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THE SWAP CONJECTURE

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Introduction. We are interested in studying relationships among the various generating sets for a finitely generated group. All groups considered in this paper are assumed to be finitely generated.

Definition. Let $\Gamma_n(G) = \{(g_1, \dots, g_n) |$ the set $\{g_1, \dots, g_n\}$ generates $G\}$ $r(G) =$ rank of $G = \min\{n | \Gamma_n(G) \neq \varnothing\}$.

The generating sets γ_1 and γ_2 are *Nielsen equivalent*, written $\gamma_1 \sim_N \gamma_2$, if there is a sequence of Nielsen transformations, without deletions or insertions, leading from γ_1 to γ_2. The generating sets γ_1 and γ_2 are *swap equivalent* if there is a sequence of elementary swaps leading from γ_1 to γ_2, where an elementary swap changes one element of $\Gamma_n(G)$ to another by changing a single entry.

It is easily checked that Nielsen equivalence implies swap equivalence but not conversely and examples abound of groups with many, even infinitely many, Nielsen classes. We are unaware of any group with more than one swap class, motivating the conjecture of the title.

The swap conjecture. Any two finite generating sets for G of the same cardinality are swap equivalent.

In this paper, we relate these notions to two well-studied invariants of a generating set, namely the relation module and the relation space group (our terminology), and verify the conjecture for certain classes of groups. In Section 1 we give topological proofs of several known properties of relation modules and relation space groups.

1. Definitions and basic properties.

Definition. For $\gamma = (g_1, \dots, g_n) \in \Gamma_n(G)$, the associated epimorphism ε_γ from the free group $F[x_1, \dots, x_n]$ to G is defined by $\varepsilon_\gamma(x_i) = g_i$. Then $N(\gamma) = \ker(\varepsilon_\gamma)$, $i_\gamma : N(\gamma) \hookrightarrow F(\gamma) = F[x_1, \dots, x_n]$,

$$\overline{N}(\gamma) = N(\gamma)/[N(\gamma), N(\gamma)] \quad \text{and} \quad \overline{F}(\gamma) = F(\gamma)/[N(\gamma), N(\gamma)].$$

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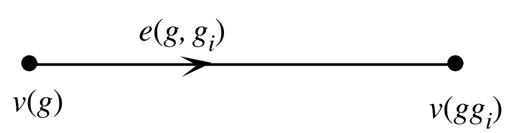
$\overline{N}(\gamma)$ is the *relation module* of γ and $\overline{F}(\gamma)$ is the *relation space group* (our terminology).

Thus $\overline{F}(\gamma)$ is an extension of the abelian group $\overline{N}(\gamma)$ by G and the action of G on $\overline{N}(\gamma)$ is the $\mathbf{Z}G$-module structure.

These objects, which we think of as invariants of γ, have been studied by a number of authors. In particular, Gruenburg [**6**], Linnell [**9**] and Williams [**16**] have studied the case of finite G and proved in particular that if $n > r(G)$, then all elements of $\Gamma_n(G)$ yield isomorphic relation modules and relation space groups. Their methods rely heavily on the finiteness of G. It is our hope to develop tools to study these issues for finitely generated G. Our basic point of view is geometric, focusing on the Cayley graph C_γ corresponding to γ; see the Definition and Proposition below, both standard. (One motivation for studying swap equivalence is the standard proof of the Tietze theorem [**11**, p. 89] which shows that two n-generator presentations of a group are related by a sequence of Tietze transformations in such a way that all the intermediate groups have at most $2n$ generators; the swap conjecture states that this can be improved to $n+1$.) For completeness and to establish the geometric point of view, we include proofs of all statements, although many are in the literature. The term *relation space group* is used for $\overline{F}(\gamma)$ because in the case of finite G, $\overline{F}(\gamma)$ is a Bieberbach group (see, e.g., Charlap [**1**]) and can be considered a space group in the classical sense.

Definition. The Cayley graph C_γ has a vertex $v(g)$ for each $g \in G$ and an edge $e(g, g_i)$ for each pair $(g, g_i) \in G \times \gamma$ with $e(g, g_i)$ joining $v(g)$ to $v(gg_i)$ as indicated below.

G acts on C_γ by $g(v(h)) = v(gh)$ and $g(e(h, g_i)) = e(gh, g_i)$.



Proposition. *$N(\gamma) \cong \pi_1(C_\gamma, v(1))$ as groups and $\overline{N}(\gamma) \cong H_1(C_\gamma, \mathbf{Z})$ as $\mathbf{Z}G$ modules.*

Proof. $C_\gamma/G \cong S^1 \vee \cdots \vee S^1$, a wedge of n circles, so $\pi_1(C_\gamma/G) \cong F[x_1, \ldots, x_n]$ where the projected image of $e(g, g_i)$ is the ith circle. The projection map $p : C_\gamma \to C_\gamma/G$ is a regular covering with the action of G as the covering translations. Furthermore, since a path in C_γ starting at $v(1)$ is closed if and only if the product of its edge labels is 1 in G, this is the covering corresponding to $N(\gamma)$. Thus,

$$\pi_1(\rho) : \pi_1(C_\gamma, v(1)) \to F[x_1, \ldots, x_n]$$

is an isomorphism onto $N(\gamma)$. The second isomorphism follows on the group level since first homology is the abelianization of first homotopy. The action of G on $\overline{N}(\gamma)$ can be described as follows: choose any setwise splitting σ_γ of ε_γ:

$$1 \to N(\gamma) \to F(\gamma) \underset{\sigma_\gamma}{\overset{\varepsilon_\gamma}{\rightleftarrows}} G \to 1$$

and let

$$g \cdot [n] = [\sigma_\gamma(g) n \sigma_\gamma(g)^{-1}].$$

This is a well-defined action modulo $[N(\gamma), N(\gamma)]$. Thus, in particular, $g_i[n] = [x_i n x_i^{-1}]$. But this is clearly the map induced on $H_1(C_\gamma, \mathbf{Z})$ by the action of g_i, which is to move forward along the ith edge. □

Theorem 1. [**8, 15**] *$\overline{F}(\gamma)$ is torsion free. If $n \geq 2$ and γ is not a free basis for G, then the action of G on $\overline{N}(\gamma)$ is effective.*

Proof. Suppose C_γ is the Cayley graph of G corresponding to the set γ, so $N(\gamma) = \pi_1(C_\gamma, *)$ and $\overline{N}(\gamma) = H_1(C_\gamma, \mathbf{Z})$. For any maximal tree in C_γ, the edges of $C_\gamma \setminus T$ determine a free generating set for $N(\gamma)$ and a free abelian generating set for $\overline{N}(\gamma)$. To analyze possible torsion in $\overline{F}(\gamma)$, we consider a particular 2-cocycle $\bar{f}_T : G \times G \to \overline{N}(\gamma)$ determined by a maximal tree T in C_γ as follows. Let $s_T : G \to F(\gamma)$ be the setwise splitting of ε_γ given by; $s_T =$ the label on the path in T from the base point to the vertex labeled by g. {The splitting s_T is another name for a Schreier system for $N(\gamma)\}$. Then

$$\begin{array}{ccc} G & \xrightarrow{s_T} & F(\gamma) \\ & \bar{s}_T \searrow & \downarrow \\ & & \overline{F}(\gamma) \end{array} \quad \text{and} \quad \bar{i}_\gamma \bar{f}_T(g,h) = \bar{s}_T(g)\bar{s}_T(h)[\bar{s}_T(gh)]^{-1}$$

$\overline{F}(\gamma)$ is then isomorphic to the set $\overline{N}(\gamma) \times G$ with multiplication given by

$$(\bar{n}, g) \cdot (\bar{m}, h) = (\bar{n} + g \cdot \bar{m} + \bar{f}_T(g,h), gh)$$

so for any k,

$$(\bar{n}, g)^k = ((1 + g + \cdots + g^{k-1})\bar{n} + \bar{f}_T(g,g) + \cdots + \bar{f}_T(g^{k-1}, g)g^k).$$

If $\overline{F}(\gamma)$ has torsion, it is possible to choose a torsion element $(\bar{n}, g)$ of order k, with g of order m dividing k, and a word w representing g so that the loop in C_γ labeled by w^m is a simple closed curve ℓ. (This is guaranteed, for example, if $(\bar{n}, g)$ has minimal order k in $\overline{F}(\gamma)$ and w is the shortest representative of any g' for which $(\bar{n}', g')$ is conjugate to $(\bar{n}, g)$.) Now choose the tree T to include all but the last arc of ℓ. It follows that $\bar{f}_T(g^j, g) = 0$ for $j < m - 1$ and $\bar{f}_T(g^{m-1}, g) = [w^m]$ is the homology class $[\ell]$ represented by ℓ. Thus, $k = mm'$ for some m' and

$$(\bar{n}, g)^k = (m'(1 + g + \cdots + g^{m-1})\bar{n} + m'[\ell], 1)$$

so

$$[\ell] = (1 + g + \cdots + g^{m-1}[-\bar{n}] \quad \text{in } H_1(C_\gamma, \mathbf{Z}).$$

This contradicts the following claim, which completes the proof that $\overline{F}(\gamma)$ is torsion free. ◻

Claim. *If T is a graph on which the cyclic group $C_m = \{1, g, \dots, g^{m-1}\}$ acts freely and ℓ is an invariant circle, then the equation*

$$[\ell] = (1 + g + \cdots + g^{m-1})[\mu]$$

has no solution $[\mu]$ in $H_1(\Gamma, \mathbf{Z})$.

Proof. We have a map of coverings

$$\begin{array}{cccccc} & S^1 & = & \ell & \hookrightarrow & \Gamma \\ p_m & \downarrow & & \downarrow & & \downarrow \\ & S^1 & = & \ell/C_m & \hookrightarrow & \Gamma/C_m \end{array}$$

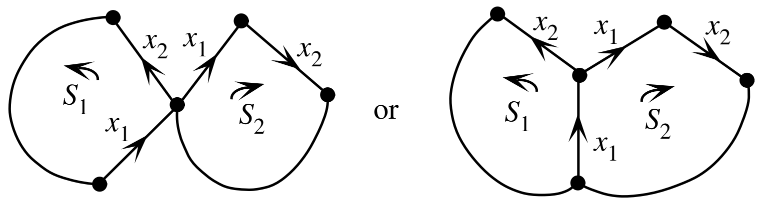
where p_m is the unique connected m-fold cover of $S^1 \cdot \Gamma/C_m$ retracts to ℓ/C_m (any connected graph retracts to any connected subgraph) and this retraction lifts to a C_m invariant retraction of Γ to ℓ. Thus, any solution in Γ gives one in ℓ, a clear impossibility. □

We now suppose that γ has at least two elements and is not a free basis for G and prove that the G-action on $\overline{N}(\gamma)$ is effective; it is easy to check that, in the excluded cases, the action is not effective.

There are two cases to consider. The first is a presentation of the form $\langle x_1, \ldots, x_n | x_1^k \rangle$ of $\mathbf{Z}_k * F[x_2, \ldots, x_n]$ which is characterized by the property that all elements of $N(\gamma)$ are conjugates of powers of a single letter. In this case, $\overline{N}(\gamma)$ has generators corresponding to the simple closed curves in C_γ, all of which are disjoint and labeled by x_1^k. Algebraically, these generators correspond to symbols g_w, w a standard normal form in $\mathbf{Z}_k * F[x_2, \ldots, x_n]$ not ending in x_1, and the action is given by $u \cdot g_w = g_{\widetilde{uw}}$, $\widetilde{uw} = uw$ with any terminal x_i's deleted. Thus, g_w is fixed only by $u = w x_1^{\pm 1} w^{-1}$ so the action is effective.

The second case includes all other presentations; thus N_γ contains a cyclically reduced element involving at least two different generators, say $x_1 x_2 w$. At each vertex of C_γ there is a closed loop labeled by $x_1 x_2 w$—the loops S_1 and S_2 shown below are then distinct loops in C_γ that meet. (We can easily arrange that S_1 and S_2 are simple closed curves.) We can then choose a maximal tree T so that two edges α_1 and α_2 in $C \backslash T$ share a vertex. Suppose now that $g \in G$ acts as the identity on $\overline{N}(\gamma)$, so that in particular g fixes $[\alpha_1]$ and $[\alpha_2]$, the homology classes corresponding to α_1 and α_2. Geometrically, $[\alpha_i]$ is represented by the simple closed curve S_i; since the action of g on $\overline{N}(\gamma) = H_1(C_\gamma, \mathbf{Z})$ is induced by covering translations and S_i is the only simple closed curve in C_γ representing $[\alpha_i]$, g must act as a nontrivial rotation on each of S_1 and S_2.

This is clearly impossible, establishing the effectiveness of the action in the second case. □



Definition. i) If $\gamma_1 \in \Gamma_n(G)$ and $\gamma_2 \in \Gamma_{n+k}(G)$, then $\gamma_1 \underset{k}{\nearrow} \gamma_2$ if $\gamma_1 \subset \gamma_2 : \gamma_1 \nearrow \gamma_2$ means $\gamma_1 \underset{1}{\nearrow} \gamma_2$. Also $\gamma^* = \{\gamma, 1\}$.

ii) If H acts on K (both groups) then $K \rtimes H$ is the split extension of K by H corresponding to this action.

Remark. The order of the elements of γ is important in defining ε_γ but not otherwise, as all reorderings of γ are Nielsen equivalent to γ.

Theorem 2. [**7**, Section 9.5] *If $\gamma_1 \nearrow \gamma_2$, then*

(i) $\overline{N}(\gamma_2) \cong \mathbf{Z}G \oplus \overline{N}(\gamma_1)$ *as* $\mathbf{Z}G$*-modules, and*

(ii) $\overline{F}(\gamma_2) \cong \mathbf{Z}G \rtimes \overline{F}(\gamma_1)$ *as groups where* $\overline{F}(\gamma_1)$ *acts on* $\mathbf{Z}G$ *through* $\varepsilon_\gamma : \overline{F}(\gamma_1) \to G$.

Proof. Statement (i) is, of course, the relation module theorem which says that $H_1(C_{\gamma^*}) \cong H_1(C_\gamma) \oplus \mathbf{Z}G$ as $\mathbf{Z}G$-modules. Geometrically, C_{γ^*} is obtained by adding a G-invariant family of edges to C_γ, one emanating from each vertex. The maximal tree T for C_γ serves for C_{γ^*} as well and the added edges generate the extra factors of $\mathbf{Z}$ that fit together under the G-action to form the copy of $\mathbf{Z}G$.

Now $\overline{F}(\gamma) = \overline{N}(\gamma) \times G$ and $\overline{F}(\gamma^*) = \overline{N}(\gamma^*) \times G$ with multiplication as described in the proof of Theorem 1. From (i) we have π_N and σ_N so that

$$1 \to \mathbf{Z}G \to \overline{N}(\gamma^*) \underset{\sigma_F}{\overset{\pi_F}{\rightleftarrows}} \overline{N}(\gamma) \to 1$$

and using a common tree

$$\begin{array}{ccc} & & \overline{N}(\gamma) \\ & \overset{\bar{f}_T}{\nearrow} & \\ G \times G & & \pi_N \uparrow\downarrow \sigma_N \\ & \underset{\bar{f}_T^*}{\searrow} & \\ & & \overline{N}(\gamma^*). \end{array}$$

Then $\pi_F = \pi_N \times id$ and $\sigma_F = \sigma_N \times id$ are homomorphisms and

$$1 \to \mathbf{Z}G \to \overline{N}(\gamma^*) \times G \underset{\sigma_F}{\overset{\pi_F}{\rightleftarrows}} \overline{N}(\gamma) \times G \to 1.$$

(*Note.* π_F and σ_F are induced by the maps

$$F[X, y] \underset{\tilde{\sigma}_F}{\overset{\tilde{\pi}_F}{\rightleftarrows}} F[X]$$

where $\tilde{\sigma}_F$ is the obvious inclusion, $\tilde{\pi}_F$ is the identity on X and $\tilde{\pi}_F(y) = w$, where w is a word in X representing the same element in g as does y.) □

Theorem 3.

(i) *If* $\gamma_1 \sim_N \gamma_2$, *then* $\gamma_1 \sim_S \gamma_2$.

(ii) *If* $\gamma_1 \nearrow \gamma_2$, *then* $\gamma_2 \sim_N \gamma_1^*$.

(iii) *If* $\gamma_1 \sim_S \gamma_2$, *then* $\gamma_1^* \sim_N \gamma_2^*$.

(iv) *If* $\gamma_1 \sim_N \gamma_2$, *then* $\overline{N}(\gamma_1) \cong \overline{N}(\gamma_2)$ *and* $\overline{F}(\gamma_1) \cong \overline{F}(\gamma_2)$.

(v) *If* $\gamma_1 \sim_S \gamma_2$, *then* $\mathbf{Z}G \oplus \overline{N}(\gamma_1) \cong \mathbf{Z}G \oplus \overline{N}(\gamma)$ *and* $\mathbf{Z}G \rtimes \overline{F}(\gamma_1) \cong \mathbf{Z}G \rtimes \overline{F}(\gamma_2)$.

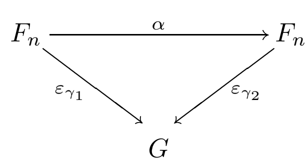
Proof. Elementary Nielsen transformations give elementary swaps, proving (i). (ii) follows from the fact that the added element in γ_2 is expressible as a word in the elements of γ_1. It suffices to prove (iii) for γ_1 and γ_2 related by an elementary swap, but then if $\gamma = \gamma_1 \cup \gamma_2$, $\gamma_1 \nearrow \gamma$ and $\gamma_2 \nearrow \gamma$ and (ii) implies that $\gamma_1^* \sim_N \gamma \sim_N \gamma_2^*$. (iv) is clear and (v) follows from Theorem 2. □

2. Primitive properties.

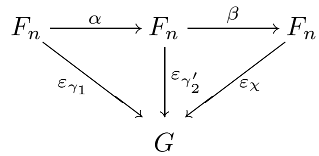
Definition. An element g of G is *primitive* if it is part of a minimal (cardinality) generating set and $Prim(G)$ is the set of primitive elements of G. G has the *primitive property* if for $\varepsilon_\gamma : F_n \to G$, $\varepsilon(\mathrm{Prim}(F_n)) = \mathrm{Prim}(G)$. More generally, a primitive set $\{g_1, \dots, g_k\}$ in G is a subset of a minimal generating set and $Prim_k G$ is the set of primitive sets of cardinality k. G has the *k-primitive property* if $\varepsilon_\gamma : \mathrm{Prim}_k(F_n) \to \mathrm{Prim}_k(G)$ is surjective.

Proposition. *The k-primitive property of G does not depend on the choice of generating set γ. Furthermore, G has the k-primitive property if and only if for any $\gamma, \gamma' \in \Gamma_n(G)$, there is a $\gamma'' \sim_N \gamma'$ so that γ'' agrees with γ in the first k entries.*

Proof. We use the standard fact that $\gamma_1 \sim_N \gamma_2$ if and only if there is an automorphism α of F_n so that the following diagram commutes



It follows easily that the k-primitive property of G relative to γ depends only on the Nielsen class of γ. Now let γ_1 and γ_2 be elements of $\Gamma_n(G)$, $\gamma_1 = (g_1, \dots, g_n)$ and suppose that G has the k-primitive property relative to γ_2. Choose a primitive set $\{w_1, \dots, w_n\} \in \Gamma(F_n)$ with $\varepsilon_{\gamma_2(w_i)} = g_i$ for $1 \le i \le k$, define $\alpha \in \mathrm{Aut}\,(F_n)$ by $\alpha(x_i) = w_i$ and let $\gamma_2' = (\varepsilon_{\gamma_2}(\alpha(x_i))$. Then γ_2' agrees with γ_1 in the first k entries and since $\gamma_2' \sim_N \gamma_2$, G has the k-primitive property relative to γ_2'. For any $\{h_1, \dots, h_k\} \subset \{h_1, \dots, h_n\} = \chi \in \Gamma_n(G)$, we can similarly get an automorphism β so that in the following diagram, the right hand triangle commutes on the subgroup $F[x_1, \dots, x_k]$.



But since α is the identity on $F[x_1, \dots, x_k]$, the large triangle commutes as well; thus $\{h_1, \dots, h_k\} = \varepsilon_{\gamma_1}\{x_1, \dots, x_k\}$ and G has the k-primitive property relative to γ_1. □

Remarks. 1) The preceding proposition says that in a group with the k-primitive property, any two generating sets are *partially* Nielsen equivalent. In particular, if $r(G) = n$, then the G has the n-primitive property if and only if it has just one Nielsen class of minimal generating sets.

2) In general, $\varepsilon_\gamma(\mathrm{Prim}F_n)$ may depend on γ. For example, in $C_5 = \{1, a, a^2, a^3, a^4\}$ with $\gamma_1 = \{a\}$ and $\gamma_2 = \{a^2\}$, $\varepsilon_{\gamma_1}(\mathrm{Prim}(F_1)) = \{a, a^4\}$ and $\varepsilon_{\gamma_2}(\mathrm{Prim}F_1) = \{a^2, a^3\}$.

Theorem 4. *If $r(G) = n$ and G has the $(n-1)$ primitive property, then any two minimal generating sets are swap equivalent.*

Proof. Suppose γ_1 and γ_2 are two minimal generating sets. The preceding proposition says that γ_2 is Nielsen equivalent (and so swap equivalent) to γ_2' so that γ_2' agrees with γ_1 in all but the last entry. Then γ_2' is swap equivalent to γ_1. □

Remarks. 1) This theorem can be particularly useful in studying the swap conjecture for groups of rank 2 since Cohen, Metzler and Zimmermann [**2**] give a nice method for recognizing primitive elements in F_2. However, many rank 2 groups having only one swap equivalence class of generating pairs do not have the primitive property, see Section 3.

2) It is known that there are groups with nonminimal generating sets not Nielsen equivalent to any set containing a minimal generating set (see Noskov [**13**]). It would be interesting to know if this is possible with swap equivalence in place of Nielsen equivalence.

3. Some examples. In this section we verify the swap conjecture for certain classes of groups. G will be said to be a *swap group* if any two sets of generators of the same cardinality are swap equivalent.

Example 1. *Free groups, free abelian groups and surface groups are swap groups.* These are examples of groups with a single Nielsen class of generators of a given cardinality. That surface groups are swap groups follows from Zieschang's generalization of the Nielsen method to amalgamated free products [**15**].

Example 2. *Finitely generated abelian groups are swap groups.* A finitely generated abelian group G has a canonical representation as

$$G \cong \mathbf{Z}^r \oplus \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \cdots \oplus \mathbf{Z}_{m_k} \quad m_{i+1}|m_i$$

and $r(G) = r + k$. A generating set γ can be thought of as a matrix $M(\gamma)$ whose rows (perhaps more than $r+k$ of them) are *vectors* with coordinates from $\mathbf{Z}$ or $\mathbf{Z}_{m_i}$ as appropriate. We proceed to row reduced $M(\gamma)$. Assuming $r \neq 0$, we perform row operations on $M(\gamma)$ (i.e., Nielsen transformations on γ) to get

$$M(\gamma') = \begin{bmatrix} 1 & p_2 & p_3 \cdots & p_{r+k} \\ 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}.$$

Since γ', defined as the set of rows in this matrix, generates G, $\left[\begin{smallmatrix} 0 & p_2 & p_3 & \cdots & p_{r+k} \end{smallmatrix}\right]$ is a combination of the rows of $M(\gamma')$. Clearly, the coefficient of row 1 is zero. It follows that $M(\gamma')$ further reduces to

$$M(\gamma'') = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}.$$

Inductively, we get

$$M(\gamma''') = \begin{bmatrix} I_r & 0 \\ 0 & M''' \end{bmatrix}$$

where the rows of M''' generate $\mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_k}$. We now proceed with the same notation assuming $r = 0$ so $G \cong \mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_k}$. (Note that the preceding argument proves that $\mathbf{Z}^r$ has one Nielsen class.)

Now $M(\gamma)$ reduces to

$$M(\gamma') = \begin{bmatrix} a & p_2 & \cdots & p_k \\ 0 & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}$$

where a generates $\mathbf{Z}_{m_1}$. Since the rows of $M(\gamma')$ generate G,

$$[\,0 \quad p_2 \quad \cdots \quad p_k\,] = \ell_1 g_1' + \cdots + \ell_k g_k', \qquad \gamma' = (g_1' \cdots g_k').$$

Considering the first coordinate, $0 \equiv \ell_1 a \pmod{m_1}$. But then $\ell_1 \equiv 0 \pmod{m_1}$ and so $\ell_1 \equiv 0 \pmod{m_i}$, $1 \leq i \leq k$. Thus, we can assume that $\ell = 0$ and further reduce $M(\gamma')$ to

$$M(\gamma'') = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \vdots & & & \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}.$$

At this point we can swap the first row $[\,1\ 0\ \cdots\ 0\,]$ and proceed as before, completing the proof by induction. This swap may be necessary. If $m \neq 2, 3, 4$ or 6, then $\mathbf{Z}_m$ has a generator $a \neq \pm 1$ and $\{1\}$ and $\{a\}$ are not Nielsen equivalent.

Example 3. *For $G = \langle x, y | x^p = y^q \rangle$ any two generating pairs are swap equivalent.* According to McCool and Pietrowski [**12**] every Nielsen class of generators of G includes one of the form $\{x^r, y^s\}$, and any two such pairs are obviously swap equivalent.

Example 3′. For

$$G = \langle a_1, a_2, \ldots, a_n | a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \ldots, a_{n-1}^{p_{n-1}} = a_n^{q_{n-1}} \rangle$$

with $p_i, q_i \geq 2$, $i = 1, \ldots, n-1$ and $\gcd(q_1 \cdots q_i, p_{i+1} \cdots p_{n-1}) = 1$, any two generating pairs are swap equivalent. This group arises in the study of one-relator groups with center. It is relatively easy to check that G is generated by a_1 and a_n, so $r(G) = 2$, and according to Collins [**3**] every generating pair is Nielsen equivalent to one of the form $\{a_1^r, a_n^s\}$ and clearly any such pair is swap equivalent to $\{a_1, a_n\}$.

Example 4. For Fuchsian groups

$$G = \langle q_1, \dots , q_m, a_1, b_1, \dots , a_g, b_g | q_1^{\alpha_1}, \dots , q_m^{\alpha_m}, q_1 q_2 \cdots q_m \prod_1^q [a_i, b_i] \rangle,$$

$r(G) = 2g+m-1$ and any two minimal generating sets are swap equivalent. Rosenberger [**13**] showed that any minimal generating set for G is Nielsen equivalent to one of the form $\{q_1^{\nu_1}, \dots , q_{i-1}^{\nu_{i-1}}, q_{i+1}^{\nu_{i+1}} q_m^{\nu_m}, a_1, b_1, \dots , a_g, b_g\}$ and any two such are clearly swap equivalent.

Example 5. The finite group describe by Dyer [**5**]

$$\langle A, B | B^{5^5} = B^{5^4} A^{-5^3} = A^{25}[B, A] = 1 \rangle$$
$$\cong \langle C, D | D^{5^5} = D^{2\cdot 5^4} C^{5^3} = C^{25}[D^{-2}, C] = 1 \rangle$$

has nonisomorphic relation space groups corresponding to the generating pairs $\{A, B\}$ and $\{C, D\}$. But $A = C$ and $B = D^{-2}$ so these presentations are clearly swap equivalent.

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