ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

EQUIVARIANT PATH FIELDS ON G-COMPLEXES

PETER WONG

ABSTRACT. A classical theorem of Wecken states that a finite connected polyhedron X satisfying the so-called Wecken condition admits a fixed point free deformation if, and only if, its Euler characteristic $\chi(X)$ vanishes. More generally, Fadell showed that $\chi(X) = 0$ implies the existence of a nonsingular simple path field on X. As an application of path fields, Schirmer showed that every nonempty closed subset of a Wecken complex is the fixed point set of a deformation. In this note, we introduce the appropriate notions of G-Wecken complexes and G-path fields, and we generalize these results to finite G-complexes.

1. Preliminaries. Throughout G will denote a finite group and X will be a finite G-(simplicial) complex (see [**1**]). For any subgroup $H \leq G$, we denote by NH the normalizer of H in G and by $WH = NH/H$, the Weyl group of H in G. The conjugacy class of H denoted by (H) is called the orbit type of H. If $x \in X$, then G_x denotes the isotropy subgroup of x, i.e., $G_x = \{g \in G | gx = x\}$. For each subgroup H of G, $X^H = \{x \in X | hx = x \text{ for all } h \in H\}$ and $X_H = \{x \in X | G_x = H\}$. Let $\{(H_j)\}$ denote the set of isotropy types of X. If (H_j) is subconjugate to (H_i), we write $(H_j) \leq (H_i)$. We can choose an *admissible* ordering on $\{(H_j)\}$ so that $(H_j) \leq (H_i)$ implies $i \leq j$. Then we have a filtration of G-subcomplexes $X_1 \subset X_2 \subset \ldots \subset X_k = X$, where $X_i = \{x \in X | (G_x) = (H_j) \text{ for some } j \leq i\}$. For each $H = H_i$, $X_{(H)} = \{x \in X | (G_x) = (H)\}$. Suppose C is a connected component of $X_{(H)}/G$, we let $X_{H,C} = p^{-1}(C) \cap X^H$ where $p : X \to X/G$ is the orbit map. Since X is a compact G-ENR, all the X^H are compact ENRs and so are $X^H - X_{H,C}$ for each C. Thus the Euler characteristic $\chi(X^H, X^H - X_{H,C}) = \chi(X^H) - \chi(X^H - X_{H,C})$ is well defined. Let $A(X, G)$ be the free abelian group generated by the set $\{((H), C) | C$ connected component of $X_{(H)}/G\}$.

Definition 1.1. The equivariant Euler characteristic $\chi_G(X)$ of X is defined to be the unique element in $A(X, G)$ whose coefficient of the

Received by the editors on February 6, 1989, and in revised form on July 12, 1989.

$((H), C)$-th entry is given by $\chi(X^H, X^H - X_{H,C})$.

Proposition 1.2. *If $f : X \to X$ is G-homotopic to the identity 1_X (i.e., a G-deformation of X) with isolated fixed points, then $\chi(X^H, X^H - X_{H,C})$ is the fixed point index of $f|X_{H,C}$. In particular, if f is a fixed point free G-deformation of X then $\chi_G(X) = 0$.*

Proof. For the proof of this, see [**6**, 2.1]. □

Let Δ be the diagonal in $X \times X$. We define a special neighborhood of Δ by $\eta(\Delta) = \{(x, y) \in X \times X | \sigma(x) \text{ and } \sigma(y) \text{ have a common vertex}\}$ where $\sigma(z)$ is the carrier of z. A map $f : X \to X$ is called a *proximity* map if $(x, f(x)) \in \eta(\Delta)$ for every $x \in X$. Since $\eta(\Delta)$ is an invariant neighborhood of Δ with the diagonal action and X^I is the space of paths on X with the compact open topology and the natural G-action given by $(g \cdot \gamma)(t) = g(\gamma(t))$ for $\gamma \in X^I$, we may state the equivariant analog of [**3**, 2.1] as follows.

Lemma 1.3. *There exists a G-map $\alpha : \eta(\Delta) \to X^I$ such that*

(1) *$\alpha(x, y)$ is a path from x to y;*

(2) *$\alpha(x, x)$ is the constant path at x;*

(3) *the track of $\alpha(x, y)$ has the form $[x, z] \cup [z, y]$ where z depends on x and y;*

(4) *$\alpha(y, x)$ is the reverse of $\alpha(x, y)$;*

(5) *if $x \neq y$, $\alpha(x, y)$ is a simple path.*

Proof. The proof is just as in [**3**, 2.1] with the observation that α as defined there is automatically equivariant. □

Definition 1.4. A G-path field on X is a G-map $\varphi : X \to X^I$ such that $\varphi(x)(0) = x$ and if $\varphi(x)(t) = x$ for some $t > 0$, then φ is the constant path. We say that φ is nonsingular if $\varphi(x)$ is never the constant path and is *simple* if $\varphi(x)$ is a simple path for each x. A *singular* orbit of φ is an orbit Gx so that $\varphi(x)$ is the constant path.

Proposition 1.5. *Suppose that there exists a fixed point-free proximity self G-map, then X admits a nonsingular simple G-path field.*

Proof. Define $\varphi(x) = \alpha(x, f(x))$ to be the nonsingular simple G-path field where α is as in Lemma 1.3 and f is a fixed point-free proximity G-map. $\square$

2. The G-Wecken condition. In this section we will show that every finite *G-Wecken complex* admits a G-path field which has at most one singular orbit in $p^{-1}(C)$ where C is a connected component of $X_{(H)}/G$ and $p : X \to X/G$ is the orbit map. We first recall the Wecken condition in classical fixed point theory. A (locally finite) simplicial complex K is said to satisfy the Wecken condition or simply be a *Wecken complex* if (i) every maximal simplex is of dimension at least two; (ii) for every two maximal simplices σ, σ' there exists a finite chain of maximal simplices $\sigma = \sigma_1, \sigma_2, \dots , \sigma_n = \sigma'$ such that $\sigma_i \cap \sigma_{i+1}$ is of dimension at least one for $i = 1, \dots , n-1$.

K is sometimes called a space of *type W* or is *two dimensionally connected* (see [**2**]). It is easy to verify that (i) and (ii) are invariant under subdivisions. Next we prove a lemma which is essential in applying the "Wecken trick" equivariantly.

Lemma 2.1. *Let x and y be two distinct points of a Wecken complex K, each lying in the interior of some maximal simplex. Then there exists a simple polygonal path from x to y such that the interior of each broken line segment lies in some maximal simplex and the endpoints lie in some simplices of dimension at least one.*

Proof. If x and y both lie inside the same maximal simplex, then the assertion is obvious. Suppose $\sigma_1, \dots , \sigma_k$ is a chain of maximal simplices with $x \in \operatorname{int} \sigma_1, y \in \operatorname{int} \sigma_k$ and $\sigma_i \cap \sigma_{i+1}$ is at least one dimensional for $i = 1, \dots , k-1$. Let v_{2i-1} and v_{2i} be the barycenters of σ_i and $\sigma_i \cap \sigma_{i+1}$, respectively. Let α be the polygonal path from v_1 to v_{2k-1} given by $[v_1, v_2] \cup \cdots \cup [v_{2k-2}, v_{2k-1}]$ where $[a, b]$ denotes the line segment from a to b. Let α' be a maximal tree of the connected graph α. For $1 < i < 2k-1$, if the vertex v_i is of degree 1 we remove v_i and the edge incident to it. By applying the procedure successively, we arrive at a subtree β of α' so that all vertices have degree 2 except for v_1 and v_{2k-1}. Thus β is a subpath of α and it is simple. The required path is obtained by adjoining β with the segments $[x, v_1]$ and $[v_{2k-1}, y]$. □

Definition 2.2. A G-complex X is said to satisfy the G-Wecken condition if for each isotropy type (H) and every connected component C of $X_{(H)}/G$, every connected component of $X_{H,C}$ is a Wecken complex. Such an X is called a G-Wecken complex.

Given any admissible ordering $(H_1) \geq \cdots \geq (H_k)$ and its corresponding filtration $X_1 \subset \ldots \subset X_k = X$, $X_{i-1}^{H_i}$ is a subcomplex of $X_i^{H_i}$ and $X_{H_i} = X_i^{H_i} - X_{i-1}^{H_i}$, where $X_i = \{x \in X | (G_x) = (H_j),\ j \leq i\}$. There is a canonical (locally finite) triangulation on X_H induced by the triangulation of X^H. Thus, the notion of G-Wecken complex is well defined. Note that if K is a Wecken complex, then so is K/G.

3. Uniting fixed orbits. Let X be a G-space, $f : X \to X$ a G-map. A *fixed orbit* is of the form Gx where $f(x) = x$. We show in this section how to coalesce two fixed orbits of a G-map which is G-homotopic to the identity map.

Lemma 3.1. *Let A be an invariant subcomplex of a finite G-complex X such that the action of G in $X - A$ is free. Suppose that each connected component of $X - A$ is a Wecken complex and $(X - A)/G$ is connected. Then given any $\varepsilon > 0$ there exists a G-map $f : (X, A) \to (X, A)$ equivariantly ε-homotopic (relative to A) to 1_X, which has at most one fixed orbit in $X - A$.*

Proof. By an equivariant version of the Hopf's construction (see [**8** or **7**]) we may assume that we have a G-map $f' : (X, A) \to (X, A)$ equivariantly ε-homotopic (relative to A) to 1_X which has a finite number of fixed points in $X - A$, each lying in the interior of some maximal simplex in $X - A$. Let $Gx_1, \ldots, Gx_n$ be the isolated fixed orbits of f' in $X - A$. Choose distinct x_i, x_j and denote by $\bar{x}$ and $\bar{y}$ the images of x_i and x_j under the orbit map $p : X \to X/G$. Since x_i, x_j lie in the interior of maximal simplices, so do $\bar{x}$ and $\bar{y}$ in $(X - A)/G$ which is also a Wecken complex. By Lemma 2.1, there exists a simple polygonal path which can be lifted to a cross-section α from x_i to gx_j

for some $g \in G$, we can coalesce these two fixed points inside a small contractible neighborhood of α in $X - A$ using the Wecken method (see [**2**]). By taking all the G-translates of this neighborhood, we unite the fixed orbits Gx_i and Gx_j. The assertion follows by repeating the above process finitely many times. □

Theorem 3.2. *Let X be a finite G-Wecken complex. There exists a proximity G-map f such that for each isotropy type (H) and each connected component C of $X_{(H)}/G$, f has at most one fixed orbit in $p^{-1}(C)$ where $p : X \to X/G$ is the orbit map.*

Proof. Let $(H_1) \geq \cdots \geq (H_k)$ be an admissible ordering on the isotropy types of X and put $X_0 = \varnothing$. We may assume inductively that there exists a proximity G-map $\hat{f}$ which has at most one fixed orbit in $p^{-1}(C)$ for each isotropy type (H_j) and each connected component C of $X_{(H_j)}/G$ for $j < i$. Let C be a connected component of $X_{(H_i)}/G \equiv X_{H_i}/WH_i$. Since the connected components of $X_{H_i,C}$ are Wecken complexes, C is also a Wecken complex. Thus we can apply Lemma 3.1 because $X_{H_i,C}$ is a free WH_i-space. Then f^{H_i} is WH_i-equivariantly ε-homotopic (relative to $X_{i-1}^{H_i}$) to a map which has at most one fixed WH_i-orbit in $p_i^{-1}(C)$ for each connected component C in X_{H_i}/WH_i where $p_i : X_{H_i} \to X_{H_i}/WH_i$ is the WH_i-orbit map. We extend this homotopy to a G-homotopy to obtain the required map. Induction completes the proof. □

Theorem 3.3. *If X is a finite G-Wecken complex, then there exists a G-path field φ on X with at most one singular orbit in $p^{-1}(C)$ for every connected component C in $X_{(H)}/G$ for each isotropy type (H). Moreover, X admits a nonsingular simple G-path field if, and only if, $\chi_G(X)$ vanishes.*

Proof. Let f be the map obtained in Theorem 3.2. Define $\varphi(x) = \alpha(x, f(x))$ to be the required G-path field where α is as in Lemma 1.3. If λ is a nonsingular simple G-path field, then we define a G-map Λ on X by $\Lambda(x) = \lambda(x)(1)$. The map Λ is a fixed point-free G-deformation and thus $\chi_G(X) = 0$ by Proposition 1.2. If $\chi_G(X) = 0$, the map from Theorem 3.2 can be G-deformed to a fixed point-free proximity map since any fixed orbit of index zero can be removed (see [**4**]). Applying Proposition 1.5 completes the proof. □

Corollary 3.4. *Let X be a finite G-Wecken complex. There exists a fixed point-free proximity self G-map of X if, and only if, $\chi_G(X) = 0$.*

Note that if M is a compact triangulable G-manifold with a *locally smooth* G-action (see [**1**]) such that every connected component of M^H is of dimension at least two, then M is a finite G-Wecken complex. Therefore, by Theorem 3.3, M admits a nonsingular simple G-path field if, and only if, $\chi_G(M) = 0$. In the smooth case, a nonvanishing G-vector field gives rise to a fixed point-free proximity G-map via the exponential map which is G-invariant. On the other hand, given a fixed point-free proximity G-map f, let γ_x be the unique geodesic in M with $\gamma_x(0) = x$ and $\gamma_x(1) = f(x)$ for each $x \in M$. Then the tangent vectors $\gamma_x'(0)$ define a nonsingular G-vector field. The above arguments together with Corollary 3.4 yield the following

Theorem 3.5. *M admits a nonsingular G-vector field if, and only if, $\chi_G(M)$ vanishes.*

Theorem 3.5 was also obtained by Wilczyński [**7**, Theorem B] using a different method.

4. Application. In [**5**] Schirmer proved, using path fields, that every Wecken complex has the *complete invariance property*, i.e., every nonempty closed subset is the fixed point set of a deformation. In this section we give conditions when a nonempty closed invariant subset of a finite G-Wecken complex can be the fixed point set of G-deformation.

Theorem 4.1. *Let A be the nonempty closed invariant subset of a finite G-Wecken complex X. Suppose that for every isotropy type (H) and connected component C of $X_{(H)}/G$, $\chi(X^H, X^H - X_{H,C}) \neq 0 \Rightarrow A^H \cap p^{-1}(C) \neq \varnothing$ where $p : X \to X/G$ is the orbit map. Then given any $\varepsilon > 0$, there exists a G ε-deformation $h : X \to X$ with* $\operatorname{Fix} h = A$.

Proof. Suppose that $\chi(X^H, X^H - X_{H,C}) \neq 0$ implies $A^H \cap p^{-1}(C) \neq \varnothing$. By Theorem 3.3, there is a G ε-deformation f with exactly one *essential* fixed orbit in $p^{-1}(C)$. Since a fixed point can be chosen arbitrarily on a two dimensionally connected space (see [**5**]), the fixed orbit of f in $p^{-1}(C)$ can be chosen so that it lies inside $A \cap p^{-1}(C)$. Thus we may assume that $\operatorname{Fix} f \subset A$. Consider the G-path field $\varphi(x) = \alpha(x, f(x))$ where α is as in Lemma 1.3. Let d be the metric of X. We may assume that d is bounded and $d < 1$. Define the G-map $h : X \to X$ with fixed point set A by $h(x) = \varphi(x)(t_x)$ where $t_x = d(x, A)$. Since the track of $\varphi(x)$ is a broken line segment from x to $f(x)$ and f is sufficiently close to 1_X, h is also close to 1_X. □

Acknowledgments. This work is based on part of an Appendix of the author's dissertation written at the University of Wisconsin-Madison under the supervision of Professor Edward R. Fadell whose support and encouragement are greatly appreciated. The author would like to thank the referee for a number of helpful suggestions.

REFERENCES

1. G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

2. R.F. Brown, *The Lefschetz fixed point theorem*, Scott-Foresman, Glenview, IL, 1971.

3. E. Fadell, *A remark on simple path fields in polyhedra of characteristic zero*, Rocky Mountain J. Math. **4** (1974), 65–68.

4. ——— and P. Wong, *On deforming G-maps to be fixed point free*, Pacific J. Math. **132** (1988), 277–281.

5. H. Schirmer, *Fixed point sets of continuous selfmaps*, in *Fixed point theory* (E. Fadell and G. Fournier, eds.), LNM 886, Springer-Verlag, New York, 1981, 417–428.

6. D. Wilczyński, *Fixed point free equivariant homotopy classes*, Fund. Math. **123** (1984), 47–60.

7. P. Wong, *Equivariant Nielsen fixed point theory for G-maps*, Ph.D. thesis, University of Wisconsin-Madison, WI, 1988.

8. ———, *Equivariant Nielsen fixed point theory for G-maps*, Pacific J. Math. **150** (1991), 179–200.

Department of Mathematics, Bates College, Lewiston, ME 04240