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AN EXISTENCE THEOREM FOR QUASILINEAR ELLIPTIC EQUATIONS ON THE N-TORUS

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1. Introduction. Let $\Omega = \{x : -\pi \leq x_j < \pi, j = 1, 2, \dots, N\}$ be the N-torus, $N \geq 2$. Also let $\phi \in C^\infty(\Omega)$ mean that $\phi \in C^\infty(\mathbf{R}^N)$ and is periodic of period 2π in each variable. $W^{m,2}(\Omega)$ will be

$$\{m \text{ times weakly differentiable } u : D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq m\},$$

where the α-th weak derivative of u is v such that $\int_\Omega \phi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx$ for all $\phi \in C^{|\alpha|}(\Omega)$. $W^{m,2}(\Omega)$ will also be denoted $H^m(\Omega)$.

Let M be the number of all derivatives D^α, for $0 \leq |\alpha| \leq m-1$. Let Du stand for the M-vector whose components are $D^\alpha u$, for all $0 \leq |\alpha| \leq m-1$. That is, for $m = 1$, $Du = (u)$; for $m = 2$, $Du = \{u, D_1 u, D_2 u, \dots, D_N u)$; and so on.

With

$$Qu = (-1)^{|\beta|} D^\beta [a_{\alpha\beta}(x, Du) D^\alpha u], \tag{1.1}$$

we shall study the equation

$$Qu = g(x, u) - h. \tag{1.2}$$

(In (1.1) we use the summation convention for $1 \leq |\alpha|, |\beta| \leq m$.) h is a distribution in $H^{-m}(\Omega)$, where $H^{-m}(\Omega) = [H^m(\Omega)]^*$.

We introduce some notions concerning the g given in (1.2). In particular, we shall assume

(g-1) $g(x, s)$ meets the usual Caratheodory conditions: For each fixed $s \in \mathbf{R}$, $g(x, s)$ is measurable on Ω; for a.e. $x \in \Omega$, $g(x, s)$ is continuous on $\mathbf{R}$.

(g-2) For $r > 0$, there is $\alpha_r \in L^2(\Omega)$ such that $|g(x, s)| \leq \alpha_r(x)$ for a.e. $x \in \Omega$ and $s \in \mathbf{R}$.

(g-3) There exists nonnegative $a(x) \in L^2(\Omega)$ such that $sg(x, s) \leq |s| a(x)$ for all $s \in \mathbf{R}$ and $x \in \Omega$.

We shall also assume with respect to the operator Q in (1.1) the following:

(Q-1) The coefficients $a_{\alpha,\beta}(x,z)$ satisfy the same Caratheodory conditions as in (g-1) above.

(Q-2) There exists a nonnegative $a(x) \in L^2(\Omega)$ and $c > 0$ such that $|a_{\alpha\beta}(x,z)| \leq a(x) + c|z|$ for every $z \in \mathbf{R}^M$ and a.e. $x \in \Omega$.

(Q-3) There exists a uniformly elliptic semilinear $Lu = (-1)^{|\beta|} D^\beta[b_{\alpha\beta}(x)D^\alpha u]$ (where the $b_{\alpha\beta}$ are real-valued functions in L^∞ and the highest order coefficients are uniformly continuous) with a symmetric bilinear form $\mathbf{L}(u,v) = \int_\Omega b_{\alpha\beta}(x)D^\alpha u D^\beta v\,dx$ with first eigenvalue equal to zero and dimension of first eigenspace equal to one (i.e., $\mathbf{L}(u,u) \geq 0$ for all $u \in H^m$ and $\mathbf{L}(v,w) = 0\,\forall w \in H^m$ if and only if $v =$ constant), such that

$$\mathbf{Q}(u,u) \geq \mathbf{L}(u,u) \quad \forall u \in C^\infty$$

where

$$\mathbf{Q}(u,v) = \int_\Omega a_{\alpha\beta}(x,Du)D^\alpha u D^\beta v\,dx.$$

(For the relevant definition concerning L, see [**2**, p. 2].)

The theorem we establish is

Theorem. *Assume* (Q-1)–(Q-3) *and* (g-1)–(g-3). *Also assume* $h \in W^{-m,2}(\Omega)$. *Then if*

$$\int_\Omega g_+(x)\,dx < h(1) < \int_\Omega g_-(x)\,dx$$

where $g_+(x) = \limsup_{s\to\infty} g(x,s)$ *and* $g_-(x) = \liminf_{s\to-\infty} g(x,s)$, *there exists* $u \in W^{m,2}(\Omega)$ *with* $g(x,u) \in L^1(\Omega)$ *which is a distribution solution of* $Qu = g(x,u) - h$.

For related results in the literature, see [**2, 4, 5, 6**].

To be quite explicit, what we mean by $u \in W^{m,2}(\Omega)$ being a distribution solution of $Qu = g(x,u) - h$ is $g(x,u) \in L^1(\Omega)$ and for all $\phi \in C^\infty(\Omega)$, we have

$$Q(u,\phi) = \int_\Omega g(x,u)\phi(x)\,dx - h(\phi).$$

2. Relevant consequences of Gårding's inequality. We will use the following form of Gårding's inequality (see [**1**, p. 170]).

On the N-torus with $\langle u, Lu\rangle = \int_\Omega \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)D^\alpha u D^\beta u$, we have that there exist $c_1, c_2 > 0$ such that $\langle u, Lu\rangle \geq c_2||u||_m^2 - c_1||u||_0^2$ where $||u||_t^2 \sim \sum (1 + l\cdot l)^t |u^\wedge(l)|^2$. Here the $a_{\alpha\beta}$ are continuous for the highest order and in L^∞ for lower order.

By Gårding's inequality, we have $c_2||u||_m^2 \leq \mathbf{L}(u,u) + c_1||u||_0^2$ where we assume $\mathbf{L}(u,v)$ is as in (Q-3). Set

$$\mathbf{L}_0(u,v) = \mathbf{L}(u,v) + c_1\langle u,v\rangle_0.$$

Now

$$c_2||u||_m^2 \leq \mathbf{L}_0(u,u) \leq c_3||u||_m^2$$

so $\mathbf{L}_0(u,v)$ is an equivalent inner product to $\langle u,v\rangle_m$, for u and $v \in H^m$.

Given $f \in \tilde{H} = \{f \in L^2(\Omega) : \int f\,dx = 0\}$, for $v \in H^m$ we have

$$|\langle f,v\rangle_0| \leq ||f||_0||v||_0 \leq ||f||_0||v||_m.$$

Therefore, $\langle f,v\rangle_0 \in [W^{m,2}(\Omega)]^*$. By Riesz [**3**, p. 121], there exists $w \in H^m(\Omega)$ such that $\mathbf{L}_0(w,v) = \langle f,v\rangle_0$ for all $v \in H^m$. Therefore, $\mathbf{L}(w,1) + c_1\langle w,1\rangle_0 = \langle f,1\rangle_0$. Since $\mathbf{L}(w,1) = 0$ and $\langle f,1\rangle_0 = 0$, it follows that $\langle w,1\rangle_0 = 0$. Therefore, $w \in \tilde{H}^m = H^m \cap \tilde{H}$. Call $w = Tf$, so $\mathbf{L}_0(Tf,v) = \langle f,v\rangle_0$ for $v \in H^m$. Therefore, $T : \tilde{H} \to \tilde{H}^m \subset \tilde{H}$.

Claim. *T is symmetric on $\tilde{H}$.*

Indeed, for $g \in \tilde{H}$, $\langle g, Tf\rangle_0 = \mathbf{L}_0(Tg,Tf) = \mathbf{L}_0(Tf,Tg) = \langle f, Tg\rangle_0$.

Claim. *T is strictly positive on $\tilde{H}$ (i.e., $\langle Tf,f\rangle_0 \geq 0$ and is $= 0 \Leftrightarrow f = 0$).*

Indeed, $\langle Tf,f\rangle_0 = \langle f,Tf\rangle_0 = \mathbf{L}_0(Tf,Tf) \geq c_2||Tf||_m^2 \geq 0$. If $f = 0$, then obviously $\langle Tf,f\rangle_0 = 0$. If $\langle Tf,f\rangle_0 = 0$, then $\mathbf{L}_0(Tf,Tf) = 0$. Therefore, $Tf = 0$. Then $0 = \mathbf{L}_0(Tf,v) = \langle f,v\rangle_0$ for all $v \in \tilde{H}^m$. $\tilde{H}^m$ is dense in $\tilde{H}$ so $\langle f,v\rangle_0 = 0$ for all $v \in \tilde{H}$. Therefore, $\langle f,f\rangle_0 = 0$. Therefore, $f = 0$.

Claim. T is compact.

Indeed, given $||f_j||_0 \leq K$ for $j = 1, 2, \ldots$; we have to show there exists a subsequence $\{Tf_{jk}\}$ which is Cauchy in $\tilde{H}$. Now $|\mathbf{L}_0(Tf_j, v)| = |\langle f_j, v\rangle_0| \leq ||f_j||_0||v||_m$. Taking $v = Tf_j$, we see that $c_2||Tf_j||_m^2 \leq \mathbf{L}_0(Tf_j, Tf_j) \leq K||Tf_j||_m$. So $||Tf_j||_m \leq K/c_2$ for $j = 1, 2, \ldots$. Now $\tilde{H}^m$ is compactly embedded in $\tilde{H}$ [**1**, p. 164]. Therefore, there exists $\{Tf_{jk}\}$ which is Cauchy in $\tilde{H}$.

Now by these last three, there exist $\{\eta_j\}_{j=2}^{\infty}$ which are positive and strictly decreasing to zero and corresponding $\{\psi_{jk}\}$ such that $T\psi_{jk} = \eta_j\psi_{jk}$ and $\{\psi_{jk}\}_{j=2}^{\infty}{}_{k=1}^{\kappa(j)}$ is a complete orthonormal system in $\tilde{H}$.

Set $\lambda_j = (1/\eta_j) - c_1$. Then $\mathbf{L}_0(\psi_{jk}, v) = (1/\eta_j)\mathbf{L}_0(T\psi_{jk}, v) = (1/\eta_j)\langle\psi_{jk}, v\rangle_0$. Therefore, $\eta_j\mathbf{L}_0(\psi_{jk}, v) = \langle\psi_{jk}, v\rangle_0 = v^\wedge(j, k)$. Hence, $\mathbf{L}(\psi_{jk}, v) = \lambda_j\langle\psi_{jk}, v\rangle_0$ for all $v \in \tilde{H}^m$. Note $0 \leq \mathbf{L}(\psi_{jk}, \psi_{jk}) = \lambda_j\langle\psi_{jk}, \psi_{jk}\rangle_0 = \lambda_j$. Therefore, $\lambda_j \geq 0$ for $j = 2, 3, \ldots$.

Now $\mathbf{L}_0(\sqrt{\eta_j}\psi_{jk}, \sqrt{\eta_j}\psi_{jk}) = \eta_j\mathbf{L}_0(\psi_{jk}, \psi_{jk}) = \langle\psi_{jk}, \psi_{jk}\rangle_0 = 1$; therefore, $\{\sqrt{\eta_j}\psi_{jk}\}$ is a complete orthonormal system with respect to $\mathbf{L}_0$ on $\tilde{H}^m$.

So $v \in \tilde{H}^m$ implies that $\mathbf{L}_0(v, v) = \sum_{j=2}^{\infty}|\mathbf{L}_0(v, \sqrt{\eta_j}\psi_{jk})|^2 = \sum_{j=2}^{\infty}(|v^\wedge(j, k)|^2/\eta_j)$.

Let $\psi_{11} = 1/(2\pi)^{N/2}$.

Claim. *$H^m = \{\psi_{11}\} \oplus \tilde{H}^m$.*

Indeed, we need to show $\{\psi_{11}\} \cup \{\psi_{jk}\}_{j=2}^{\infty}{}_{k=1}^{\kappa(j)}$ is a complete orthonormal system with respect to $\mathbf{L}_0$. Suppose $\mathbf{L}_0(v, \psi_{11}) = 0$ and $\mathbf{L}_0(v, \psi_{jk}) = 0$ for $j = 2, 3, \ldots$ and $k = 1, 2, \ldots, \kappa(j)$, where $v \in H^m$. Hence, $\mathbf{L}(v, \psi_{11}) + c_1\langle v, \psi_{11}\rangle_0 = 0$ but $\mathbf{L}(v, \psi_{11}) = 0$. Therefore, $\langle v, \psi_{11}\rangle_0 = 0$. Therefore, $v \in \tilde{H}^m$. Since $\mathbf{L}_0(v, \psi_{jk}) = 0$ for $j = 2, 3, \ldots$ and $k = 1, 2, \ldots, \kappa(j)$, we have $v = 0$ establishing the claim.

Now $\mathbf{L}(\psi_{jk}, w) = \lambda_j\langle\psi_{jk}, w\rangle_0$ for all $w \in \tilde{H}^m$. Therefore, $\mathbf{L}(\psi_{jk}, \psi_{11}) = \lambda_j(\psi_{jk}, \psi_{11}\rangle_0$ for $j \geq 2$. Given $v \in H^m$, $v = v^\wedge(1, 1)\psi_{11} + w$ where $w \in \tilde{H}^m$. Therefore, $\mathbf{L}(\psi_{jk}, v) = \mathbf{L}(\psi_{jk}, w) = \lambda_j\langle\psi_{jk}, w\rangle_0 =$

$\lambda_j \langle \psi_{jk}, v\rangle_0$. Thus, ψ_{jk} is an eigenfunction with respect to λ_j and ψ_{jk} is not identically zero because $\langle \psi_{jk}, \psi_{11}\rangle_0 = 0$ and $\langle \psi_{jk}, \psi_{jk}\rangle_0 = 1$. Therefore, $\lambda_j \neq 0$ for $j \geq 2$. Therefore, $\lambda_j > 0$ for $j \geq 2$. Thus, for $v \in \tilde{H}^m$,

$$\mathbf{L}(v,v) = \mathbf{L}_0(v,v) - c_1\langle v, v\rangle_0 = \sum_{j=2}^{\infty} |v^\wedge(j,k)|^2 \left(\frac{1}{\eta_j} - c_1\right)$$
$$= \sum_{j=2}^{\infty} \lambda_j |v^\wedge(j,k)|^2 \geq \lambda_2 \sum_{j=2}^{\infty} |v^\wedge(j,k)|^2.$$

Lemma A. *If* $\mathbf{L}(v^n, v^n) \to 0$ *where* $v^n \in H^m$ *and* $v^n \to v$ *in* L^2, *then* $v = C$, *a constant.*

Proof. Set $w^n = v^n - v^{n\wedge}(1,1)\psi_{11} \in \tilde{H}^m$. Now $\mathbf{L}(w^n, w^n) = \mathbf{L}(v^n - c_1\psi_{11}, v^n - c_1\psi_{11}) = \mathbf{L}(v^n, v^n) \to 0$. So $\mathbf{L}(w^n, w^n) \to 0$. Since $\mathbf{L}(w^n, w^n) \geq \lambda_2\langle w^n, w^n\rangle_0$, we have that $w^n \to 0$ in L^2. So $v^n - v^{n\wedge}(1,1)\psi_{11} \to 0$ in L^2. Thus, $v^n \to v^\wedge(1,1)\psi_{11} =$ constant. Therefore, v is constant. □

Lemma B. *With the conditions as above and the assumption that* $||v^n||_m^2 = 1$, C *is nonzero.*

Proof. $1 = ||v^n||_m^2$ and by Gårding, this is $\leq c_2^{-1}[\mathbf{L}(v^n, v^n) + c_1||v^n||_0^2]$. Now as $n \to \infty$, we have that $\mathbf{L}(v^n, v^n) \to 0$ and $||v^n||_0 \to ||v||_0$, so $1 \leq (c_1/c_2)||v||_0^2$. Therefore, v is nonzero. □

3. Fundamental lemmas.

Lemma 1. *Let* $B \geq 0$ *be an* L^2 *function,* g *satisfy* (g-1), Q *satisfy* (Q-1)–(Q-3), $h \in H^{-m}(\Omega)$, *and* $|g(x,s)| \leq B(x)$ *for* $s \in \mathbf{R}$, *a.e.* $x \in \Omega$. *If* n *is a positive integer, there exists* $u^n = \gamma_1^n\psi_1 + \cdots + \gamma_n^n\psi_n$ *such that*

(3.1)
$$\int_\Omega \sum \left[a_{\alpha\beta}(x, Du)D^\alpha u^n D^\beta \psi_k + \frac{u^n\psi_k}{n}\right] dx = \int_\Omega \psi_k g(x, u^n)\, dx - h(\psi_k).$$

Here, $\{\psi_k\}_{k=1}^{\infty}$ *is a complete orthonormal sequence in* $L^2(\Omega)$ *with each* $\psi_k \in C^\infty(\Omega)$ *and* $\psi_1 = (2\pi)^{-N/2}$*. Furthermore, given* $\phi \in C^\infty(\Omega)$*, there exists a sequence of constants* $\{c_k\}_{k=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} c_k\psi_k(x) = \phi(x)$$

uniformly for $x \in \Omega$*.*

Proof. Let $f_k(\alpha) = \mathbf{Q}(\alpha_p\psi_p, \psi_k) + (\langle \alpha_p\psi_p, \psi_k\rangle/n) - \int_\Omega \psi_k g(x, \alpha_p\psi_p) + h(\psi_k)$ for $k = 1, \dots, n$. Note that $f_k(\alpha)\cdot\alpha_k \geq \mathbf{L}(\alpha_p\psi_p, \alpha_k\psi_k) + (|\alpha|^2/n) - \int_\Omega B(x)|\alpha_k\psi_k| - h(\alpha_k\psi_k)| \geq 0 + (|\alpha|^2/n) - K_1|\alpha| - K_2|\alpha| \geq (|\alpha|^2/n) - K_0|\alpha| > 0$ for $|\alpha|$ large, say $|\alpha| = p$. Define $F(x,\lambda) = \lambda f(x) + (1-\lambda)x$ for $0 \leq \lambda \leq 1$. Let $\overline{D} = \overline{B}(0,p)$.

Now $f(x)\cdot x > 0$ for $|x| = p$ and indeed $f(x)\cdot x \geq \varepsilon > 0$ for $|x| = p$.

Then $F(x,\lambda)\cdot x = \lambda f(x)\cdot x + (1-\lambda)|x|^2 \geq \lambda\varepsilon + (1-\lambda)|x|^2 > 0$. Therefore, $F(x,\lambda) \neq 0$ for $0 \leq \lambda \leq 1$ and $|x| = p$.

Now, using topological degree theory, $d(f, D, 0) = d(F(x,1), D, 0) = d(F(x,0), D, 0)$ (due to invariance with respect to homotopy) $= d(I, D, 0) = 1$. So by the Kronecker existence theorem, there exists $x^* \in \mathbf{R}^n$ such that $f(x^*) = 0$. Letting $\alpha = x^*$, we have (3.1). □

The next lemma we prove is

Lemma 2. *Let* n *be a given positive integer. Also, let* g *satisfy* (g-1)–(g-3)*. Suppose that* Q *satisfies* (Q-1)–(Q-3)*. Then there is a function* $u = \gamma_1\psi_1 + \dots + \gamma_n\psi_n$*, where* $\gamma_1, \dots, \gamma_n$ *are constants, such that*

$$\int_\Omega \sum_{1\leq|\alpha|,|\beta|\leq m} \left[a_{\alpha\beta}(x, Du)D^\alpha u D^\beta\psi_k + \frac{u\psi_k}{n}\right] dx = \int_\Omega \psi_k(x)g(x,u)\,dx - h(\psi_k).$$

Proof. For each positive integer p, set

$$g^p(x,s) = \begin{cases} g(x,p), & s \geq p; \\ g(x,s), & -p \leq s \leq p; \\ g(x,-p), & s \leq -p. \end{cases}$$

Then it follows from (g-2) that there is an $\alpha_p(x) \in L^2(\Omega)$ such that $|g^p(x,s)| \leq \alpha_p(x)$ for $s \in \mathbf{R}$ and a.e. $x \in \Omega$.

Consequently, it follows from Lemma 1 that there exist constants $\{\gamma_i^p\}_{i=1}^n$ such that

$$u^p = \gamma_1^p \psi_1 + \cdots + \gamma_n^p \psi_n \tag{3.2}$$

and satisfies (3.1) with g replaced by g^p, i.e.,
(3.3)

$$\langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du^p) D^\alpha u^p \rangle_0 + \frac{\langle \psi_k, u^p \rangle_0}{n} = \langle \psi_k, g^p(\cdot, u^p) \rangle_0 - h(\psi_k),$$
$$\text{for } k = 1, \dots, p.$$

Now it follows from the definition and (g-3) that $sg^p(x,s) \leq |s|a(x)$ for all $s \in \mathbf{R}$ and a.e. $x \in \Omega$. A similar inequality will prevail a.e. in Ω if we replace s by the u^p given in (3.2). Consequently, if we multiply both sides of (3.3) by γ_k^p and sum on k, we obtain by (Q-3) that for all $p \in \mathbf{Z}^+$, $0 + (\langle u^p, u^p \rangle_0 / n) \leq \langle u^p, g^p(\cdot, u^p) \rangle_0 - h(u^p) \leq \langle |u^p|, a \rangle_0 - h(u^p) \leq \langle u^p, u^p \rangle_0^{1/2} \langle a, a \rangle_0^{1/2} - h(u^p)$. Now, since $h \in H^{-m}(\Omega)$, $(\langle u^p, u^p \rangle_0 / n) \leq \langle u^p, u^p \rangle_0^{1/2} K + K' \langle u^p, u^p \rangle_m^{1/2}$. It is clear that there is a constant depending on n such that $||u||_m \leq K^n ||u||_0$; therefore, $(\langle u^p, u^p \rangle_0 / n) \leq \langle u^p, u^p \rangle_0^{1/2} K + K'' \langle u^p, u^p \rangle_m^{1/2}$. Therefore, $\langle u^p, u^p \rangle_0^{1/2} \leq n(K + K'')$. Thus, by (3.2) and the orthonormality of the ψ_j's, $(\psi_1^p)^2 + \cdots + (\psi_n^p)^2 \leq$ a constant depending on n.

Therefore, there exists a subsequence $\{\gamma_k^p\}$ which converges for each $k = 1, \dots, n$. For ease of notation, say it is the full sequence and write

$$\lim_{p \to \infty} \gamma_k^p = \gamma_k^n \qquad \text{for } k = 1, \dots, n. \tag{3.4}$$

We set $u = \gamma_1^n \psi_1 + \cdots + \gamma_n^n \psi_n$ and see by the definition of u^p and (3.4) that

$$\lim_{p \to \infty} u^p(x) = u(x) \qquad \text{uniformly for } x \in \Omega \tag{3.5a}$$

and

(3.5b)
$$\lim_{p\to\infty} D^\alpha u^p(x) = D^\alpha u(x) \qquad \text{uniformly for } x \in \Omega \text{ and } 1 \le |\alpha| \le m.$$

From this and (Q-1), we see that $\lim_{p\to\infty} a_{\alpha\beta}(x, Du^p(x)) = a_{\alpha\beta}(x, Du(x))$ for a.e. $x \in \Omega$ and $1 \le |\alpha|$, $|\beta| \le m$. From this with (Q-2), (3.2), (3.4), and (3.5) using the generalized Lebesgue Convergence Theorem, we see that

$$\lim_{p\to\infty} \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du^p) D^\alpha u^p\rangle_0 = \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du) D^\alpha u\rangle_0 \quad \text{for } k = 1, \ldots, n. \tag{3.6}$$

Then we see from (3.2) and (3.4) that $\{u^p\}_{p=1}^\infty$ is uniformly bounded on Ω and is in $C^\infty(\Omega)$ for each p.

Thus, from the definition of g^p, there exists p_0 such that $p \ge p_0$ implies that $g^p(x, u^p(x)) = g(x, u^p(x))$ for $x \in \Omega$. Then by (g-1), (g-2) and (3.5) we see that $\lim_{p\to\infty} \langle \psi_k, g^p(\cdot, u^p)\rangle_0 = \langle \psi_k, g(\cdot, u)\rangle_0$ for $k = 1, \ldots, n$. Now from this with (3.3), (3.5) and (3.6), we obtain our conclusion. □

The next lemma we prove is the following

Lemma 3. *Suppose Q satisfies* (Q-1)–(Q-3)*, $h \in H^{-m}(\Omega)$, and that g satisfies* (g-1)–(g-3)*. Suppose also that for every positive integer n, there is a $u^n = \gamma_1^n \psi_1 + \cdots + \gamma_n^n \psi_n$, where $\gamma_1^n, \ldots, \gamma_n^n$ are constants, which satisfies for $k = 1, \ldots, n$,*

$$\int_\Omega \sum_{1\le|\alpha|,|\beta|\le m} \left[a_{\alpha\beta}(x, Du^n) D^\alpha u^n D^\beta \psi_k + \frac{u^n \psi_k}{n} \right] dx = \int_\Omega \psi_k(x) g(x, u)\, dx - h(\psi_k). \tag{3.7}$$

Assume furthermore that there is a constant K such that

$$||u^n||_m \le K \qquad \text{for } n = 1, 2, \ldots . \tag{3.8}$$

Then there is a constant K^* such that $\langle |g(\cdot, u^n)|, |u^n|\rangle_0 \le K^*$ for $n = 1, 2, \ldots$.

Proof. Multiplying both sides of (3.7) by γ_k^n and summing over $k = 1, \dots, n$, we obtain

$$\langle D^\beta u^n, a_{\alpha\beta}(\cdot, Du^n) D^\alpha u^n \rangle_0 + \frac{\langle u^n, u^n \rangle_0}{n} = \langle u^n, g(\cdot, u^n) \rangle_0 - h(u^n).$$

Consequently, we have from (Q-3) that

$$0 \le \langle u^n, g(\cdot, u^n) \rangle_0 - h(u^n). \tag{3.9}$$

Next we set

$$A_n = \{x \in \Omega : u^n g(x, u^n) \ge 0\} \tag{3.10a}$$

and

$$B_n = \{x \in \Omega : u^n g(x, u^n) < 0\} \tag{3.10b}$$

and observe from (Q-3) that $\int_{A_n} u^n g(x, u^n)\, dx \le ||u^n||_0 ||a||_0$ for $n = 1, 2, \dots$. Therefore, it follows from (3.8) that there is a constant K_1 such that

$$\int_{A_n} u^n g(x, u^n)\, dx \le K_1 \qquad \text{for } n = 1, 2, \dots . \tag{3.11}$$

Owing to (3.8), (3.9) and the fact that $\Omega = A_n \cup B_n$, $-\int_{B_n} u^n g(x, u^n)\, dx \le \int_{A_n} u^n g(x, u^n)\, dx + K_2$ follows. But then from (3.11) we have

$$-\int_{B_n} u^n g(x, u^n)\, dx \le K_1 + K_2 \qquad \text{for } n = 1, 2, \dots .$$

This fact, in conjunction with (3.10) and (3.11), gives us $\int_\Omega |u^n||g(x,u^n)|\, dx \le 2K_1 + K_2$ for $n = 1, 2, \dots$. However, this is the conclusion with $K^* = 2K_1 + K_2$ so the proof is complete. □

Lemma 4. *Suppose the conditions in the hypothesis of Lemma* 3 *hold. Then the sequence* $\{g(x, u^n)\}_{n=1}^\infty$ *is absolutely equi-integrable.*

To be precise, what we mean by absolutely equi-integrable is the following: given $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that if* $E \subset \Omega$ *with*

$\mu(E) < \delta$, then $\int_E |g(x,u^n)|\,dx < \varepsilon$ for $n = 1, 2, \dots$, where μ is N-dimensional Lebesgue measure.

Proof. First we choose $r > 0$ so that

$$\frac{K^*}{r} < \frac{\varepsilon}{2}, \tag{3.12}$$

where K^* is the constant in Lemma 3. Next, using (g-2), we choose $\alpha_r \in L^2(\Omega)$ such that

$$|g(x,s)| \le \alpha_r(x) \qquad \text{for a.e. } x \in \Omega \text{ and } |z| < r.$$

Also, we set

$$A_n = \{x \in \Omega : |u^n| \le r\}$$

and

$$B_n = \{x \in \Omega : |u^n| > r\}$$

and choose $\delta > 0$ such that $\mu(E) < \delta$ implies that $\int_E \alpha_r(x)\,dx < \varepsilon/2$. Now suppose $\mu(E) < \delta$ as in this last statement. Then it follows from Lemma 3 and these last three formulae that

$$\begin{aligned}\int_E |g(x,u^n(x))|\,dx &\le \int_{E\cap A_n} \alpha_r(x)\,dx + r^{-1}\int_{E\cap B_n} |u^n(x)g(x,u^n(x))|\,dx \\ &\le \frac{\varepsilon}{2} + \frac{K^*}{r} \qquad \text{for } n = 1, 2, \dots .\end{aligned}$$

From (3.12) we see that the right-hand side of this last established inequality is less than ε. Consequently, $\{g(x,u^n)\}_{n=1}^{\infty}$ is absolutely equi-integrable, and the proof of the lemma is complete. □

4. Proof of Theorem. Note that the hypotheses of the theorem imply those of Lemma 2. So for $n \in \mathbf{Z}^+$, there exists u^n as in the conclusion of Lemma 2.

We claim there is a constant K such that

$$||u^n||_m \le K \qquad \text{for } n = 1, 2, \dots \tag{4.1}$$

where

$$||u||_m^2 = \sum_{|\alpha| \le m} ||D^\alpha u||_0^2. \tag{4.2}$$

Say not, i.e., (4.1) is false. Then (for ease of notation) $\lim_{n\to\infty} ||u^n||_m = \infty$; and setting

$$v^n = \frac{u^n}{||u^n||_m} \tag{4.3}$$

we get that [**1**, p. 169, Lemma 10 with $H_0 = L^2(\Omega)$ and $H_m = W^{m,2}(\Omega)$]:

$$\begin{aligned} ||v^n - v||_0 \to 0 \qquad &\text{as } n \to \infty \quad \text{for some } v \in W^{m,2}(\Omega); \\ v^n \to v \qquad &\text{for a.e. } x \in \Omega; \end{aligned} \tag{4.4}$$

and

$$\begin{gathered} \lim_{n\to\infty} \int_\Omega w D^\alpha v^n \, dx = \int_\Omega w D^\alpha v \, dx \\ \text{for all } w \in L^2(\Omega) \text{ and } 0 \le |\alpha| \le m. \end{gathered} \tag{4.5}$$

Therefore, $v^n \to v$ weakly in H^m. The conclusion of Lemma 2 now gives

$$\begin{aligned} &\langle D^\beta v^n, a_{\alpha\beta}(x, Du^n) D^\alpha v^n \rangle_0 + \langle v^n, v^n \rangle_0 n^{-1} \\ &\qquad = [\langle u^n, g(\cdot, u^n)\rangle_0 - h(u^n)] ||u^n||_m^{-2}. \end{aligned} \tag{4.6}$$

By (Q-3), the left-hand side of (4.6) is greater than zero. Now by (g-3), there exists a nonnegative $a(x) \in L^2(\Omega)$ such that $sg(x,s) \le |s| a(x)$ for all $s \in \mathbf{R}$ and $x \in \Omega$. Thus, we see from (Q-3) and (4.6) that

$$\mathbf{L}(v^n, v^n) \le \frac{\int_\Omega |u^n||a(x)| \, dx}{||u^n||_m^2} - \frac{h(v^n)}{||u^n||_m} \le \frac{||u^n||_0 ||a||_0}{||u^n||_m^2} - \frac{h(v^n)}{||u^n||_m}.$$

Thus we have from (Q-3) that

$$\lim_{n\to\infty} \mathbf{L}(v^n, v^n) = 0. \tag{4.7}$$

Therefore, by Lemma A and Lemma B, $v =$ constant, which is different from zero. Thus, we see that $v = k > 0$ for a.e. $x \in \Omega$ or $v = -k$ for a.e. $x \in \Omega$. Suppose that $v = k$ for a.e. $x \in \Omega$. (The case $v = -k$ for a.e. $x \in \Omega$ is similar.)

Now by (Q-3) and (4.6), $0 \leq \langle u^n, g(\cdot, u^n)\rangle_0 - h(u^n)$. Therefore, $h(u^n) \leq \langle u^n, g(\cdot, u^n)\rangle_0 = \int_\Omega u^n g(\cdot, u^n)\,dx$. By the linearity of h we have

$$h(v^n) \leq \int_\Omega v^n g(x, u^n)\,dx. \tag{4.8}$$

Now (again using the linearity of h) $h(v^n) \to kh(1)$ as $n \to \infty$. We let $g_{++}(x) = \limsup_{n\to\infty} g(x, u^n)$ and we observe that $g_{++}(x) \leq g_+(x)$. From (4.8),

$$-h(v^n) \geq \int_\Omega (-v^n g(x, u^n) + a(x)|v^n|)\,dx - \int_\Omega a(x)|v^n|\,dx.$$

Now, by Fatou,

$$\begin{aligned} -kh(1) &\geq -\int_\Omega (v g_{++}(x) - a(x)|v|)\,dx - \int_\Omega a(x)|v|\,dx \\ &= -\int_\Omega v g_{++}(x)\,dx \\ &= -k\int_\Omega g_{++}(x)\,dx. \end{aligned}$$

Therefore, $h(1) \leq \int_\Omega g_{++}(x)\,dx \leq \int_\Omega g_+(x)\,dx$ which is a contradiction to the hypotheses. Thus, we conclude our claim (4.1) is true.

Then [**1**] there exists a subsequence (for ease of notation the full sequence) of $\{u^n\}_{n=1}^\infty$ and a function $u \in W^{m,2}(\Omega)$ such that:

$$\lim_{n\to\infty} ||D^\alpha u^n - D^\alpha u||_0 = 0 \qquad \text{for } |\alpha| \leq m-1; \tag{4.9}$$

$$\lim_{n\to\infty} D^\alpha u^n = D^\alpha u \qquad \text{for a.e. } x \in \Omega \text{ and } |\alpha| \leq m-1; \tag{4.10}$$

and

$$\lim_{n\to\infty} \int_\Omega w D^\alpha u^n\,dx = \int_\Omega w D^\alpha u\,dx \qquad \text{for all } w \in L^2(\Omega) \text{ and } |\alpha| = m. \tag{4.11}$$

Therefore $u^n \to u$ weakly in H^m.

From (4.10) and (Q-1), we see that $\lim_{n\to\infty} a_{\alpha\beta}(x, Du^n(x)) = a_{\alpha\beta}(x, Du(x))$ for a.e. $x \in \Omega$.

With this, (4.9), (Q-2), and the generalized Lebesgue Convergence Theorem [**3**, p. 89], we obtain

$$\lim_{n\to\infty} ||a_{\alpha\beta}(\cdot, Du^n) - a_{\alpha\beta}(\cdot, Du)||_0 = 0.$$

Indeed, $2a(x) + c[|Du| + |Du^n|] \to 2a(x) + 2c|Du|$ and by (Q-2) $\lim_{n\to\infty} \int \{2a(x)+c[|Du|+|Du^n|]\}^2\, dx = \int \{2a(x)+2c|Du|\}^2\, dx$. Also, $|a_{\alpha\beta}(x, Du^n) - a_{\alpha\beta}(x, Du)| \leq 2a(x) + c[|Du| + |Du^n|]$ so the theorem applies and the result is obtained.

Now this with (4.1) (which implies that $[\int |D^\alpha u^n|^2\, dx]^{1/2} <$ constant) and Schwarz give that for k fixed

$$\lim_{n\to\infty} \langle D^\beta \psi_k, [a_{\alpha\beta}(\cdot, Du^n) - a_{\alpha\beta}(\cdot, Du)] D^\alpha u^n\rangle_0 = 0. \tag{4.12}$$

From (4.11) and (Q-2) (which implies that $[D^\beta\psi_k][a_{\alpha\beta}(x, Du)] \in L^2(\Omega)$) we get

$$\lim_{n\to\infty} \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du) D^\alpha u^n\rangle_0 = \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du) D^\alpha u\rangle_0.$$

Then with this and (4.12) we get

$$\lim_{n\to\infty} \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du^n) D^\alpha u^n\rangle_0 = \langle D^\beta \psi_k, a_{\alpha\beta}(\cdot, Du) D^\alpha u\rangle_0. \tag{4.13}$$

Next from (4.10) and (g-1) we see that

$$\lim_{n\to\infty} g(x, u^n(x)) = g(x, u(x)) \qquad \text{for a.e. } x \in \Omega. \tag{4.14}$$

By (g-3), we can apply Lemma 4 to get that

(4.15) $\{g(x, u^n(x))\}$ is absolutely equi-integrable.

(Note: $u^n \in C^\infty(\Omega)$.) Thus, there exists K such that $\int_\Omega |g(x, u^n)|\, dx \leq K$ for $n = 1, 2, \ldots$. Then using (4.14), we see that $\int_\Omega |g(x, u)|\, dx \leq K$; therefore, $g(x, u) \in L^1(\Omega)$.

Now for k fixed, $\lim_{n\to\infty} g(x,u^n)\psi_k = g(x,u)\psi_k$ for a.e. $x\in\Omega$. So

$$\{g(x,u^n)\psi_k\} \text{ is absolutely equi-integrable.} \tag{4.16}$$

Now, given $\varepsilon > 0$, there exists δ such that $E\subset\Omega$ and $\mu(E)<\delta$ imply that $\int_E |g(x,u^n)\psi_k|dx < \varepsilon$ for $n = 1,2,\ldots$. By Egoroff, given δ there exists $E\subset\Omega$ with $\mu(E)<\delta$ such that $[g(x,u^n)-g(x,u)]\psi_k \to 0$ uniformly on $\Omega - E$. Then

$$\begin{aligned}\limsup_{n\to\infty}&\left|\int_\Omega \psi_k[g(x,u^n)-g(x,u)]\,dx\right| \\ &\le \limsup_{n\to\infty}\left|\int_{\Omega-E} \psi_k[g(x,u^n)-g(x,u)]\,dx\right| \\ &\quad + \limsup_{n\to\infty}\left|\int_E \psi_k[g(x,u^n)]\,dx\right| \\ &\quad + \limsup_{n\to\infty}\left|\int_E \psi_k[g(x,u)]\,dx\right| \\ &\le 0+\varepsilon+\varepsilon \\ &\le 2\varepsilon.\end{aligned} \tag{4.17}$$

ε was arbitrary so $\lim_{n\to\infty}\int_\Omega \psi_k g(x,u^n)\,dx = \int_\Omega \psi_k g(x,u)\,dx$. Thus,

$$\lim_{n\to\infty}\langle\psi_k, g(\cdot,u^n)\rangle_0 - h(\psi_k) = \langle\psi_k, g(\cdot,u)\rangle_0 - h(\psi_k). \tag{4.18}$$

From (4.9), Lemma 2, (4.13), and (4.18) we get

$$\langle D^\beta\psi_k, a_{\alpha\beta}(x,Du)D^\alpha u\rangle_0 = \langle\psi_k, g(x,u)\rangle_0 - h(\psi_k). \tag{4.19}$$

Now, given $\phi\in C^\infty(\Omega)$, from the uniform approximation property of the ψ's, there exist real $\{c_q^n\}_{q=1}^n$ and $\{\phi_n\}_{n=1}^\infty$ with

$$\phi_n = c_1^n\psi_1 + \cdots + c_n^n\psi_n \tag{4.20}$$

such that

$$\lim_{n\to\infty}\phi_n(x) = \phi(x) \qquad \text{uniformly for } x\in\Omega \tag{4.21}$$

and

(4.22)
$$\lim_{n\to\infty} D^\alpha \phi_n(x) = D^\alpha \phi(x) \qquad \text{uniformly for } x \in \Omega \text{ and } 1 \le |\alpha| \le m.$$

Since $u \in W^{m,2}(\Omega)$, from (Q-2) and Schwarz we see

$$a_{\alpha\beta}(x,u) D^\alpha u \in L^1(\Omega) \qquad \text{for } 1 \le |\alpha|, |\beta| \le m. \tag{4.23}$$

From (4.22) and (4.23), we obtain

$$\lim_{n\to\infty} \langle D^\beta \phi_n, a_{\alpha\beta}(\cdot, Du) D^\alpha u\rangle_0 = \langle D^\beta \phi, a_{\alpha\beta}(\cdot, Du) D^\alpha u\rangle_0. \tag{4.24}$$

Also from (4.21)

$$\lim_{n\to\infty} \langle \phi_n, g(\cdot, u)\rangle_0 - h(\phi_n) = \langle \phi, g(\cdot, u)\rangle_0 - h(\phi). \tag{4.25}$$

Now, from (4.20), we see that (4.19) holds with ψ_k replaced by ϕ_n. Then from (4.24) and (4.25) we see that (4.19) holds with ψ_k replaced by ϕ, i.e.,

$$\langle D^\beta \phi, a_{\alpha\beta}(x, Du) D^\alpha u\rangle_0 = \langle \phi, g(x,u)\rangle_0 - h(\phi).$$

ϕ is arbitrary in $C^\infty(\Omega)$ so this shows there exists a distribution solution of $Qu = g(x,u) - h$. $\square$

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