ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

ON ISOMORPHISM OF GROUP ALGEBRAS OF TORSION ABELIAN GROUPS

WILLIAM ULLERY

1. Introduction. Suppose R is a commutative ring with 1 and G is a torsion abelian group. Suppose further that H is a group such that the group algebras $R(G)$ and $R(H)$ are R-isomorphic. The theme of this paper is to determine conditions under which such an R-isomorphism implies isomorphism of the groups G and H.

Of course, one cannot always expect isomorphism of $R(G)$ and $R(H)$ to imply that the torsion groups G and H are isomorphic. For example if R is the field of complex numbers, it is known that the isomorphism class of $R(G)$ is completely determined by $|G|$ (see [**2**]). The basic philosophy of the proof of this result is that the presence in R of invertible rational primes p, for which the p-component G_p of G is nontrivial, introduces idempotents into $R(G)$. These idempotents, in turn, tend to obscure the structure of G.

Returning to the general case, denote by $\operatorname{inv}(R)$ the set of rational primes p which invert in R (i.e., such that $p \cdot 1$ is a unit) and set $G_R = \amalg\{G_p : p \in \operatorname{inv}(R)\}$. We call G *R-favorable* if G_R is the trivial subgroup of G. In view of the above example, we restrict our attention to the case when G is R-favorable.

Consider first the special case in which R is a field of characteristic $p \neq 0$. In this case, an R-favorable torsion group G is a p-group. If $R(G) \cong R(H)$ for some group H, it is known that $G \cong H$ if G is totally projective [**4**], an N-group [**7, 8**], or an elementary A-group [**8**]. Recently, all of these cases were extended to the class of $A_n(\mu)$-groups [**9**], where n is a positive integer and μ is a limit ordinal. The precise definition of the classes $A_n(\mu)$ is unimportant in the present context. However, we mention the facts that the totally projective groups and N-groups are examples of elementary A-groups, and an elementary A-group is an $A_1(\mu)$-group for a suitable ordinal μ. These examples lead us to state the following.

Received by the editors on May 2, 1990.

Problem 1. Suppose R is a field of characteristic $p \neq 0$ and G is a p-group. If H is a group such that $R(G) \cong R(H)$, is G isomorphic to H?

It has been conjectured (see [**1**, Conjecture B, p. 174]) that the answer to Problem 1 is "Yes" in all possible cases. We also remark that the positive solutions in the above mentioned special cases rely heavily on group-theoretic techniques.

Suppose now that R has characteristic 0. In this case, if G is an R-favorable torsion group, $R(G) \cong R(H)$ is known to imply that $G \cong H$ if R is an integral domain [**3**] and, more generally, if R is indecomposable [**5**]. We remark that the proofs of these results, in contrast to the characteristic p case, rely heavily on ring-theoretic techniques. We now state a second problem.

Problem 2. Suppose R is a commutative ring with 1 and of characteristic 0. If G and H are R-favorable torsion groups and if $R(G) \cong R(H)$, is G isomorphic to H?

In the next section of this paper, we show that Problems 1 and 2 are actually equivalent (Theorem 2.3). Thus, for example, one cannot obtain a positive result in the characteristic p case and, at the same time, a counterexample in characteristic 0. So that we do not leave the impression that Theorem 2.3 is merely a case of *ignotum per ignotius*, we obtain another class of rings R for which Problem 2 has a positive answer. Precisely, if the additive group of R is assumed to be torsion-free (in which case R is necessarily of characteristic 0), it is shown that the isomorphism class of $R(G)$ determines the isomorphism class of G, provided of course that G is R-favorable (Theorem 3.2). Finally, in the last section, we construct a ring R of characteristic 0, whose additive subgroup has nontrivial p-torsion for all primes p, but for which Problem 2 has an affirmative answer (Example 4.6). Thus, it is shown that the hypothesis of torsion-free R in Theorem 3.2 is not necessary.

In the sequel, all rings considered will be commutative with 1 and R will always denote such a ring. Moreover, all groups considered are abelian and will be written multiplicatively (with the exception of the additive groups of rings). If G is an arbitrary abelian group, and if p is a prime number, G_p denotes the p-component of the torsion subgroup of G, and $G_R = \amalg\{G_p : p \in \text{inv}\,(R)\}$.

2. The problems are equivalent. We begin with two lemmas which are generalizations of Lemmas 2 and 3 in [**2**]. W. May proved these under the assumption that the ring R is an algebraically closed field. However, as we shall see, the hypothesis that R is algebraically closed is not necessary provided that R contains sufficiently many roots of unity. As a reference for the character theory used in the proof of our first lemma, we refer the reader to [**1**, Section 3.2].

Lemma 2.1. *Suppose H is a subgroup of a torsion group G of finite index n. Suppose further that R is an integral domain such that for all primes p with G_p nontrivial, $p \in \text{inv}\,(R)$ and R contains a primitive p^k-th root of unity for all positive integers k. Then $R(G)$ and $R(H)^n = R(H) \times \cdots \times R(H)$ (n factors) are isomorphic as $R(H)$-algebras. Moreover, the isomorphism can be chosen so that $\alpha \in R(H) \subseteq R(G)$ corresponds to $(\alpha, \dots, \alpha) \in R(H)^n$.*

Proof. Select a finite subgroup G_1 of G such that $G = G_1 H$ and set $H_1 = G_1 \cap H$. Observe that $|G_1/H_1| = n$.

We claim that $R(G_1) \cong R(H_1)^n$ as $R(H_1)$-algebras. To see this, let F be the quotient field of R, set $m = |H_1|$, and let $X_1, \dots, X_m$ denote the distinct characters of H_1 over F. For $1 \le i \le m$, set

$$e_i = (1/m) \sum\{X_i(h^{-1}) \cdot h : h \in H_1\}.$$

Then, $\{e_1, \dots, e_m\}$ is a complete set of pairwise orthogonal primitive idempotents for $F(H_1)$. For $1 \le i \le m$ and $1 \le j \le n$, set

$$e_{ij} = (1/mn) \sum\{X_{ij}(g^{-1}) \cdot g : g \in G_1\},$$

where $X_{i1}, \dots, X_{in}$ are the distinct characters of G_1 over F extending X_i. Thus, each e_{ij} is a primitive idempotent of $F(G_1)$ with $e_i = \sum\{e_{ij} : 1 \le j \le n\}$. By our hypotheses on R, it is evident that each e_i and e_{ij} are actually elements of $R(H_1)$ and $R(G_1)$, respectively. Moreover, each element of $R(H_1) \cdot e_i$ (respectively, $R(G) \cdot e_{ij}$) can be written uniquely in the form re_i (respectively, re_{ij}) for some $r \in R$. Thus, it is easily checked that the correspondence $re_{ij} \mapsto re_i$ is an isomorphism of $R(H_1)$-algebras from $R(G_1) \cdot e_{ij}$ onto $R(H_1) \cdot e_i$. Now, for $1 \leq j \leq n$, set

$$f_j = \sum\{e_{ij} : 1 \leq i \leq m\}.$$

Then $f_1, \dots, f_n$ are pairwise orthogonal idempotents of $R(G_1)$ with $\sum\{f_j : 1 \leq j \leq n\} = 1$. Consequently, $R(G_1) \cdot f_j \cong R(H_1)$ as $R(H_1)$-algebras. From this, it easily follows that $R(G_1) \cong R(H_1)^n$ as $R(H_1)$-algebras, and the claim is established.

Finally, by applying $\otimes_{R(H_1)} R(H)$ to both sides of the isomorphism $R(G_1) \cong R(H_1)^n$, we obtain $R(G) \cong R(H)^n$ as $R(H_1)$-algebras. Moreover, it is not difficult to see that the isomorphism is constructed so that $\alpha \in R(H) \subseteq R(G)$ is carried to $(\alpha, \dots, \alpha) \in R(H)^n$. Therefore, the isomorphism is also an $R(H)$-algebra isomorphism. □

Lemma 2.2. *Suppose that G and H are p-groups, R is an integral domain with $p \in \text{inv}\,(R)$, and that R contains a primitive p^k-th root of unity for every positive integer k. If $|G| = |H|$, then $R(G) \cong R(H)$ as R-algebras.*

Proof. It suffices to show that $R(G) \cong R(K)$ where K is a direct sum of cyclic groups of order p. Select a smooth chain $G_1 < G_2 < \cdots < G_i < \cdots (i < \lambda)$ of subgroups of $G = \cup_{i<\lambda} G_i$ such that

(A) $|G_1| = p$.

(B) For each ordinal $i < \lambda$, $|G_{i+1}/G_i| = p$.

Let L be a direct sum of cyclic groups of order p with $|L| > |G|$. Select a smooth chain $K_1 < K_2 < \cdots < K_i < \cdots (i \leq \lambda)$ of subgroups of L satisfying:

(A′) $|K_1| = p$.

(B′) For each ordinal $i < \lambda$, $|K_{i+1}/K_i| = p$.

By (A) and (A′), there is an isomorphism $\psi_1 : R(G_1) \to R(K_1)$. Suppose $1 < i < \lambda$ and, by induction, for each $j < i$, an isomorphism $\psi_j : R(G_j) \to R(K_j)$ has been constructed such that ψ_j extends ψ_k for each $k < j$. We show how to construct an isomorphism

$\psi_i : R(G_i) \to R(K_i)$ which extends each ψ_j.

If i is a limit ordinal, the construction of such a ψ_i is easily obtained by taking unions and using the smoothness of the sequences of G_i's and K_i's. Now suppose i is isolated. Say $i = j+1$. By Lemma 2.1 and by (B) and (B′), there exist isomorphisms $g_j : R(G_i) \to R(G_j)^p$ and $k_j : R(K_j)^p \to R(K_i)$, where $g_j(\alpha) = (\alpha, \dots, \alpha)$ for all $\alpha \in R(G_j)$ and $k_j(\beta, \dots, \beta) = \beta$ for all $\beta \in R(K_j)$. Let $f = \prod \psi_j : R(G_j)^p \to R(K_j)^p$, where $\prod \psi_j = \psi_j \times \cdots \times \psi_j$ (p factors). Setting $\psi_i = k_j \circ f \circ g_j : R(G_i) \to R(K_i)$ yields an isomorphism extending ψ_j.

Therefore, by taking $K = \cup_{i<\lambda} K_i$, we obtain an isomorphism $\psi : R(G) \to R(K)$. Moreover, since K is a subgroup of L, K is also a direct sum of cyclic groups of order p. □

In [**6**] it was shown that if F is a field of characteristic p and if G and H are abelian groups with $F(G) \cong F(H)$ as F-algebras, then $F(G/G_F) \cong F(H/H_F)$. This means that if G is a torsion group, then $F(G_p) \cong F(H_p)$, whenever $F(G) \cong F(H)$. Using this fact in conjunction with Lemma 2.2, we obtain the main result of this section.

Theorem 2.3. *Problems* 1 *and* 2 *are equivalent. More precisely, the following statements are equivalent.*

(1) *For every field F of nonzero characteristic p and for every p-group G, $F(G) \cong F(H)$ for some group H implies that $G \cong H$.*

(2) *For every ring R of characteristic 0 and for all R-favorable torsion groups G and H, $R(G) \cong R(H)$ implies that $G \cong H$.*

Proof. (1) $\Rightarrow$ (2). Suppose R has characteristic 0 and G and H are R-favorable torsion groups with $R(G) \cong R(H)$. Suppose p is a prime number such that G_p is nontrivial. Thus, $p \notin \operatorname{inv}(R)$ and there exists a maximal ideal M of R with $p \in M$. Set $F = R/M$, a field of characteristic p. Applying $F \otimes_R$ to both sides of $R(G) \cong R(H)$, we obtain $F(G) \cong F(H)$ as F-algebras. By the remark preceding the statement of the Theorem, $F(G_p) \cong F(H_p)$. Thus, an application of (1) implies that $G_p \cong H_p$, for every $p \notin \operatorname{inv}(R)$. Therefore, $G \cong H$.

(2) $\Rightarrow$ (1). Suppose F is a field of characteristic p and $F(G) \cong F(H)$ for some p-group G and group H. Observe that H must also be a

p-group. Moreover, $|G| = |H|$. Let C_p denote the set of all complex p^k-th roots of unity, $k \geq 1$. If $S = \mathbf{Z}[1/p, C_p]$, Lemma 2.2 implies that $S(G) \cong S(H)$. Thus, if we set $R = S \times F$, note that $R(G) \cong R(H)$ and that both G and H are R-favorable torsion groups. Therefore, (2) implies $G \cong H$. $\square$

3. Torsion-free R. In this section we show that Problem 2 has an affirmative answer if the additive group of R is torsion-free. Clearly, such a ring has characteristic 0. For convenience, let $\mathrm{zd}\,(R)$ denote the set of prime numbers that are zero divisors in R. Thus, R is torsion-free if and only if $\mathrm{zd}\,(R) = \varnothing$. We begin with some elementary facts concerning the relationship between the sets $\mathrm{inv}\,(R)$ and $\mathrm{zd}\,(R)$ and the minimal prime ideal structure of R.

Lemma 3.1. (a) *$p \in \mathrm{inv}\,(R)$ if and only if $p \in \mathrm{inv}\,(R/P)$ for every minimal prime ideal P of R.*

(b) *If R/P has nonzero characteristic p for some minimal prime ideal P, then $p \in \mathrm{zd}\,(R)$. The converse holds if R has trivial nilradical.*

Proof. (a) If $p \in \mathrm{inv}\,(R)$, then p inverts in every homomorphic image. In particular, $p \in \mathrm{inv}\,(R/P)$ for every minimal prime ideal P. Conversely, if $p \notin \mathrm{inv}\,(R)$, there exists a maximal ideal M of R and a minimal prime ideal P with $p \in M$ and $P \subseteq M$. Therefore, $p \notin \mathrm{inv}\,(R/P)$, since R/M is a homomorphic image of R/P and $p \notin \mathrm{inv}\,(R/M)$.

(b) If R/P has nonzero characteristic p for some minimal prime ideal P, then $p \in P$. Since every minimal prime ideal consists of zero divisors, $p \in \mathrm{zd}\,(R)$. Conversely, if R has trivial nilradical, the set of zero divisors of R is the union of the minimal prime ideals of R. Thus, if $p \in \mathrm{zd}\,(R)$, $p \in P$ for some minimal prime ideal P. In this case R/P has characteristic p. $\square$

Theorem 3.2. *Problem 2 has an affirmative answer for a torsion-free ring R. More precisely, if the additive group of R is torsion-free and if G and H are R-favorable torsion groups with $R(G) \cong R(H)$, then $G \cong H$.*

Proof. Suppose $p \notin \operatorname{inv}(R)$. By Lemma 3.1(a), $p \notin \operatorname{inv}(R/P)$ for some minimal prime ideal P. Moreover, since R is torsion-free, $\operatorname{zd}(R) = \varnothing$. Therefore, Lemma 3.1(b) implies that R/P has characteristic 0.

If $R(G) \cong R(H)$ for R-favorable torsion groups G and H, then $(R/P)(G) \cong (R/P)(H)$. Since R/P is an integral domain of characteristic 0, it follows from [**3** or **5**] that $G/G_{R/P} \cong H/H_{R/P}$. As $p \notin \operatorname{inv}(R/P)$, we conclude that $G_p \cong H_p$ for every $p \notin \operatorname{inv}(R)$. Since G and H are both R-favorable torsion groups, $G \cong H$. □

Suppose that G and H are (possibly mixed) abelian groups. As shown in [**5**], for $R(G) \cong R(H)$ to always imply $G/G_R \cong H/H_R$, it is necessary that R be an *ND-ring*. This means that no matter how R is decomposed as a finite direct product, at least one of the factors has the same invertible primes as R itself.

Example 3.3. Suppose $p_1, p_2, \dots, p_n$, $n \geq 2$, are distinct prime numbers and set $R = \mathbf{Z}[1/p_1] \times \cdots \times \mathbf{Z}[1/p_n]$. Note that $\operatorname{inv}(R) = \varnothing$ and so R is not an ND-ring, since each factor contains an invertible prime. Therefore, by Theorem 2 in [**5**] and its proof, there exist nonisomorphic mixed abelian groups G and H with $R(G) \cong R(H)$. On the other hand, suppose $R(G) \cong R(H)$ where G and H are torsion groups. Since G and H are R-favorable and since R is torsion-free, Theorem 3.2 implies $G \cong H$.

4. Mixed R and further examples. In Theorem 3.2 it was shown that the question asked in Problem 2 has an affirmative answer if R is torsion-free. This section is devoted to constructing an example of a mixed ring R of characteristic 0 which also yields an affirmative answer to Problem 2. In fact, the R we produce will be seen to have nontrivial p-torsion for every prime p. Therefore, the hypothesis in Theorem 3.2 that R is torsion-free is sufficient, but not necessary. Along the way, we shall also obtain some results concerning isomorphism of group algebras of arbitrary abelian groups.

At this point we direct our attention to several needed technicalities. First, by an idempotent of R, we mean a *nonzero* idempotent of R. Call a set of idempotents $\{e_1, \dots, e_m\} \subseteq R$ a *complete orthogonal set* if $e_i e_j = 0$ for $i \neq j$ and $e_1 + \cdots + e_m = 1$. In this case $R = Re_1 \oplus \cdots \oplus Re_m$

and each Re_i may be viewed as a ring with identity e_i. As usual, an idempotent e is called *primitive* if Re is indecomposable.

Lemma 4.1. *Suppose R and S are rings with complete orthogonal sets $\{e_1, \dots, e_m\} \subseteq R$ and $F = \{f_1, \dots, f_n\} \subseteq S$. If each f_j, $1 \le j \le n$, is primitive and if $\psi : R \to S$ is an injective ring-homomorphism, then*:

(a) *For $1 \le i \le m$, $E_i = \{f \in F : \psi(e_i)f = f\}$ is nonempty and $\psi(e_i) = \sum\{f : f \in E_i\}$.*

(b) *$E_1, \dots, E_m$ partition F.*

(c) *Each E_i is uniquely determined by e_i. That is, if $F_i \subseteq F$ with $\psi(e_i) = \sum\{f : f \in F_i\}$, then $F_i = E_i$.*

Proof. (a) Note that $\psi(e_i) = \psi(e_i)f_1 + \dots + \psi(e_i)f_n$. Since $\psi(e_i) \neq 0$ and is idempotent, each f_j primitive implies that $\psi(e_i)f_j = f_j$ for some j. Therefore, $E_i \neq \varnothing$ and $\psi(e_i) = \sum\{f : f \in E_i\}$.

(b) Suppose $m \ge 2$ and $f \in E_i \cap E_j$ with $1 \le i < j \le m$. Then, $\psi(e_i) = f + e$ and $\psi(e_j) = f + e'$, where $e = \sum\{s : s \in E_i - \{f\}\}$ and $e' = \sum\{s : s \in E_j - \{f\}\}$. Hence, $0 = \psi(e_i e_j) = f + ee'$ and $0 = f^2(f + ee') = f + (ef)(e'f) = f$, contradicting $f \neq 0$. Consequently, $E_i \cap E_j = \varnothing$ whenever $i \neq j$. Also, $1 = \psi(1) = \psi(e_1 + \dots + e_n) = \sum_i \sum\{f : f \in E_i\} = \sum\{f : f \in \cup_i E_i\}$. Therefore, $F = \cup_i E_i$ since $f_1, \dots, f_n$ are pairwise orthogonal.

(c) This follows easily from the fact that F is a complete orthogonal set for S. We omit the details. □

For a directed set I, a direct system over I will be written as $\{S_\alpha(\alpha \in I); \psi_{\alpha\beta}\}$. Here the S_α are objects in some category (for us, either rings or sets) and, for each $\alpha \le \beta$, $\psi_{\alpha\beta} : S_\alpha \to S_\beta$ is a morphism satisfying $\psi_{\alpha\alpha} = \mathrm{id}_\alpha$, the identity map on S_α, and $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$ whenever $\alpha \le \beta \le \gamma$. We write $\lim_{\to I} S_\alpha$ for the direct limit of $\{S_\alpha(\alpha \in I); \psi_{\alpha\beta}\}$. Inverse systems are defined dually, and we employ the notation $\lim_{\leftarrow I} S_\alpha$ for the inverse limit of the inverse system $\{S_\alpha(\alpha \in I); \psi_{\beta\alpha}\}$.

As the following result is a routine exercise, we omit the proof.

Lemma 4.2. *Let* $\{S_\alpha(\alpha \in I); \psi_{\alpha\beta}\}$ *be a direct system of rings and ring-homomorphisms over a directed set* I.

(a) *If each* S_α *has characteristic* 0, *then* $\lim_{\to I} S_\alpha$ *has characteristic* 0.

(b) *If each* S_α *is indecomposable, so is* $\lim_{\to I} S_\alpha$.

In Theorem 3 of [**5**], ND-rings were characterized as those rings R which possess an indecomposable homomorphic image S with $\operatorname{inv}(S) = \operatorname{inv}(R)$. The following result shows that S may be taken to have characteristic 0, if R is an ND-ring of a special type.

Theorem 4.3. *Suppose* R *has characteristic* 0, *and whenever* R *is written as a finite direct sum* $R = R_1 \oplus \cdots \oplus R_n$ *of nonzero ideals* $R_1, \dots, R_n$, *there exists an* i, $1 \le i \le n$, *such that* R_i *has characteristic* 0 *and* $\operatorname{inv}(R_i) = \operatorname{inv}(R)$. *Then* R *has an indecomposable characteristic* 0 *homomorphic image* S *with* $\operatorname{inv}(S) = \operatorname{inv}(R)$.

Proof. Let $\{R_\alpha : \alpha \in I\}$ be the set of finitely generated subrings of R indexed by a set I. Partially order I by defining $\alpha \le \beta$ for $\alpha, \beta \in I$ if and only if $R_\alpha \subseteq R_\beta$. Note then that I is a directed set. If $\alpha \le \beta$, let $\psi_{\alpha\beta} : R_\alpha \to R_\beta$ be the inclusion map. Thus, a direct system $\{R_\alpha(\alpha \in I); \psi_{\alpha\beta}\}$ is obtained. Moreover, we may view $R = \lim_{\to I} R_\alpha$.

Since each R_α is Noetherian, we have $R_\alpha = R_\alpha e_{\alpha 1} \oplus \cdots \oplus R_\alpha e_{\alpha n(\alpha)}$, where $n(\alpha)$ is a positive integer and $E_\alpha = \{e_{\alpha 1}, \dots, e_{\alpha n(\alpha)}\}$ is a complete orthogonal set for R_α (and hence R) with each $R_\alpha e_{\alpha i}$ indecomposable ($1 \le i \le n(\alpha)$). For each pair $\alpha \le \beta$ in I, use parts (a) and (b) of Lemma 4.1 to obtain a partition $E^\alpha_{\beta 1}, \dots, E^\alpha_{\beta n(\alpha)}$ of E_β such that $\psi_{\alpha\beta}(e_{\alpha i}) = \sum\{f : f \in E^\alpha_{\beta i}\}$, for $1 \le i \le n(\alpha)$.

For each $\beta \ge \alpha$, define a mapping $\lambda_{\beta\alpha} : E_\beta \to E_\alpha$ as follows: Given $e \in E_\beta$, select the unique $E^\alpha_{\beta j}$, $1 \le j \le n(\alpha)$, with $e \in E^\alpha_{\beta j}$. Define $\lambda_{\beta\alpha}(e) = e_{\alpha j}$. Note that the mapping is well-defined by Lemma 4.1. It is now easily verified that $\{E_\alpha(\alpha \in I); \lambda_{\beta\alpha}\}$ is an inverse system of nonempty finite sets.

For each $\alpha \in I$, set $F_\alpha = \{e \in E_\alpha : \operatorname{inv}(R_\alpha e) \subseteq \operatorname{inv}(R)$ and $R_\alpha e$ has characteristic $0\}$. The hypotheses on R guarantee that each F_α is nonempty. If we set $\mu_{\beta\alpha} = \lambda_{\beta\alpha} | F_\alpha$, an inverse system $\{F_\alpha(\alpha \in I); \mu_{\beta\alpha}\}$

of nonempty finite sets is obtained. Since $\lim_{\leftarrow I} F_\alpha$ is known to be nonempty, for each α we may select an idempotent $e_\alpha \in F_\alpha$ with $\mu_{\beta\alpha}(e_\beta) = e_\alpha$ for all $\beta \geq \alpha$.

For each α, let $\pi_\alpha : R_\alpha \to R_\alpha e_\alpha$ be the projection map and, if $\alpha \leq \beta$, set $\eta_{\alpha\beta} = \pi_\beta \circ (\pi_{\alpha\beta} | R_\alpha e_\alpha)$. In this way, a direct system $\{R_\alpha e_\alpha (\alpha \in I); \eta_{\alpha\beta}\}$ of rings and ring-homomorphisms is obtained, where each $R_\alpha e_\alpha$ is indecomposable of characteristic 0 and $\mathrm{inv}\,(R_\alpha e_\alpha) \subseteq \mathrm{inv}\,(R)$. Set $S = \lim_{\to I} R_\alpha e_\alpha$. By Lemma 4.2, S is indecomposable of characteristic 0. Moreover, for each $\alpha \leq \beta$, the diagram

$$\begin{array}{ccc} R_\alpha & \xrightarrow{\psi_{\alpha\beta}} & R_\beta \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi_\beta} \\ R_\alpha e_\alpha & \xrightarrow{\eta_{\alpha\beta}} & R_\beta e_\beta \end{array}$$

commutes. Recalling that $R = \lim_{\to I} R_\alpha$, we have an induced homomorphism $\pi : R \to S$, which is surjective, since each π_α is. The existence of such a homomorphism $\pi : R \to S$ implies that $\mathrm{inv}\,(R) \subseteq \mathrm{inv}\,(S)$. Finally, since $\mathrm{inv}\,(R_\alpha, e_\alpha) \subseteq \mathrm{inv}\,(R)$ for each α, $\mathrm{inv}\,(S) \subseteq \mathrm{inv}\,(R)$. $\square$

Before proceeding to our final example, it is desirable to recall some notions from [**5**]. First a ring R is said to satisfy the *Isomorphism Theorem* if whenever $R(G) \cong R(H)$ for (possibly mixed) abelian groups G and H, then $G/G_R \cong H/H_R$. In [**5**] it was shown that any indecomposable ring of characteristic 0 satisfies the Isomorphism Theorem. Combining this with Theorem 4.3, we obtain

Corollary 4.4. *If R has the hypotheses of Theorem* 4.3*, then R satisfies the Isomorphism Theorem.*

Proof. By Theorem 4.3, there exists an indecomposable homomorphic image S of R such that S has characteristic 0 and $\mathrm{inv}\,(R) = \mathrm{inv}\,(S)$. Suppose $R(G) \cong R(H)$ for abelian groups G and H. Viewing S as an R-algebra, we have $S(G) \cong S \otimes_R R(G) \cong S \otimes_R R(H) \cong S(H)$. Therefore, by the above mentioned result of [**5**], $G/G_R = G/G_S \cong H/H_S = H/H_R$. $\square$

In [**5**] it was shown that a ring R which satisfies the Isomorphism Theorem must be an ND-ring. Moreover, the converse was shown to hold if $\mathrm{inv}\,(R)$ is not the complement of a single prime number. As a preliminary fact needed in our final example, we now give an example of an ND-ring with exactly one noninvertible prime which satisfies the Isomorphism Theorem. Our example will not be a finite product of indecomposable rings of characteristic 0. This latter class contains all previously known examples of rings R with exactly one noninvertible prime for which the statements "R is an ND-ring" and "R satisfies the Isomorphism Theorem" are equivalent.

Example 4.5. Fix a prime number p. For each positive integer n, let (p^n) be the ideal of $\mathbf{Z}$ generated by p^n and set

$$R_p = \mathbf{Z}/(p) \times \mathbf{Z}/(p^2) \times \cdots \times \mathbf{Z}/(p^n) \times \cdots .$$

Clearly, $\mathrm{inv}\,(R_p)$ contains all primes but p. Moreover, R_p is a characteristic 0 ND-ring satisfying the hypotheses of Theorem 4.3 (because whenever R_p is written as a finite product, at least one of the factors is isomorphic to a product of finitely many $\mathbf{Z}/(p^n)$'s for various n's). Therefore, Corollary 4.4 implies that R_p satisfies the Isomorphism Theorem.

We can now give our final example.

Example 4.6. There exists a ring R such that the additive group of R has nontrivial p-torsion for every prime p, yet R yields a positive response to Problem 2.

Proof. For a prime number p, let R_p be as in Example 4.5. If $p_1, p_2, \ldots, p_n \ldots$ are the prime numbers, set

$$R = R_{p_1} \times R_{p_2} \times \cdots \times R_{p_n} \times \cdots .$$

Note that R has nontrivial p-torsion for all primes p and $\mathrm{inv}\,(R) = \varnothing$, so that every torsion group is R-favorable. Suppose now that G and H are torsion groups with $R(G) \cong R(H)$. Then $R_{p_n}(G) \cong R_{p_n}(H)$ for every $n \geq 1$, and Example 4.5 shows that $G_{p_n} \cong H_{p_n}$. Therefore, $G \cong H$. $\square$

REFERENCES

1. G. Karpilovsky, *Commutative group algebras*, Marcel Dekker, New York, 1983.

2. W. May, *Invariants for commutative group algebras*, Illinois J. Math. **15** (1971), 525–531.

3. ———, *Isomorphism of group algebras*, J. Algebra **40** (1976), 10–18.

4. ———, *Modular group algebras of simply presented abelian groups*, Proc. Amer. Math. Soc. **104** (1988), 403–409.

5. W. Ullery, *Isomorphism of group algebras*, Comm. Algebra **14** (1986), 767–785.

6. ———, *A conjecture relating to the isomorphism problem for commutative group algebras*, in *Group and semigroup rings*, North-Holland Math. Studies No. 126, North-Holland, Amsterdam, 1986, 247–252.

7. ———, *Modular group algebras of N-groups*, Proc. Amer. Math. Soc. **103** (1988), 1053–1057.

8. ———, *Modular group algebras of isotype subgroups of totally projective p-groups*, Comm. Algebra **17** (1989), 2325–2332.

9. ———, *An isomorphism theorem for commutative modular group algebras*, Proc. Amer. Math. Soc., to appear.

Department of Mathematics, Auburn University, Auburn, Alabama 36849-5307