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A TOPOLOGICAL APPROACH TO MORITA EQUIVALENCE FOR RINGS WITH LOCAL UNITS

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ABSTRACT. In [**1**] and [**3**] a theory of Morita equivalence has recently been developed for certain not necessarily unital rings called rings with local units. In this article we prove that the special Hom-sets which figure in the description of equivalence functors are actually the sets of continuous homomorphisms from a locally projective generator (endowed with a suitable topology) into discrete modules. The main result of this paper says that two rings with local units which fulfill a topological condition of projectivity are Morita equivalent if and only if suitable matrix rings over them are isomorphic to each other.

Following the terminology of [**3**], a ring R is said to have *local units* if there is a set E of idempotents in R such that for any $r, s \in R$ there is an $e \in E$ which acts as a two-sided identity for both r and s; in particular, any unital ring has local units with $E = \{1\}$. Note that if R has local units, then $R = \cup_{e\in E} eRe$. For any ring R with local units and any (infinite) set I, denote by R_I^f the ring of $I \times I$ matrices over R which contain at most finitely many nonzero entries. Clearly, R_I^f also has local units.

Throughout this article all modules are assumed to be left modules (unless otherwise indicated), and all module homomorphisms will be written on the right. For any set I and any module M, we denote by $M^{(I)}$ the (discrete) direct sum of I copies of M.

Let R be a ring with local units. As in [**3**], we denote by RMod the category of unitary modules ${}_RM$ (those with $RM = M$) together with all R-homomorphisms. Recall [**3**, Definition 2] that a module $P \in R$Mod is said to be *locally projective* if there is a direct system $\{P_i\}_{i\in I}$ of finitely generated projective direct summands of P (so that

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I is an upward directed set, and P_i is a summand of P_j whenever $i \leq j$) together with projections $\psi_i : P \to P_i$ satisfying $\psi_j \psi_i = \psi_i$ whenever $i \leq j$, and such that $\varinjlim\{P_i\} = P$. If we order the idempotents in a set E of local units for R by defining $e_i \leq e_j$ whenever $e_i e_j = e_j e_i = e_i$, then ${}_R R$ is seen to be locally projective, with summands $P_i = Re_i$ and projections $\psi_i : R \to Re_i$ defined via right multiplication by e_i.

If R is a ring with a set E of local units, then a natural topology can be introduced in R by taking the sets $R(1-e) = \{r - re \mid r \in R, e \in E\}$ as a base of open neighborhoods of 0. This topology is Hausdorff because $\cup_{e \in E} Re = R$. More generally, if ${}_R P$ is any locally projective module with a direct system P_i and projections ψ_i, then the sets $\ker \psi_i$ form a base of open neighborhoods of 0 in P. The corresponding topology, which is Hausdorff, will be called the *locally projective topology* for P, and $\overline{P}$ will denote the locally projective module P endowed with the locally projective topology. Note that the topology of $\overline{P}$ induces the discrete topology on each P_i.

As we shall see below in Example 3, a locally projective module may carry various locally projective topologies, depending on its defining system of projections. Nevertheless, we shall speak of *the* locally projective topology, because whenever we have a locally projective module, we assume tacitly that a defining system has been fixed for it; however, in order to avoid cumbersome notations, we shall continue to write simply P for a locally projective module. When speaking of ${}_R\overline{R}$, we always mean the locally projective structure induced by the set of all idempotents of R as was described above; in fact, any set of local units induces this same topology on R.

For any $M \in R$Mod, we denote by $\underline{M}$ the module M endowed with the discrete topology. If $P, M \in R$Mod and P is locally projective, then a mapping $\alpha : P \to M$ factors through one of the canonical projections ψ_i of P if and only if it is continuous as a mapping from $\overline{P}$ to $\underline{M}$, because both conditions mean that $\ker \alpha$ contains $\ker \psi_i$ for some $i \in I$.

For any two topological left R-modules X and Y we denote the abelian group of continuous R-homomorphisms from X to Y by $\mathrm{ContHom}_R(X, Y)$ (the subscript R will be suppressed when the underlying ring is clear from context). Then the following can be added to the list of properties of equivalences given in [**3**, Theorem 2.1]:

Proposition 1. *Let R and S be equivalent rings with local units via inverse equivalences $G : R\mathrm{Mod} \to S\mathrm{Mod}$ and $H : S\mathrm{Mod} \to R\mathrm{Mod}$, and put $P = H({}_SS)$ and $Q = G({}_RR)$. Then we have, under suitable locally projective topologies on ${}_RP$ and ${}_SQ$,*

$$G \cong \mathrm{ContHom}_R(\overline{P}, \text{---}\,) \text{ and } H \cong \mathrm{ContHom}_S(\text{---}\overline{Q}, \underline{\ \ }).$$

Proof. As was seen before the formulation of Theorem 2.5 in [**3**], for any $M \in R\mathrm{Mod}$, $S\mathrm{Hom}_R(P, M)$ consists of those homomorphisms from P to M which factor through one of the ψ_i. Since these homomorphisms are exactly the elements of $\mathrm{ContHom}_R(\overline{P}, \text{---}\,)$, by [**3**, Theorem 2.1(3)] we obtain the validity of our first claim. The second is proven dually. □

Remark . In particular, the ring $\mathrm{ContHom}_R(\overline{P}, \underline{P})$ consists of those endomorphisms of P which factor through one of the ψ_i. On the other hand, we have that

$$\mathrm{ContHom}(\overline{P}, \underline{P}) = \{\alpha \in \mathrm{ContHom}(\overline{P}, \overline{P}) \mid \alpha \text{ is of finite rank}\},$$

where "α is of finite rank" means that the image of α is contained in a finitely generated submodule. To verify this, note that if $\alpha \in \mathrm{ContHom}(\overline{P}, \overline{P})$ is of finite rank, then $\alpha(P)$ is contained also in some P_i; then, for this i, $\ker \psi_i \cap \alpha(P) = 0$ and thus $\alpha^{-1}(0) = \alpha^{-1}(\ker \psi_i)$. Now $\ker \psi_i$ is open in $\overline{P}$ and $\alpha : \overline{P} \to \overline{P}$ is continuous, whence $\alpha^{-1}(0)$ is open in $\overline{P}$ and thus $\alpha : \overline{P} \to \underline{P}$ is continuous. Conversely, if $\alpha : \overline{P} \to \underline{P}$ is continuous, then, of course, it is also continuous as $\alpha : \overline{P} \to \overline{P}$, and it is of finite rank because by an earlier observation it factors through some ψ_i, whose image is the finitely generated submodule P_i.

Proposition 1 and the above remark as well as [**3**, Theorem 2.1 and 2.5] now yield the following.

Theorem 2. *Two rings R and S with local units are Morita equivalent if and only if there exists a locally projective generator ${}_RP$ such that S is isomorphic to the ring of continuous endomorphisms of finite rank of $\overline{P}$.*

Example 3. Consider a division ring D and a dual pair $({}_DM, N_D)$ of vector spaces over D (i.e., a bilinear product $\langle\ ,\ \rangle : M \times N \to D$

is defined which is nondegenerate in the sense that $\langle m, y\rangle = 0$ for all $y \in N$ implies $m = 0$ and $\langle x, n\rangle = 0$ for all $x \in M$ implies $n = 0$). As is described in Chapter IV of [6], the given bilinear product induces a topology on M (which is called the *finite topology*). Denote by S the ring of continuous endomorphisms of finite rank of this topological module M. S is a regular ring, hence a ring with local units (see [**3**, Section 3, Example 1]), and the set of all idempotents of S induces a locally projective structure on the left vector space M by putting ${}_D M = \varinjlim \{Mf \mid f^2 = f \in S\}$, with order inherited from $f \leq g$ if and only if $fg = gf = f$ and with the canonical projections $\psi_f : m \mapsto mf$. It is clear from the considerations in [**6**] that the locally projective topology of M agrees with the original topology of M.

Conversely, if $M = \varinjlim M_i$ is a locally projective left vector space over D with projections ψ_i we denote by N the right vector space of all continuous homomorphisms from ${}_D\overline{M}$ to ${}_D\underline{D}$. Putting $\langle m, n\rangle = (m)n$ we obtain a dual pair (M, N) over D, and it is clear that the topology induced by this bilinear product on M agrees with the original locally projective topology.

By [**6**, Ch. IV, Section 6, Prop. 3] we also know that, for a dual pair (M, N) over D, all submodules of M are closed if and only if $N = M^*$ (= all D-homomorphisms from M to D; obviously, if (M, N) is a dual pair, then N can be considered as a subspace of M^*). We have seen above that the topologies induced by dual pairs on a vector space M are the same as the locally projective topologies on M; hence, if (M, N) is a dual pair with $N \neq M^*$, then the dual pairs (M, N) and (M, M^*) yield two different locally projective topologies on M.

Now let ${}_R P$ be a locally projective module and Γ an arbitrary set. Then $P^{(\Gamma)}$ is also locally projective, being the direct limit of the system $\{\oplus_{\gamma\in\Gamma} A_\gamma\}$ where only finitely many A_γ are different from 0 and each of the latter is a P_i in P_γ. The locally projective topology of $P^{(\Gamma)}$ is just the restriction of the product topology in $\prod_{\gamma\in\Gamma} \overline{P}$, so that $\overline{P}^{(\Gamma)} \cong \overline{P^{(\Gamma)}}$ as topological modules. Furthermore, if $\alpha : \overline{P} \to \overline{Q}$ is a topological isomorphism between two locally projective R-modules, then α induces in the obvious way an isomorphism between the rings $\mathrm{ContHom}(\overline{P}, \underline{P})$ and $\mathrm{ContHom}(\overline{Q}, \underline{Q})$. Therefore, we have a ring isomorphism

$$\mathrm{ContHom}(\overline{P^{(\Gamma)}}, \underline{P^{(\Gamma)}}) \cong [\mathrm{ContHom}(\overline{P}, \underline{P})]^f_\Gamma.$$

Let R be a ring with a set E of local units. Throughout the remainder of this article we assume (unless otherwise indicated) that any direct sum of modules is endowed with the topology induced by the product topology; in particular, we endow the module ${}_RU = \oplus_{e\in E}\underline{Re}$ with the topology induced by the product topology. Then we have a direct sum $\overline{R^{(E)}} = U \oplus L$ also in the topological sense, where $L = \oplus_{e\in E}R(1-e)$. Now let Γ be an arbitrary set whose cardinality is larger than that of E. Then we have topological isomorphisms

$$\begin{aligned}\overline{R}^{(\Gamma)} &\cong \overline{R^{(\Gamma)}} \cong \overline{R^{(E\times\Gamma)}} \cong (\overline{R^{(E)}})^{(\Gamma)} \cong (U\oplus L)^{(\Gamma)} \cong U^{(\Gamma)} \oplus L^{(\Gamma)}\\ &\cong U^{(\Gamma)} \oplus U^{(\Gamma)} \oplus L^{(\Gamma)} \cong U^{(\Gamma)} \oplus \overline{R^{(\Gamma)}} \cong U^{(\Gamma)} \oplus \overline{R}^{(\Gamma)} \cong (U\oplus \overline{R})^{(\Gamma)}.\end{aligned}$$

Definition. A unitary topological module ${}_RP$ is said to be *topologically projective* if it is a topological direct summand of a direct sum of discrete modules of the form Re, $e \in E$. A ring R with local units is called topologically projective if ${}_R\overline{R}$ is topologically projective.

Obviously, every topologically projective module is projective, but (as we will show in Example 11 below) the converse need not hold. It is also clear that a module is topologically projective if and only if it is a topological direct summand of $U^{(K)}$ for a suitable set K.

In spite of the one-sided definition, the property of being topologically projective is a two-sided property for rings with local units. In fact, suppose that R is a ring with a set E of local units and ${}_R\overline{R}$ is topologically projective. Then there is a topological isomorphism $U^{(K)} \cong_R \overline{R} \oplus R'$, for a suitable topological module $R' \in R$Mod and some set K. If we endow the ContHom-sets with the finite topology, then we have topological isomorphisms

$$\begin{array}{ccc}
(\underset{e\in E}{\oplus}\underline{eR})^{(K)} & \cong & (\underset{e\in E}{\oplus}\mathrm{Hom}(\underline{Re},\underline{R}))^{(K)}\\
 & & \big\downarrow{\cong}\\
\mathrm{ContHom}(U^{(K)},\underline{R}) & \cong & (\mathrm{ContHom}(U,\underline{R}))^{(K)}\\
\big\downarrow{\cong} & & \\
\mathrm{ContHom}(\overline{R}\oplus R',\underline{R}) & \cong & \mathrm{ContHom}(\overline{R},\underline{R}) \oplus \mathrm{ContHom}(R',\underline{R}) \cong \overline{R}_R \oplus R'',
\end{array}$$

whence the topological isomorphism $\underline{(\oplus_{e\in E}\mathrm{eR})^{(K)}} \cong \overline{R}_R \oplus R''$ is valid. Thus $\overline{R}_R$ is indeed topologically projective.

Proposition 4. *Let* R *and* S *be rings with local units, and* $G : R\mathrm{Mod} \to S\mathrm{Mod}$ *be an equivalence functor. Then* G *preserves topological projectivity.*

Proof. We first prove the statement for modules $M \in R\mathrm{Mod}$ which are topological direct sums of discrete modules of the form Re, $e \in E$. Indeed, we have $\overline{G(\oplus Re)} = \oplus \underline{G(Re)}$ algebraically and topologically; here every $G(Re)$ is finitely generated and projective, so there is an idempotent $f = f(e)$ in S and an integer n such that $G(Re)$ is a direct summand of $(Sf)^n$.

Suppose now that $P = \varinjlim P_i$ is a locally projective module with projections ψ_i. Then every $G(P_i)$ is a finitely generated projective S-module, hence $G(P) = \varinjlim G(P_i)$ is a locally projective module with projections $G(\psi_i)$. Assume now that P is topologically projective; i.e, there is a module ${}_RQ$ such that $P \oplus Q = \oplus_{j\in J} Re$ algebraically and topologically. Since the topology on $\oplus_{j\in J} Re$ is linear, the above condition means that the topology on Q is linear. In addition, for any open submodules $P_1 \subseteq P$ and $Q_1 \subseteq Q$ there is a finite subset $K \subset J$ with $\oplus_{J\setminus K} Re \subset P_1 \oplus Q_1$; also, for every finite subset $L \subset J$ there are open submodules $P_2 \subseteq P$ and $Q_2 \subseteq Q$ with $P_2 \oplus Q_2 \subset \oplus_{J\setminus L} Re$. Since G preserves both direct sums and submodule inclusions, the topologies on $G(P) \oplus G(Q) = G(\oplus_{j\in J} Re)$ defined by carrying over the linear topologies of $G(P)$ and $G(Q)$ (respectively, $G(\oplus_{j\in J} Re)$) from P and Q (respectively, $\oplus_{j\in J} Re$) are identical. This means that $G(\overline{\oplus_{j\in J} Re}) = \oplus_{j\in J} G(Re)$ is a topological direct sum of $G(P)$ and $G(Q)$. Since a topological direct summand of a topologically projective module is clearly topologically projective, the proposition is established by using the first paragraph of the proof. □

Suppose now that R is topologically projective. Then there is a topological module ${}_RR'$ such that $\overline{R} \oplus R' \cong U^{(K)}$ for some infinite set K. This induces topological isomorphisms

$$\begin{aligned} U^{(K)} &\cong U^{(K\times K)} \cong (U^{(K)})^{(K)} \cong (\overline{R}\oplus R')^{(K)} \cong \overline{R}^{(K)} \oplus R'^{(K)} \\ &\cong \overline{R}^{(K)} \oplus \overline{R}^{(K)} \oplus R'^{(K)} \cong \overline{R}^{(K)} \oplus U^{(K)} \cong (U\oplus \overline{R})^{(K)}, \end{aligned}$$

and similarly, if ${}_RP$ is an arbitrary topologically projective module then we obtain a topological isomorphism $U^{(K)} \cong (U\oplus P)^{(K)}$ for some infinite set K.

If such a P is in addition a locally projective generator for RMod, then for each $e \in E$ there is an integer $n = n(e)$ such that $\underline{Re}$ is a topological direct summand of P^n. Since Re is finitely generated and P is locally projective, we can assume that Re is contained in a finitely generated projective direct summand of P^n, and therefore we can choose a direct complement of Re in P^n such that this decomposition is topological. Hence U is a topological direct summand of $P^{(J)}$ for a suitable infinite set J, and we obtain, in the same way as above, a topological isomorphism $P^{(J)} \cong (U\oplus P)^{(J)}$.

After all these preparations we prove the main result of this paper.

Theorem 5. *Let R and S be topologically projective rings with local units. Then R and S are Morita equivalent if and only if there is a set I for which $R_I^f \cong S_I^f$.*

Proof. Suppose that R and S are Morita equivalent topologically projective rings. Since equivalence functors preserve topological projectivity (Proposition 4) and S is topologically projective, the equivalence of R and S is induced, in view of [**3**, Theorem 2.5], by a bimodule ${}_RP_S$ such that ${}_R\overline{P}$ is a topologically projective generator. Since R is topologically projective, we may use the topological isomorphisms obtained above to ensure the existence of an infinite set I such that

$$\overline{R^{(I)}} \cong \overline{R}^{(I)} \cong (\overline{R}\oplus U)^{(I)} \cong U^{(I)} \cong (U\oplus \overline{P})^{(I)} \cong \overline{P}^{(I)} \cong \overline{P^{(I)}}$$

with topological isomorphisms. Therefore, there is a ring isomorphism

$$\mathrm{ContHom}(\overline{R^{(I)}}, \underline{R^{(I)}}) \cong \mathrm{ContHom}(\overline{P^{(I)}}, \underline{P^{(I)}}).$$

We have seen that

$$\mathrm{ContHom}(\overline{P^{(I)}}, \underline{P^{(I)}}) \cong [\mathrm{ContHom}(\overline{P}, \underline{P})]_I^f,$$

and that $\mathrm{ContHom}(\overline{P}, \underline{P})$ consists exactly of those endomorphisms of P which factor through one of the ψ_i. By Theorem 2.5 in [**3**], the ring of these endomorphisms is just S, hence

$$\mathrm{ContHom}(\overline{P^{(I)}}, \underline{P^{(I)}}) \cong S_I^f.$$

Similarly, we have

$$\mathrm{ContHom}(\overline{R^{(I)}}, \underline{R^{(I)}}) \cong [\mathrm{ContHom}(\overline{R}, \underline{R})]_I^f,$$

and clearly $\mathrm{ContHom}(\overline{R}, \underline{R}) \cong R$. Thus we obtain $R_I^f \cong S_I^f$.

The converse follows from the following more general statement.

Lemma 6. *If R is any ring with local units, then R and R_I^f are Morita equivalent for any set I.*

Proof. Let P' denote ${}_RR^{(I)}$. Since ${}_RR$ is locally projective, P' is as well; further, P' is obviously a generator. Now it is easy to see that R_I^f is isomorphic to the ring of those endomorphisms of P' which factor through one of the projections belonging to the canonical direct system of P', whence [**3**, Theorem 2.5] yields that R and R_I^f are Morita equivalent. This completes the proof of both the Lemma and Theorem 5. □

As a special case of the above theorem we have the following unpublished result from the Ph.D. thesis of W. Stephenson [**9**, Theorem 3.6]:

Corollary 7. *Let R and S be rings with identity, and let I be any infinite set. Then R and S are Morita equivalent if and only if R_I^f and S_I^f are isomorphic.*

Remarks. 1. If R and S are topologically projective and the cardinalities of the sets of local units $E(R)$ and $E(S)$ are at most c where c is infinite, then the proof of the theorem yields that $R_c^f \cong S_c^f$ whenever R and S are Morita equivalent.

2. If a ring R has an orthogonal set E of idempotents such that $R = \sum_{e\in E} Re = \sum_{e\in E} eR$, then the set of all finite sums of elements

from E is a set of local units for R and R is topologically projective; hence, the theorem applies to these rings. These rings are called *rings with enough idempotents* in [**4**] and subsequent papers. In [**3**] it is shown (see the Remark at the end of Section 2) that every Morita equivalence class of rings with local units contains rings with enough idempotents. We have no example of a topologically projective ring which is not a ring with enough idempotents. However, even if we restrict the rings R and S of Theorem 5 to be rings with enough idempotents, we know of no direct (i.e., nontopological) way to establish the indicated ring isomorphism.

Now we apply Theorem 5 to find, in a functorial way, canonical representatives of Morita equivalence classes. Intuitively, a functor F which "chooses" exactly one representative from each Morita equivalence class should satisfy the following two properties for all rings R and S with local units:

(a) R and $F(R)$ are Morita equivalent, and

(b) if R and S are Morita equivalent, then $F(R)$ and $F(S)$ are isomorphic.

In what follows, such a functor F will be called a *choice functor*. It is not hard to see that for unital rings there is no faithful choice functor. Specifically, let K be a finite field of prime order. By the Noether-Skolem theorem (see, e.g., [**5**, Theorem 4.3.1]) every automorphism of the full matrix ring K_n is inner. Since for $n > m$ the number of units in K_n is clearly greater than the number of units in K_m, we may conclude that $|\mathrm{Aut}(K_n)| > |\mathrm{Aut}(K_m)|$. Now if F is a choice functor, then properties (a) and (b) together with Morita's theorem imply the existence of an integer q such that $F(K_n)$ is isomorphic to K_q for every n. But any functor preserves automorphisms, so in particular $F(\mathrm{Aut}(K_{q+1})) \subseteq \mathrm{Aut}(K_q)$. By the above numerical observation we see that F cannot be faithful.

Let $\mathbf{S}$ denote the full subcategory of the category of all rings whose objects are those rings which contain an at most countable set of local units; in particular, $\mathbf{S}$ contains all unital rings. If R is a ring from $\mathbf{S}$, then the idempotents of R can be orthogonalized; hence, R is a ring with enough idempotents. Therefore, Theorem 5 applies to all rings in $\mathbf{S}$, and in this case all the index sets occurring in the proof of the theorem can be chosen to be countable. We let $FM : \mathbf{S} \to \mathbf{S}$ denote the functor $FM(R) = R_N^f$ (with coordinatewise induced morphisms as usual), where N is any countably infinite set. Then we obtain

Corollary 8. *The functor $FM : \mathbf{S} \to \mathbf{S}$ is a faithful choice functor.*

Remark . Based on Remark 1 after Corollary 7, if we put any upper bound on the cardinalities of the sets of local units, we obtain faithful choice functors in the corresponding categories of not necessarily unital topologically projective rings similar to the one constructed above.

Finally we present two examples to illustrate that topological projectivity is in fact a restrictive condition on rings with local units. We begin by constructing a ring R with local units such that ${}_RR$ is not projective.

Lemma 9. *Let S be a regular ring with identity and let R be an ideal in S. Then ${}_RR$ is projective if and only if ${}_SR$ is projective.*

Proof. Since S is regular, R is also, and we have $SR = R^2 = R$. Therefore, every unitary R-module can be considered as an S-module. Conversely, if M is an S-module, then RM is a unitary R-module. In addition, if $f :_R R \to_R X$ is a homomorphism of left R-modules, then f is in fact an S-homomorphism, since (for $s \in S$, $r \in R$, and $e \in R$ with $re = r$) we have

$$(sr)f = (sre)f = sr \cdot (e)f = s \cdot r \cdot (e)f = s \cdot (re)f = s \cdot (r)f.$$

A straightforward check now completes the proof of the claim. □

Example 10. Define S to be the direct product of infinitely many copies of an arbitrary field. Then S is a self-injective nonartinian regular ring. Since S is not artinian, it is not hereditary either (see [**8**]), hence it has an ideal R such that ${}_SR$ is not projective. By Lemma 9 and [**3**, Section 3, Example 1] this R is a ring with local units such that ${}_RR$ is not projective.

Finally, we construct a ring S with local units for which ${}_SS$ is projective but not topologically projective. For the necessary elementary properties of linearly compact vector spaces, see, e.g., [**7**].

Example 11. Let M be an infinite dimensional right vector space over a division ring D, and consider the locally projective structure induced by the dual pair (M^*, M) on M. Then $\overline{M^*}$ is isomorphic, both algebraically and topologically, to a direct product of one-dimensional discrete spaces, hence $\overline{M^*}$ is linearly compact. Suppose that $\overline{M^*}$ is a topological direct summand of a topological vector space ${}_DV$ which decomposes into a topological direct sum $\oplus_{i\in I}\underline{Dx_i}$ for a basis $\{x_i \mid i \in I\}$. Then there is a subset $J \subseteq I$ such that $V = \overline{M^*} \oplus (\oplus_{i\in J}\underline{Dx_i})$ algebraically, and here $N = \oplus_{i\in J}\underline{Dx_i}$ is a closed subspace of V. Furthermore, since M is infinite dimensional, M^* is also, and therefore the set $I\backslash J$ must be infinite. Since addition is continuous, the identical mapping $\overline{M^*} \oplus N \to V$ is a continuous function. On the other hand, if $X \subseteq \overline{M^*}$ and $Y \subseteq N$ are open subspaces, then they must be closed, hence they are both closed subspaces of V. Here X is linearly compact, being a closed subspace of a linearly compact vector space, therefore the image X' of X in the quotient vector space V/Y is also linearly compact, hence it is closed. Thus the preimage of X', which is $X+Y$, is a closed subspace of V. Furthermore, X and Y are of finite codimension in $\overline{M^*}$ and N, respectively; hence, $X \oplus Y$ is of finite codimension in $V = \overline{M^*} \oplus N$. Therefore, $X \oplus Y$ is an open subspace of V, which proves that the identical mapping $V \to \overline{M^*} \oplus N$ is also continuous. Thus $V \cong \overline{M^*} \oplus N$ both algebraically and topologically. An application of the quotient mapping $V \to V/N$ yields now that $\oplus_{i\in I\backslash J}\underline{Dx_i} \cong V/N \cong \overline{M^*}$ holds both algebraically and topologically. Being the direct sum of infinitely many discrete spaces, $\oplus_{i\in I\backslash J}\underline{Dx_i}$ cannot be linearly compact, though we have seen that $\overline{M^*}$ is so. This contradiction proves that $\overline{M^*}$ cannot be a topological direct summand of V; in other words, $\overline{M^*}$ is not topologically projective.

Consider now the ring S of continuous endomorphisms of finite rank of $\overline{M^*}$ (in the notation of [**2**], $S \cong M \otimes_D M^*$). Then S is Morita equivalent to D by Theorem 2, and this equivalence is induced by the bimodule ${}_DM^*_S$. Therefore, ${}_SS$, being the image of ${}_DM^*$ under this equivalence, is projective but not topologically projective by Proposition 4.

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