ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

ON CONTINUED FRACTIONS $K(a_n/1)$, WHERE ALL a_n ARE LYING ON A CARTESIAN OVAL

HAAKON WAADELAND

ABSTRACT. In a recent paper L. Jacobsen and W.J. Thron have proved results on oval convergence regions and circular limit regions for continued fractions. In the present paper is discussed what happens to the limit region when the oval region is replaced by its boundary. This extends earlier results on boundary versions of Worpitzky's theorem and the parabola theorem.

1. Introduction. The present paper deals with continued fractions

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n}{1}, \tag{1.1}$$

and the type of problem to be discussed is: What can be said about the values of (1.1) when all elements a_n are in a prescribed convergence region E? For analytic theory of continued fractions generally, as well as for special concepts like, for instance, "convergence region," we refer to the monograph [**2**]. Following the tradition there, the word "region," used in concepts such as "element region," "convergence region," "value region," "limit region," simply means a *set* of complex numbers. The definition of those concepts are also contained in Section 2 of [**1**].

Our starting point is one of the theorems in [**1**]. We first need to introduce two special types of sets, whose role in continued fraction theory goes back to Lane [**4**].

Let Γ be a complex number with

$$|\Gamma| < |1 + \Gamma| \tag{1.2}$$

Received by the editors on October 10, 1989, and in revised form on December 1, 1989.
1980 *Mathematics Subject Classification.* 30B70, 40A15.
Key words and phrases. Continued fractions, element region, limit region.
This work was supported by the Norwegian Research Council for Science and the Humanities (NAVF).

(or equivalently $\operatorname{Re}\Gamma > -1/2$), and let ρ be a positive number $< |1+\Gamma|$. Then $V(\Gamma,\rho)$ shall denote the closed circular disk

$$V(\Gamma,\rho) := \{w : |w-\Gamma| \le \rho\}, \tag{1.3}$$

and $E(\Gamma,\rho)$ shall denote the set

$$E(\Gamma,\rho) := \{w : |w(1+\overline{\Gamma}) - \Gamma(|1+\Gamma|^2-\rho^2)| + \rho|w| \le \rho(|1+\Gamma|^2-\rho^2)\}. \tag{1.4}$$

We could also call $E(\Gamma,\rho)$ a disk, but not a circular disk. Its boundary, $\partial E(\Gamma,\rho)$ is a Cartesian oval, which, at least in the cases we are interested in here, is a simple, closed curve with *some* resemblance to an ellipse.

Theorem 1.1. [**1**, Theorem 3.1]. *If $\rho > |\Gamma|$, then $E(\Gamma,\rho)$ is a convergence region and $V(\Gamma,\rho)$ a corresponding limit region for continued fractions* (1.1).

Theorem 1.1 tells that any continued fraction (1.1) with all a_n in $E(\Gamma,\rho)$ will converge, and that its value will be located in $V(\Gamma,\rho)$. (It actually tells more, but right now this is what we need.)

We shall here restrict ourselves to a special family of oval convergence regions, given by the conditions

$$\Gamma \ge 0 \quad \text{and} \quad \rho = \Gamma + \frac{1}{2} - \varepsilon(\Gamma), \tag{1.5}$$

where ε is a positive (continuous) function with the properties

$$\varepsilon(\Gamma) \to 0 \quad \text{when} \quad \Gamma \to 0 \quad \text{and when} \quad \Gamma \to \infty, \qquad |\varepsilon(\Gamma)| < \frac{1}{2}. \tag{1.6}$$

(Actually, we shall always think of ε as being "very small.") We have in this case that

$$V\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon(\Gamma)\right) = \left\{w : |w-\Gamma| \le \Gamma + \frac{1}{2} - \varepsilon(\Gamma)\right\} \tag{1.3$'$}$$

and

(1.4′)

$$E\left(\Gamma,\Gamma+\frac{1}{2}-\varepsilon(\Gamma)\right)=\left\{w:\left|(1+\Gamma)w-\Gamma\left(\Gamma+\frac{3}{4}-\frac{\varepsilon}{2}\right)(1+2\varepsilon)\right|\right.$$
$$+\left(\Gamma+\frac{1}{2}-\varepsilon\right)|w|$$
$$\left.\leq\left(\Gamma+\frac{1}{2}-\varepsilon\right)\left(\Gamma+\frac{3}{4}-\frac{\varepsilon}{2}\right)(1+2\varepsilon)\right\}.$$

Here and in the following we write ε instead of $\varepsilon(\Gamma)$. Where no confusion is possible, we shall also write V instead of $V(\Gamma,\Gamma+(1/2)-\varepsilon(\Gamma))$ and E instead of $E(\Gamma,\Gamma+(1/2)-\varepsilon(\Gamma))$.

The boundary ∂V of V intersects the real axis at

$$-\frac{1}{2}+\varepsilon \quad \text{and at} \quad 2\Gamma+\frac{1}{2}-\varepsilon.$$

The boundary ∂E of E intersects the real axis at

$$-\frac{1}{4}+\varepsilon^2 \quad \text{and at} \quad \left(\Gamma+\frac{1}{4}-\frac{\varepsilon}{2}\right)(1+2\varepsilon).$$

Keeping in mind that ε was chosen such that $\varepsilon\to 0$ when $\Gamma\to 0$ or $\Gamma\to\infty$ we find in the extreme cases

$\Gamma=0$

$$E\left(0,\frac{1}{2}\right)=\left\{w:|w|\leq\frac{1}{4}\right\}, \tag{1.6}$$

$$V\left(0,\frac{1}{2}\right)=\left\{w:|w|\leq\frac{1}{2}\right\}, \tag{1.7}$$

and Theorem 1.1 is in this case the famous Worpitzky theorem [**9**].

$\Gamma\to\infty$

In this case, if we take the closure, the oval region degenerates to the parabolic region

$$(E=)P=\left\{w:|w|\leq\operatorname{Re}w+\frac{1}{2}\right\}, \tag{1.8}$$

and the limit region, the circular disk, degenerates to the half plane

$$(V =)H = \left\{ w : \operatorname{Re} w \geq -\frac{1}{2} \right\}. \tag{1.9}$$

The last statement is almost immediate, the first one is not so obvious, but also not difficult. It is proved in [**1**, Theorem 8.1] for the case $\varepsilon = 0$, but the ε we have here does not make any essential change in the proof. The case $\Gamma \to \infty$ thus gives the Scott-Wall parabolic region [**6**]. Since this is not a convergence region, only a conditional convergence region, we do not get a special case of Theorem 1.1 here.

These considerations justify the statement of Jacobsen and Thron in [**1**]: "If $\Gamma \geq 0$ the ovals represent a connecting link between the Worpitzky disk and the Scott and Wall parabola." They actually also have a bridge [**1**, Theorem 8.2] between the Worpitzky disk and the more general parabolic region

$$|w| - \operatorname{Re}(we^{-2i\alpha}) \leq \frac{1}{2}\cos^2\alpha,$$

[**7**], but we shall not be needing that in the present paper.

Before presenting the problem to be discussed in this paper we need to point out two properties of the limit regions in the Worpitzky case and the Scott-Wall case. In addition to knowing that in the first case $0 < |a_n| \leq 1/4$ for all n implies $0 < |f| \leq 1/2$ for all values of the continued fractions we also know that all values in $0 < |w| \leq 1/2$ are taken. In the second case we similarly know that for all a_n in the parabolic region minus the origin any point in the half plane $\operatorname{Re} w \geq -1/2$ minus the origin is taken on as a value of some continued fraction in the family. With E as the element region and V as the set of values we shall in this paper phrase it as follows: V is the *range* of the family of continued fractions (1.1) defined by the condition $a_n \in E$ for all n. The other property to be pointed out is the following, which is an immediate consequence of the first property in both cases (and even more generally): The range of the family of continued fractions (1.1) defined by the condition $a_n \in \partial E$ for all n contains ∂V.

We shall need to know that these properties also hold for the oval-disk theorems in our case and will return to this later.

2. The problem. In the paper [**8**] the following question was raised and discussed in the Worpitzky case and the Scott-Wall case: What happens to the range when E is replaced by ∂E? In the first case the answer was that the range changes from

$$0 < |w| \le \frac{1}{2}$$

to

$$\frac{1}{6} \le |w| \le \frac{1}{2},$$

and in the second case the answer was that the range remained unchanged, i.e., it was the half plane $\operatorname{Re} w \ge -1/2$ minus the origin.

The problem to be discussed in the present paper is the same type of question for the oval-disk case in the version defined by (1.3′) and (1.4′) in Theorem 1.1. But before this can be raised as a meaningful question, we need to know the range.

Observation A. *Let Γ be a fixed positive number. The range of the family of continued fractions* (1.1) *defined by the condition that all $a_n \in E(\Gamma, \Gamma + (1/2) - \varepsilon)$ is the closed disk $V(\Gamma, \Gamma + (1/2) - \varepsilon)$ minus the orign.*

Proof of Observation A. The oval region is determined as the set of all a, such that

$$\frac{a}{1+V} \subseteq V.$$

Simple computation shows that

$$\frac{a}{1+V} \tag{2.1}$$

is a closed disk with center at

$$\frac{4\Gamma + 4}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)} a \tag{2.2}$$

and radius

$$\frac{4\Gamma + 2 - 4\varepsilon}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)} |a|. \tag{2.3}$$

Fixing the argument of a and letting $|a|$ increase until the boundary of (2.1) hits the circle

$$|w-\Gamma| = \Gamma + \frac{1}{2} - \varepsilon \tag{2.4}$$

we get a union of disks (2.1). All the circles (boundaries of (2.1)) have the same two tangents from the origin, and the union consists of all points on the two tangent line segments to the last (largest) circle and the largest connecting arc and all points inside. (The origin is not a point in the set, which looks like a drawing of an ice cream cone.) If we vary $\arg a$ over all angles in $[0, 2\pi)$ we see that the whole disk $|w-\Gamma| \le \Gamma + (1/2) - \varepsilon$ is covered, except the origin. Take an arbitrary $w^{(0)}$ in the punctured disk

$$0 < |w-\Gamma| \le \Gamma + \frac{1}{2} - \varepsilon. \tag{2.5}$$

Since $w^{(0)}$ is covered by some disk (2.1), there must be an a_1 in the oval region and a $w^{(1)}$ in the punctured disk, such that

$$w^{(0)} = \frac{a_1}{1+w^{(1)}}. \tag{2.6}$$

(If $w^{(0)}$ happens to be on the boundary, a_1 is unique and is on the boundary of the oval region, as seen from the argument above. $w^{(1)}$ is thus also unique, and it is easy to see that it cannot be an interior point of (2.5).) We can continue this process without stopping and will thus, to any arbitrary point $w^{(0)}$ in (2.5), create a continued fraction (1.1) with the property that for all n

$$w^{(0)} = \frac{a_1}{1} \, {}_{+} \, \frac{a_2}{1} \, {}_{+} \cdots {}_{+} \, \frac{a_n}{1+w^{(n)}}. \tag{2.6$'$}$$

Since, according to Theorem 1.1, $E(\Gamma, \Gamma + (1/2) - \varepsilon)$ is a convergence region and $V(\Gamma, \Gamma+(1/2)-\varepsilon)$ a corresponding limit region the continued fraction converges and has its value f in (2.5). The question is: Does it converge to $w^{(0)}$? The answer is *yes*, and it can be proved in different ways. Since this paper pretty much is based upon [**1**], we use that paper to establish this answer. In [**1**, Theorem 6.1] is proved an estimate, which in our case takes the form (with standard notation from [**2**])

(2.7)

$$|f-S_n(w_n)| \le 2\left(\Gamma + \frac{1}{2} - \varepsilon\right)\left(\frac{4\Gamma+1-2\varepsilon}{1+2\varepsilon}\right)\left(1 - \frac{2\varepsilon}{(2\Gamma+\frac{3}{2}-\varepsilon)^2}\right)^{\frac{n-1}{2}}$$

for *any* sequence from V, in particular for the sequence we get in creating the continued fraction (tail sequence). Hence,

$$w^{(0)} = \lim_{n\to\infty} S_n(w_n) = f, \tag{2.8}$$

and Observation A is proved. □

Observation B. *If all a_n are restricted to the boundary $\partial E(\Gamma, \Gamma + (1/2) - \varepsilon)$, the range will contain the whole circle*

$$|w - \Gamma| = \Gamma + \frac{1}{2} - \varepsilon \tag{2.4}$$

Proof. We know that the circle (2.4) is part of the range when $a_n \in E(\Gamma, \Gamma + (1/2) - \varepsilon)$. If $K(a_n/1) = f$, where f is on (2.4), then no a_n can be an interior point of the oval region, since then a neighborhood of f would have to be in V, contradicting the assumption that $f \in \partial V$. □

3. Some results. The basic principle for handling the problem in the previous section is as follows.

Since the circle (2.4) is contained in the range of the family given by

$$a_n \in \partial E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right) \tag{3.1}$$

any circle

$$\frac{a_n}{1 + \partial V(\Gamma, \Gamma + \frac{1}{2} - \varepsilon)}, \qquad a_n \in \partial E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right),$$

and hence the set

$$\bigcup_{a_n \in \partial E} \frac{a_n}{1 + \partial V(\Gamma, \Gamma + \frac{1}{2} - \varepsilon)} \tag{3.2}$$

is in the range of the family of continued fractions (1.1) given by (3.1). On the other hand, the disk $V(\Gamma, \Gamma + (1/2) - \varepsilon)$ contains the range of this family, and hence

$$\bigcup_{a_n \in \partial E} \frac{a_n}{1 + V(\Gamma, \Gamma + \frac{1}{2} - \varepsilon)} \tag{3.3}$$

contains the range. Since the center moves on a simple, closed continuous curve surrounding the origin, and the radii are less than the absolute value of the centers, then the sets (3.2) and (3.3) are the same. Hence, (3.2) is actually the range we are looking for.

In order to handle this computationally, it seems to be of advantage to represent the oval in polar coordinates. This is done in [**1**] more generally. In our case, where $\Gamma \geq 0$ and $\rho > \Gamma$ (actually $\rho = \Gamma + (1/2) - \varepsilon$), the formula (4.4) in [**1**] takes the form

$$\begin{aligned} r(\theta) = &\sqrt{[\rho^2 - \Gamma(\Gamma+1)\cos\theta]^2 + ((1+\Gamma)^2 - \rho^2)(\rho^2 - \Gamma^2)} \\ &- [\rho^2 - \Gamma(\Gamma+1)\cos\theta], \qquad 0 \leq \theta < 2\pi. \end{aligned} \tag{3.4}$$

Another useful form is

$$r(\theta) = \frac{((1+\Gamma)^2 - \rho^2)(\rho^2 - \Gamma^2)}{\sqrt{} + [\rho^2 - \Gamma(\Gamma+1)\cos\theta]}, \tag{3.4$'$}$$

where $\sqrt{}$ is the same square root as in (3.4). For later use, we make the observations: When θ increases from 0 to π, then $r(\theta)$ decreases from

$$\left(\Gamma + \frac{1}{4} - \frac{\varepsilon}{2}\right)(1 + 2\varepsilon) \quad \text{to} \quad \frac{1}{4} - \varepsilon^2$$

(when we insert $\rho = \Gamma + (1/2) - \varepsilon$).

We find in the two extreme cases when we take into account the ε-conditions (1.6) (and maintain $\rho = \Gamma + (1/2) - \varepsilon$):

$\Gamma = 0$

$$r(\theta) = \frac{1}{4}, \qquad 0 \leq \theta < 2\pi \qquad \text{(Worpitzky circle).}$$

$\Gamma \to \infty$

We divide numerator and denominator in (3.4$'$) by Γ^2, and let $\Gamma \to \infty$. This gives

$$r(\theta) = \frac{1}{2(1 - \cos\theta)}, \qquad 0 \leq \theta < 2\pi \qquad \text{(Scott-Wall parabola).}$$

Back to the general case: From what is said before the range of the family of continued fractions (1.1) with all $a_n \in \partial E(\Gamma, \Gamma + (1/2) - \varepsilon)$ is the set of all points (values) on all circles with center at

$$\frac{4\Gamma + 4}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)} r(\theta) e^{i\theta}$$

and radius

$$\frac{4\Gamma + 2 - 4\varepsilon}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)} r(\theta),$$

when θ varies in the interval $[0, 2\pi)$ (see (2.2) and (2.3)). If we choose to describe the circles in cartesian coordinates we get

$$\text{(3.5)} \quad \left(x - \frac{(4\Gamma + 4)r(\theta)\cos\theta}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)}\right)^2 + \left(y - \frac{(4\Gamma + 4)r(\theta)\sin\theta}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)}\right)^2 = \left(\frac{4\Gamma + 2 - 4\varepsilon}{(4\Gamma + 3 - 2\varepsilon)(1 + 2\varepsilon)}\right)^2 r(\theta)^2.$$

One way of approaching the problem of determining the union of a certain one-parameter family of "nice" sets (provided that they depend sufficiently smoothly upon that parameter) is to find the envelope of the family of curves bounding the sets. This is done by eliminating between the equation for the curve family and the partial derivative with respect to the parameter. The role of the envelope is that part of it (or all of it) often is the boundary of the union set we are trying to determine. We can in principle do this with (3.5), but essentially it will not lead us to results we cannot obtain more easily. It is, in particular, easy to carry it out in the two extreme cases and thereby reestablish the results on boundary versions of the (Scott-Wall) parabola theorem and Worpitzky's theorem. However, for space reasons we shall leave this out.

We already know that the set (3.3) must be contained in the disk $V(\Gamma, \Gamma + (1/2) - \varepsilon)$ and that the circles in (3.2) must touch the circle (2.4). This means that the circle (2.4) must be part of the envelope of (3.5), we could call it "outer envelope." This may be written in the following way.

Observation C. *Let $0 < \Gamma < \infty$, and let $r(\theta)$ be given as in* (3.4). *Then the following holds for all θ.*

$$\left|\frac{4\Gamma+4}{(4\Gamma+3-2\varepsilon)(1+2\varepsilon)}r(\theta)e^{i\theta}-\Gamma\right|+\frac{4\Gamma+2-4\varepsilon}{(4\Gamma+3-2\varepsilon)(1+2\varepsilon)}r(\theta) = \Gamma+\frac{1}{2}-\varepsilon. \tag{3.6}$$

Some words on computational verification. Without any computation we already know, from the construction, that this is true. The easiest way to check it computationally is to move the second term on the left to the right hand side and square on both sides. Some rearranging gives the quadratic equation for the right $r(\theta)$. The logic of the argument is taken care of.

We conclude this section by a descriptive theorem for the range.

Theorem 1. *Let Γ be a nonnegative number, $0 < \varepsilon < 1/2$, and let $\mathcal{F}$ be the family of all continued fractions*

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{1} \tag{3.7}$$

which are such that for all n

$$\left|(1+\Gamma)a_n-\Gamma\left(\Gamma+\frac{3}{4}-\frac{\varepsilon}{2}\right)(1+2\varepsilon)\right|+\left(\Gamma+\frac{1}{2}-\varepsilon\right)|a_n| = \left(\Gamma+\frac{1}{2}-\varepsilon\right)\left(\Gamma+\frac{3}{4}-\frac{\varepsilon}{2}\right)(1+2\varepsilon). \tag{3.8}$$

Let $\mathcal{L}$ be the range of this family, i.e., the set of all possible values f a continued fraction in $\mathcal{F}$ can take on.

Then

a) *$\mathcal{L}$ is contained in the disk, given by*

$$|w-\Gamma| \le \Gamma+\frac{1}{2}-\varepsilon. \tag{3.9}$$

b) *$\mathcal{L}$ contains the annulus, given by*

$$\Gamma+\frac{1}{2}-\varepsilon-\frac{(2\Gamma+1-2\varepsilon)(1-2\varepsilon)}{4\Gamma+3-2\varepsilon} \le |w-\Gamma| \le \Gamma+\frac{1}{2}-\varepsilon. \tag{3.10}$$

c) *The disk, given by*

$$|w| < \frac{1-4\varepsilon^2}{6+8\Gamma-4\varepsilon} \tag{3.11}$$

does not overlap $\mathcal{L}$.

The bounds in a), b), c) *are best.*

Proof. a) The inclusion statement is already proved, since we know that the larger family, where all a_n are in an oval region (where $=$ is replaced by $\le$ in (3.8)), has a range contained in (3.9). The bestness follows from the fact that any value on $|w-\Gamma| = \Gamma + (1/2) - \varepsilon$ is taken at least once by some continued fraction (1.1) with all $a_n \in \partial E(\Gamma, \Gamma + (1/2) - \varepsilon)$.

b) The circles (3.5) all touch the circle $|w-\Gamma| = \Gamma + (1/2) - \varepsilon$. Let d be the minimum diameter of the circles. Once we have found this minimum and seen that it conforms with the left-hand side of (3.10), b) is proved. The diameter is (see (3.5))

$$2\frac{4\Gamma+2-4\varepsilon}{(4\Gamma+3-2\varepsilon)(1+2\varepsilon)}r(\theta).$$

Since the minimum value of $r(\theta)$ is $r(\theta) = (1/4) - \varepsilon^2$, b) follows by simple verification.

c) The shortest distance from the origin to any point on a fixed circle (3.5) is

$$\frac{4\Gamma+4}{(4\Gamma+3-2\varepsilon)(1+2\varepsilon)}r(\theta) - \frac{4\Gamma+2-4\varepsilon}{(4\Gamma+3-2\varepsilon)(1+2\varepsilon)}r(\theta) = \frac{2}{4\Gamma+3-2\varepsilon}r(\theta).$$

Since the minimum of $r(\theta)$ is $(1/4) - \varepsilon^2$ the shortest distance to any point on any circle is

$$\frac{1-4\varepsilon^2}{6+8\Gamma-4\varepsilon}.$$

This proves c). (The bestness in b) left and c) follows from the fact that any point on any circle (3.5) is in the range, in particular the ones for $\theta = \pi$.) □

Remarks. 1. Observe that for $\Gamma = 0$ the theorem gives back the result on the boundary version of Worpitzky's theorem. (Keep in mind, here and in Remark 2, the property (1.6).)

2. Observe that for $\Gamma \to \infty$ the disk in c) shrinks to the origin (Scott-Wall parabola case). We also see from b) that the parallel strip $-(1/2) \leq \operatorname{Re} w < 0$ is in $\mathcal{L}$ in this case, but this is not a very good result.

3. Finally, observe that the disk (3.11) touches the inner circle of the annulus (3.10) in the point

$$-\frac{1-4\varepsilon^2}{6+8\Gamma-4\varepsilon}.$$

(Common point of "bestness.")

We present some computer graphics illustrations of the way the range is obtained as a union of circles. In the illustrations ε is so small that it is negligible.

FIGURE 1. $\Gamma = 0$.

FIGURE 2. $\Gamma = 0.3$.

4. The significance of a_1. By analyzing the method to establish the range, we find exactly the same range by requiring

$$a_1 \in \partial E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right)$$
$$a_n \in E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right) \qquad \text{for } n \geq 2$$

as the one we get if the condition is

$$a_n \in \partial E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right) \qquad \text{for all } n.$$

Compared to the case $a_n \in E(\Gamma, \Gamma + (1/2) - \varepsilon)$ we may therefore say that a_1 alone decreases the range from the disk $V(\Gamma, \Gamma + (1/2) - \varepsilon)$ to the "disk with a hole."

FIGURE 3. $\Gamma = 2$.

On the other hand, if

$$a_1 \in E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right)$$

whereas

$$a_n \in \partial E\left(\Gamma, \Gamma + \frac{1}{2} - \varepsilon\right),$$

then the range is again all of $V(\Gamma, \Gamma + (1/2) - \varepsilon)$, a new illustration of the significance of the first parameter.

Acknowledgment. The author gratefully acknowledges the able assistance of Dr. Frode Rönning in producing computer graphics for this investigation.

REFERENCES

1. L. Jacobsen and W.J. Thron, *Oval convergence regions and circular limit regions for continued fractions* $K(a_n/1)$, *Analytic theory of continued fractions* II (W.J. Thron, ed.), Lecture Notes in Mathematics **1199**, Springer-Verlag, Berlin (1986), 90–126.

2. W.B. Jones and W.J. Thron, *Continued fractions*: *Analytic theory and applications*, Encyclopedia of Mathematics and its Applications **11**, Addison-Wesley (now available from Cambridge University Press), 1980.

3. W.B. Jones, W.J. Thron and H. Waadeland, *Value regions for continued fractions* $K(a_n/1)$ *whose elements lie in parabolic regions*, Math. Scand. **56** (1985), 5–14.

4. R.E. Lane, *The value region problem for continued fractions*, Duke Math. J. **12** (1945), 207–216.

5. W. Leighton and W.J. Thron, *On value regions of continued fractions*, Bull. Amer. Math. Soc. **48** (1942), 917–920.

6. W.T. Scott and H.S. Wall, *A convergence theorem for continued fractions*, Bull. Amer. Math. Soc. **47** (1941), 580–585.

7. W.J. Thron, *On parabolic convergence regions for continued fractions*, Math. Z. **69** (1958), 173–182.

8. H. Waadeland, *Boundary versions of Worpitzky's theorem and of parabola theorems*, in *Analytic theory of continued fractions* III (L. Jacobsen, ed.), Lecture Notes in Mathematics **1406**, Springer-Verlag, Berlin, (1989), 135–142.

9. J.D.T. Worpitzky, *Untersuchungen über die Entwickelung der monodromen und monogenen Funktionen durch Kettenbrüche*, Friedrichs-Gymnasium und Realschule Jahresbericht, Berlin (1865), 3–39.

Department of Mathematics and Statistics, University of Trondheim–AVH, N-7055 Dragvoll, Norway