge 0$, $g_2(S, I) \le g_1(S, I)$.

If $F(\ \ )$ is an arbitrary real function such that $F \in C^2(R)$, $F \ge 0$ and $F' > 0$, we may observe via tedious calculation that

$$\frac{d}{dt}\int_\Omega F(I)\,dx \le -\int_\Omega F''(I)\psi(I)|\nabla I|^2\,dx + \int_\Omega g_1(S,I)F'(I)\,dx \tag{2.9a}$$

such that

(2.9b)

$$\begin{aligned}\frac{d}{dt}\int_\Omega (S+S^2)F(I)\,dx \le &\int_\Omega [(S+S^2)F'(I) - (1+2S)F(I)]g_1(S,I)\,dx\\ &- \int_\Omega [(1+2S)(\phi(S)+\psi(I)]F'(I)\langle \nabla I, \nabla S\rangle\,dx\\ &- \int_\Omega 2F(I)\phi(S)|\nabla S|^2\\ &- \int_\Omega \psi(I)(S+S^2)F''(I)|\nabla I|^2\,dx.\end{aligned}$$

We recall that there are positive constants $0 < \underline{a} < \overline{a}$ which bound $\phi(\ \ )$ and $\psi(\ \ )$ from above and below. We choose $F(v) = e^{\varepsilon v}$, where $\varepsilon > 0$ will be determined later and set

$$K = 2\overline{a}(1 + 2||S_0(\ \ )||_\infty). \tag{2.10}$$

We may observe that

(2.11a)

$$\begin{aligned}\frac{d}{dt}\int_\Omega (S+S^2)e^{\varepsilon I}\,dx \le &\int_\Omega \{\varepsilon(S+S^2) - (1+2S)\}e^{\varepsilon I}g_1(S,I)\,dx\\ &+ \int_\Omega \frac{\varepsilon^2 K^2}{8\underline{a}}|\nabla I|^2 e^{\varepsilon I}\,dx\end{aligned}$$

and

$$\frac{d}{dt}\int_\Omega e^{\varepsilon I}\,dx \le -\int_\Omega \omega^2 \underline{a} e^{\varepsilon I}|\nabla I|^2\,dx + \int_\Omega \varepsilon e^{\varepsilon I} g_1(S,I)\,dx. \tag{2.11b}$$

Letting $\delta = 8\underline{a}^2/K^2$, we see that

(2.12)

$$\begin{aligned}\frac{d}{dt}\int_\Omega &[1+\delta(S+S^2)]e^{\varepsilon I}\,dx\\ &\le \int_\Omega \{\varepsilon + \delta[\varepsilon(S+S^2) - (1+2S)]\}e^{\varepsilon I} g_1(S,I)\,dx\\ &\le \int_\Omega \{\varepsilon + \delta[\varepsilon(||S_0||_{\infty,\Omega} + ||S_0||^2_{\infty,\Omega}) - (1+2S)]\}e^{\varepsilon I} g_1(S,I)\,dx.\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small so that $\varepsilon \le \delta/(1+\delta[||S_0||_{\infty,\Omega} + ||S_0||^2_{\infty,\Omega}])$, we may use the inequality in (2.12) to deduce that

$$\frac{d}{dt}\int_\Omega [1+\delta(S(x,t)+S^2(x,t))]e^{\varepsilon I(x,t)}\,dx \le 0 \quad \text{on } (0,\infty). \tag{2.13}$$

From (2.13) we see that there is a constant M so that for all $t > 0$

$$\int_\Omega e^{\varepsilon I(x,t)}\,dx \le M. \tag{2.14}$$

Consequently,

$$\begin{aligned}||I(\cdot,\cdot)||_{p,\Omega\times(\tau,T)} &= \left[\int_\tau^T \int_\Omega I^p(x,t)\,dx\,dt\right]^{\frac{1}{p}}\\ &\le \left[\int_\tau^T \int_\Omega \left(\frac{p}{\varepsilon}\right)^p e^{\varepsilon I(x,t)}\,dx\,dt\right]^{\frac{1}{p}}\\ &\le \left[\left(\frac{p}{\varepsilon}\right)^p M(T-\tau)\right]^{\frac{1}{p}}\\ &\le \frac{p}{\varepsilon} M^{\frac{1}{p}} (T-\tau)^{\frac{1}{p}}\end{aligned} \tag{2.15}$$

and the result follows. □

The next result will provide a priori gradient bounds for S in $L_2(\Omega)$.

Proposition 2.16. *If* $(S(\ ,\ ), I(\ ,\ ))$ *is the solution to* (2.5a–d) *guaranteed by Theorem* 2.2 *on* $\overline{\Omega} \times [0, \infty)$, *then there exists a constant* $C_2 > 0$ *so that for all* $T > 0$,

$$||\nabla S(\cdot, T)||_{2,\Omega} \le C_2. \tag{2.17}$$

Proof. We multiply (2.5a) by S and integrate over the space time cylinder $\Omega \times (k, k+1)$, where $k \in \{0, 1, 2, \dots\}$, to obtain

$$\int_\Omega \frac{1}{2} S^2(x, k+1)\, dx + \int_k^{k+1} \int_\Omega \phi(S(x,t)) |\nabla S(x,t)|^2\, dx\, dt \le \int_\Omega \frac{1}{2} S^2(x,k)\, dx. \tag{2.18}$$

By virtue of (1.2) and (2.8), we observe that

$$\int_k^{k+1} \int_\Omega |\nabla S(x,t)|^2\, dx\, dt \le \frac{1}{2\underline{a}} \int_\Omega S^2(x,0)\, dx = \frac{1}{2\underline{a}} ||S_0||^2_{2,\Omega}. \tag{2.19}$$

From the mean value theorem for integrals we are guaranteed the existence of a $\tau_k \in (k, k+1)$ so that

$$\int_\Omega |\nabla S(x, \tau_k)|^2\, dx \le \frac{1}{2\underline{a}} ||S_0||^2_{2,\Omega}. \tag{2.20}$$

Consequently, for each $k \in \{0, 1, 2, \dots\}$ we have a τ_k so that $k < \tau_k < k+1$ and (2.20) is satisfied. Now, let $T \in (0, \infty)$. We multiply (2.5a) by $\phi(S)\partial S/\partial T$ and integrate over the space time cylinder $\Omega \times (\tau_k, T)$, where $\tau_k < T \le \tau_{k+1}$ to obtain

$$\begin{aligned} &\int_{\tau_k}^T \int_\Omega \phi(S(x,t)) \left(\frac{\partial S(x,t)}{\partial t} \right)^2 dx\, dt + \int_\Omega |\phi(S(x,T)) \nabla S(x,T)|^2\, dx \\ &\quad \le \mu \int_{\tau_k}^T \int_\Omega \left| \phi(S(x,t)) \frac{\partial S(x,t)}{\partial t} \right| S(x,t) I(x,t)\, dx\, dt \\ &\qquad + \int_\Omega |\phi(S(x,\tau_k)) \nabla S(x,\tau_k)|^2\, dx. \end{aligned} \tag{2.21}$$

Applying Young's inequality we have for $\varepsilon > 0$

(2.22)
$$\int_{\tau_k}^T \int_\Omega \left[\phi(S(x,t)) - \frac{\mu\varepsilon}{2}\phi^2(S(x,t))\right]\left(\frac{\partial S(x,t)}{\partial t}\right)^2 dx\,dt + \int_\Omega |\phi(S(x,T))\nabla S(x,T)|^2\,dx \le \frac{\mu}{2\varepsilon}\int_{\tau_k}^T \int_\Omega S^2(x,t)I^2(x,t)\,dx\,dt + \int_\Omega |\phi(S(x,\tau_k))\nabla S(x,\tau_k)|^2\,dx.$$

We now choose $\varepsilon > 0$ so that $(\phi(S) - (\mu\varepsilon/2)(\phi(S))^2) > 0$. Using (1.2), (2.7), and (2.20), we observe

$$\int_\Omega |\nabla S(x,T)|^2\,dx \le \frac{\mu}{2\varepsilon \underline{a}^2}||S_0||_{\infty,\Omega}^2 K_2(T-\tau_k) + \frac{\overline{a}^2}{2\underline{a}^3}||S_0||_{2,\Omega}^2. \tag{2.23}$$

Since $0 < T - \tau_k < 2$, the result follows. □

Proposition 2.24. *If* $(S(\ ,\ ), I(\ ,\ ))$ *is the solution to* (2.5a–c) *guaranteed by Theorem* 2.2 *on* $\overline{\Omega} \times [0,\infty)$, *then there exists a constant* $C_3 > 0$ *so that for all* $T > 0$,

$$||\nabla I(\cdot,T)||_{2,\Omega} \le C_3. \tag{2.25}$$

Proof. We proceed in a manner similar to that in Proposition (2.16), multiplying (2.5b) by I and performing an integration to obtain

$$\int_k^{k+1}\int_\Omega |\nabla I(x,t)|^2\,dx\,dt \le \frac{\mu}{\underline{a}}||S_0||_{\infty,\Omega}K_2^2 + \frac{1}{2}\int_\Omega I^2(x,k)\,dx. \tag{2.26}$$

Consequently, for each $k \in \{0,1,2,\dots\}$, we have a τ_k so that $k < \tau_k < k+1$ and

$$\int_\Omega |\nabla I(x,\tau_k)|^2\,dx \le \frac{\mu}{\underline{a}}||S_0||_{\infty,\Omega}K_2 + \frac{K_2}{2\underline{a}}. \tag{2.27}$$

Multiplying (2.5b) by $\psi(I)\partial I/\partial t$ and integrating over $\Omega\times(\tau_k,T)$ where $\tau_k<T\le\tau_{k+1}$, we obtain

(2.28)
$$\int_{\tau_k}^{T}\int_\Omega \psi(I(x,t))\left(\frac{\partial I(x,t)}{\partial t}\right)^2 dx\,dt+\int_\Omega |\psi(I(x,T)\nabla I(x,T))|^2\,dx$$
$$\le \int_{\tau_k}^{T}\int_\Omega \left|\psi(I(x,t))\frac{\partial I(x,t)}{\partial t}\right| |\mu S(x,t)I(x,t)-\lambda I(x,t)|\,dx\,dt$$
$$+\int_\Omega |\psi(I(x,\tau_k))\nabla I(x\tau_k)|^2\,dx.$$

Applying Young's inequality we have for $\varepsilon>0$

(2.29)
$$\int_{\tau_k}^{T}\int_\Omega \left[\psi(I(x,t))-\frac{\varepsilon}{2}\psi^2(I(x,t))\right]\left(\frac{\partial I(x,t)}{\partial t}\right)^2 dx\,dt$$
$$+\int_\Omega |\psi(I(x,t)\nabla I(x,T)|^2\,dx$$
$$\le \frac{\mu^2}{2\varepsilon}||S_0||^2_{\infty,\Omega}K_2^2(T-\tau_k)^2+\frac{\lambda^2}{2\varepsilon}K_2^2(T-\tau_k)^2$$
$$+\frac{\overline{a}^2\mu}{2\underline{a}}||S_0||_{\infty,\Omega}K_2+\frac{K_2\overline{a}^2}{2\underline{a}}.$$

Choosing $\varepsilon>0$ so that $(\psi(I)-(\varepsilon/2)\psi^2(I))>0$ and observing that $0<T-\tau_k<2$, we obtain our desired result. □

3. Asymptotic behavior. In this section we complete our study by investigating the asymptotic behavior of our system. Although we follow techniques developed by Webb [**14**] and utilized by Fitzgibbon and Morgan [**1**], our convergence results are not as strong as in either of the papers. Our asymptotic results will provide convergence in $L_2(\Omega)$, not $C(\Omega)$.

We shall let X denote the positive cone of $H^1(\Omega)\times H^1(\Omega)$. We define a family of operators $\{U(t)|\,t\ge 0\}$ by

(3.1)
$$U(t)(S_0,I_0)=(S(\ ,t),I(\ ,t))$$

where $(S(\ ,t),I(\ ,t))$ is the globally defined solution to (2.5a–d) guaranteed by Theorem 2.2. It is well known that $\{U(t)|\,t\ge 0\}$ is a strongly continuous semigroup on X. We shall also need to consider the positive cone Y in the weaker space $L_2(\Omega) \times L_2(\Omega)$. We have

Proposition 3.2. *If* $(S_0, I_0) \in X$, *then* $O(S_0, I_0) = \{(S(\ ,t), I(\ ,t)) | (S(\ ), I(\ ))$ *is a solution to* (2.5a–d), $S(\ ,0) = S_0(\ )$, $I(\ ,0) = I_0$ *and* $t \geq 0\}$ *is precompact in* Y. *Moreover,* $O(S_0, I_0)$ *has a nonempty, compact, connected,* ω*-limit set,* $\omega(S_0, I_0)$ *in* Y *such that* *dist* $(U(t)(S_0, I_0), \omega(S_0, I_0)) \to 0$ *as* $t \to \infty$ *where the distance is taken in the norm of* Y.

Proof. Propositions 2.18 and 2.17 insure that the trajectory $O(S_0, I_0)$ lies in a bounded subset of $H^1(\Omega)$ and therefore the Rellich lemma may be applied to establish precompactness. The remaining assertions of the proposition are standard results in the theory of dynamical systems, [**5**]. □

We specify a nonnegative real-valued functional on Y by setting

$$W(S, I) = ||G(S, I)||_1 = \int_\Omega (S + I). \tag{3.3}$$

Clearly, $W(\ )$ is continuous on Y. If we sum the components of (2.5a–b) and integrate on the space time cylinder $Q(0,t) = \Omega \times (0,t)$, we get

$$W(S(\ ,t), I(\ ,t)) + \int_0^t \lambda ||I(\ ,s)||_1 \, ds = W(S_0(\ ), I_0(\ )). \tag{3.4}$$

We may therefore consider $W(\ ,\ )$ to be a Lyapunov function for our dynamical system. We let Π_1 and Π_2 be the projections of X onto its first and second coordinates, respectively.

Proposition 3.5. *If* $(S_0(\ ), I_0(\ )) \in X$ *and* $(\Phi, \Psi) \in \omega(S_0, I_0)$, *then* $\Psi \equiv 0$ *and the projection* $\Pi_2(U(t), (\Phi, \Psi)) = 0$ *for all* $t \geq 0$.

Proof. Because $W(U(t)(S_0, I_0))$ is nonincreasing in t and bounded below by zero we can let $k = \lim_{t\to\infty} W(U(t)(S_0, I_0))$. The continuity of $W(\ ,\ )$ implies that $W(\Phi, \Psi) = k$. Because ω-limit sets are forward invariant $U(t)(\Phi,\Psi) \in \omega(S_0, I_0)$, and, hence, $W(U(t)(\Phi,\Psi)) = k$ for all $t > 0$. By virtue of (3.4), we must have $||\Pi_2(U(s),(\Phi,\Psi))|| = 0$ for all $s \geq 0$ and we obtain our desired conclusion. □

We introduce a family of Lyapunov-like functions $W_\varepsilon(\ ,\ ): Y \to \mathbf{R}$ by

$$W_\varepsilon(\Phi,\Psi) = \int_\Omega \left\{ \Phi + \Psi - \frac{\lambda}{\mu}\log(\Phi+\varepsilon)\right\}. \tag{3.6}$$

Clearly, $W_\varepsilon(\ ,\ )$ is continuous and bounded below.

Lemma 3.7. *If $(S_0, I_0) \in X$, then $W_\varepsilon(U(t)(S_0, I_0))$ is nonincreasing in t.*

Proof. Let $(S(\ ,\ ), I(\ ,\ ))$ be solutions to (2.5a–d) with initial data S_0, I_0. We compute

$$\begin{aligned}\frac{d}{dt}W_\varepsilon(U(t)(S_0,I_0)) &= \int_\Omega \partial_t\left\{S + I - \frac{\lambda}{\mu}\log(S+\varepsilon)\right\}\\ &= -\lambda\int_\Omega \left(I\left(1 - \frac{S}{S+\varepsilon} + \frac{1}{\mu}\frac{\phi(S)|\nabla S|^2}{(S+\varepsilon)^2}\right)\right.\\ &\leq 0\end{aligned}$$

and thereby conclude that $W_\varepsilon(U(t)(S_0, I_0))$ is decreasing in t. □

Proposition 3.9. *If $(S_0, I_0) \in X$ and $(\Phi,\Psi) \in \omega(S_0, I_0)$, there is a unique nonnegative constant C so that $\Phi \equiv C$.*

Proof. Following the reasoning of Proposition 3.2, we argue that there exists a k so that $k = \lim_{t\to\infty} W_\varepsilon(U(t)(S_0, I_0))$. Thus, if $(\Phi,\Psi) \in \omega(S_0, I_0)$, then $W_\varepsilon(\Phi,\Psi) = k$. The forward invariance of $\omega(S_0, I_0)$ implies that $W_\varepsilon(U(t)(\Phi,\Psi)) = k$ for $t > 0$. Moreover, if $S(t) = \Pi_1(U(t)(\Phi,\Psi))$ and $I(t) = \Pi_2(U(t)(\Phi,\Psi))$ we may integrate (3.8) to observe $\lambda/\mu \int_0^t ||\phi(S)|\nabla S|^2/(S+\varepsilon)^2||_1 = 0$ and conclude that $|| \, |\nabla S|^2/(S+\varepsilon)||_1 = 0$, and we may conclude via the absolute continuity of S that $S \equiv C$ for some $C \geq 0$. To see that C is unique, we let $t_n \to \infty$

be such that $U(t_n)(S_0, I_0) \to (\Phi, \Psi) \in Y$. We then have $\Phi \equiv C$ and $\Psi \equiv 0$. If we take the limit on each side of (3.4), we may observe that

$$\begin{aligned} W(C,0) &+ \int_0^\infty \lambda ||\Pi_2(U(s)(S_0, I_0))||_1 \\ &= m(\Omega)C + \int_0^\infty \lambda ||\Pi_2(U(s)(S_0, I_0))||_1 \\ &= W(S_0, I_0) \end{aligned} \tag{3.10}$$

where $m(\Omega)$ denotes the m-dimensional Lebesgue measure of Ω. Because (3.10) may be satisfied for only one value of C, C is unique. □

Proposition 3.11. *If* $(S_0(\ ,\ ), I_0(\ ,\ )) \in X$, *then* $\lim_{t\to\infty} ||S(\ ,t) - \Phi||_{H_1(\Omega)} = 0$.

Proof. We have shown that $S(\ ,t)$ converges to Φ in $L_1(\Omega)$ and together with the fact that $||S(\ ,t)||_{\infty,\Omega} < ||S_0||_{\infty,\Omega}$ implies that $S(\ ,t)$ converges to Φ in $L_2(\Omega)$. Consequently, if we multiply (2.5a) by S and integrate by parts we may observe that

$$\frac{1}{2}m(\Omega)\Phi^2 + \int_0^\infty \int_\Omega \phi(S)|\nabla(S)|^2 \, dx \, dt \le \frac{1}{2}||S_0||^2_{2,\Omega}. \tag{3.12}$$

Therefore, we may obtain a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \le t_{k+1}$, $t_k \to \infty$, $t_{k+1} - t_k < 2$, and

$$\lim_{k\to\infty} || \, |\nabla S(\ ,t_k)|^2||_{2,\Omega} = 0. \tag{3.13}$$

If $t_k \le t \le t_{k+1}$, we apply the argument producing inequality (2.22) to produce

$$\begin{aligned} \int_{t_k}^t \int_\Omega \left[\phi(S(x,t)) - \frac{\mu\varepsilon}{2}\phi^2(S(x,t))\right] \left(\frac{\partial S(x,t)}{\partial t}\right)^2 \, dx \, dt \\ + \int_\Omega |\phi(S(x,t))\nabla S(x,t)|^2 \, dx \\ \le \frac{\mu}{2\varepsilon}\int_{t_k}^t \int_\Omega S^2(x,t)I^2(x,t) \, dx \, dt + \int_\Omega |\phi(S(x,t_k))\nabla S(x,t_k)|^2 \, dx. \end{aligned} \tag{3.14}$$

Because

$$\int_{t_k}^{t}\int_{\Omega} S^2(x,t)I^2(x,t)\,dx\,dt \le ||S_0||_{2,\Omega}^2 \int_{t_k}^{t}\int_{\Omega} I^2(x,t)\,dx\,dt, \tag{3.15}$$

we may deduce via standard convergence arguments that the first term on the right-hand side of (3.14) converges to zero. We notice that

$$\int_{\Omega} |\phi(S(x,t_k))\nabla S(x,t_k)|^2\,dx \le \overline{a}^2 \int_{\Omega} |\nabla S(x,t_k)|^2\,dx \tag{3.16}$$

and conclude via (3.13) that the second term converges to zero. Consequently, from (3.14), we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned}\lim_{t\to\infty} ||S(\ ,t) - \Phi||_{H_1(\Omega)} \\ = \lim_{t\to\infty} (||S(\ ,t) - \Phi||_{2,\Omega} + ||\nabla S(\ ,t)||_{2,\Omega}) = 0. \quad \square\end{aligned} \tag{3.17}$$

We now use the Sobolev imbedding theorem to conclude that if $n = 1$, then

$$\sup_{x\in\Omega} |S(x,t) - \Phi| \to 0. \tag{3.18}$$

Using (3.18) together with the results of [**1**, Theorem 4.9], we obtain

Proposition 3.19. *If* $(S_0(\ ), I_0(\ )) \in X$ *and* $n = 1$*, then* $||S(\cdot,t) - \Phi||_{\infty,\Omega} \to 0$ *and if* $S(x,0) > 0$ *on a set of positive measure, then* $\Phi > 0$.

From an epidemiological point of view, we have examined the geographical spread of an infectious disease and have shown that the infective population decays to zero while the susceptible population converges to a constant (positive if $S_0 > 0$ and $n = 1$) value over the spatial region. This agrees with standard results for the spatially independent case, cf. [**14**].

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