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MEAGER-NOWHERE DENSE GAMES (I): n-TACTICS

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ABSTRACT. In the introduction to this article we give a brief survey of a problem in the theory of Banach-Mazur games. We introduce two games, $MG(J)$ and $SMG(J)$ (where J is a free ideal on some set), which evolved from a study of an example relevant to this problem. The second player has a winning perfect information strategy in both of these games and we examine under what conditions it suffices for the second player to remember only the most recent n or fewer moves of the opponent (n some fixed positive integer) in order to insure a win. Strategies depending on only this information are called n-tactics.

The subject of this article belongs to the areas of combinatorial games and of topological games of length ω. In this rather lengthy introduction we give a short survey of the problem that motivated the work to be presented here. Readers who are interested in more details could consult Telgarsky's survey paper [**11**] and its extensive bibliography to the source literature.

The Scottish Book [**14**, Prob. 3] is probably the earliest popular record of the Banach-Mazur game. This game on a topological space (X,τ) is denoted by $BM(X,\tau)$ and is played as follows. First, player ONE picks a nonempty open subset E_1 of X, after which TWO picks a nonempty open subset N_1 of E_1. Next, ONE picks a nonempty open subset E_2 of N_1 and TWO responds with a nonempty open subset N_2 of E_2, and so on. In this manner, the players construct a sequence $(E_1, N_1, \dots, E_k, N_k, \dots)$ where for each positive integer k,

(i) E_k denotes ONE's k'th move and N_k, TWO's k'th move.

(ii) E_{k+1} is a subset of N_k which in turn is a subset of E_k, and these are all nonempty open subsets of X.

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Such a sequence is called *a play of* $BM(X, \tau)$. This play is won by TWO if the intersection of the N_k's is nonempty, and otherwise it is won by ONE.

A strategy, F, *for TWO* is a function whose domain is the set of finite, monotonically nonincreasing (with respect to set inclusion) sequences of nonempty open sets, and F has the property that if $(U_1, \dots, U_k)$ is such a finite sequence, then $F(U_1, \dots, U_k)$ is a subset of U_k and is a nonempty open subset of X. Such a strategy F for TWO is *a winning perfect information strategy* if each play $(E_1, N_1, \dots, E_k, N_k, \dots)$ of $BM(X, \tau)$ for which $N_k = F(E_1, \dots, E_k)$ for each positive integer k, is won by TWO.

The notions of a strategy and of a winning perfect information strategy for player ONE are defined analogously.

It is part of the folklore of the subject that there are spaces (X, τ) such that ONE has a winning perfect information strategy in $BM(X, \tau)$, there are spaces (X, τ) such that TWO has a winning perfect information strategy in $BM(X, \tau)$, and, with the aid of the axiom of choice, there are spaces (X, τ) such that neither player has a winning perfect information strategy in $BM(X, \tau)$, in which case we say $BM(X, \tau)$ is undetermined.

Consider a space (X, τ) for which $BM(X, \tau)$ is not undetermined. These games, being infinite, already put severe requirements on the endurance and patience of the players. To compound things, a perfect information strategy in $BM(X, \tau)$ has severe requirements on the memory of a player, and one might wonder if a player with a winning perfect information strategy doesn't perhaps have a winning strategy requiring less memory. Fix a positive integer k. A strategy of a player which requires knowledge of only at the most the k most recent moves of the opponent is called a *k-tactic* (we are extending the terminology of Choquet [**1**, p. 116, Definition 7.11] who calls a 1-tactic a tactic).

The situation for player ONE is as simple as possible due to the following theorem of Oxtoby, dating back to the 1950's [**9**].

Oxtoby's theorem. *The following statements are equivalent for topological space* (X, τ):

(a) *ONE has a winning perfect information strategy in* $BM(X, \tau)$.

(b) (X, τ) *is not a Baire space.*

It then follows from (b) that ONE in fact has a winning 1-tactic in $BM(X, \tau)$.

In the 1970's Fleissner and Kunen [**3**, p. 238, Question 3] asked if it is also true that if player TWO has a winning perfect information strategy in $BM(X, \tau)$, then TWO has a winning 1-tactic in $BM(X, \tau)$. In the early 1980's Debs [**2**] answered this question in the negative. It turns out (in one instance under an additional set theoretic hypothesis which is known to be independent of ZFC) that in each of Debs' examples, TWO has a winning 2-tactic. As far as we know, this is the present state of knowledge concerning *Telgarsky's conjecture* [**11**, p. 236]. This conjecture states that for every positive integer k there is a topological space (X_k, τ_k) such that TWO does not have a winning k-tactic in $BM(X_k, \tau_k)$, but does have a winning $(k+1)$-tactic.

Other kinds of winning strategies for TWO which require less than perfect information have also been studied by Debs [**2**] and by Galvin and Telgarsky [**4**].

As we mentioned, in one of Debs' examples, TWO has a winning 2-tactic if an additional set theoretic hypothesis is assumed. Thus, this example is a candidate for giving (via consistency results, perhaps) more insight into Telgarsky's conjecture. We now briefly describe the example, an analysis of which led to the work to be presented here.

The underlying set for our topological space is $\mathbf{R}$, the real line. σ will denote the usual topology on $\mathbf{R}$. Whenever we talk about an "open subset of $\mathbf{R}$" or a "meager subset of $\mathbf{R}$" or a "nowhere dense subset of $\mathbf{R}$" without further qualifying "open," "meager" or "nowhere dense," these properties are to be understood in the sense of σ. Define a new topology, τ, on $\mathbf{R}$ by putting a set V in τ, if it is of the form $V = U \setminus M$ where U is an open subset of $\mathbf{R}$ and M is a meager subset of $\mathbf{R}$. Player TWO has a winning perfect information strategy in $BM(\mathbf{R}, \tau)$, but does not have a winning 1-tactic in $BM(\mathbf{R}, \tau)$. Under an additional assumption about the collection of meager subsets of $\mathbf{R}$, it follows that TWO has a winning 2-tactic in $BM(\mathbf{R}, \tau)$.

When reconstructing the proofs of these facts, it seemed to us that $BM(\mathbf{R}, \tau)$ is in some sense a combination of two games being played simultaneously, namely, $BM(\mathbf{R}, \sigma)$ and "some game in which the two players do various things with meager and nowhere dense subsets of

R," and that it was the part in quotation marks and the way these games combine to give $BM(\mathbf{R}, \tau)$ that must be understood better.

The game $MG(J)$ (defined and discussed in part 1 of the paper) was our first attempt at making the part in quotation marks precise. The game $SMG(J)$ (defined and discussed in part 2 of the paper) turned out, as explained in part 2, to be a more successful attempt and recaptured the abovementioned facts about $BM(\mathbf{R}, \tau)$.

Aside from the potential application to Banach-Mazur games, we consider the problem of existence of winning k-tactics for player TWO in the games $MG(J)$ and $SMG(J)$ interesting in their own right. Other strategies of TWO (requiring less than perfect information) have been studied, but since the techniques involved are significantly different from those for k-tactics, these results will be presented elsewhere.

Our notation and terminology is standard. The reader is assumed to be familiar with basic facts about sets, cardinal numbers, partial orders, topology and the partition calculus. Beyond possibly consulting [**6, 12, 13**] on these matters, the reader should also consult [**10**] for the definitions and proofs of various partition relations which will be used in this article. Except where we explicitly make additional assumptions, we work in the framework of the traditional Zermelo-Fraenkel set theory, including the axiom of choice. We denote this theory by ZFC.

The article is divided into two parts which can be read independently without much loss of crucial information. We recommend though that part 1 be read first. It introduces the game $MG(J)$ and various examples that are used throughout to illustrate various aspects of the theory. In the introduction to part 1, we explain how this part is organized and give some samples of results we obtain.

Part 2 introduces the game $SMG(J)$ and contains the application to $BM(\mathbf{R}, \tau)$. The introduction to part 2 explains how this part is organized and gives some samples of results obtained there.

Thirteen open problems are mentioned at appropriate places throughout the text.

The article is a modified version of Chapter 2 of the author's dissertation, written at the University of Kansas under the supervision of Professor Fred Galvin. We thank Professor Galvin for introducing the

theory of Banach-Mazur games to us and for his enthusiastic guidance during this project.

Part 1—The game $MG(J)$. Let (S, τ) be a T_1-space without isolated points. J denotes the collection of nowhere dense subsets of S and $\langle J \rangle$ denotes the collection of meager (also known as first category) subsets of S. $MG(J)$, which we call "the monotonic meager-nowhere dense game on J," is played as follows.

First, player ONE picks a meager set M_1, then player TWO picks a nowhere dense set N_1. Then, in the second inning, ONE picks a meager set M_2 with $M_1 \subset M_2$ (unless explicitly indicated otherwise, "$\subset$" means "is a proper subset of") and TWO responds with a nowhere dense set N_2, and so on. The players construct a sequence $(M_1, N_1, M_2, N_2, \dots, M_k, N_k, \dots)$ where for each positive integer k

(i) M_k denotes ONE's meager set picked during the kth inning,

(ii) N_k denotes TWO's nowhere dense set picked during the kth inning, and

(iii) $M_k \subset M_{k+1}$.

Such a sequence is a play of $MG(J)$ and TWO wins this play if $\cup_{k=1}^{\infty} M_k$ is contained in $\cup_{k=1}^{\infty} N_k$. The notions of a winning perfect information strategy and of a winning k-tactic for TWO are defined as before. We will often use the easily verified fact that player TWO has a winning perfect information strategy in $MG(J)$.

We used topological terminology to describe $MG(J)$ only because it was convenient. The mathematical structure which is really relevant here is the notion of a free ideal and its σ-completion. Let S be a set. Recall that a family J of subsets of S is a free ideal on S if: S is not in J, every finite subset of S is in J, if A is in J and B is a subset of A, then B is in J, and if A and B are both in J, then $A \cup B$ is in J. We let $\langle J \rangle$ denote the smallest family of subsets of S which contains J and which is closed under countable unions. We call $\langle J \rangle$ *the σ-completion of J*. Note that if J is a free ideal on S and if S is not in $\langle J \rangle$, then $\langle J \rangle$ is also a free ideal on S. Also note that the following statements are equivalent for a family J of subsets of S:

(a) J is a free ideal on S,

(b) There is a T_1-topology τ on S such that

(i) (S,τ) has no isolated points and

(ii) J is the collection of nowhere dense subsets of S.

$(J,\subset)$ and $(\langle J\rangle,\subset)$ are partially ordered sets and hence we will freely use concepts defined for partially ordered sets in this context.

For the remainder of this article, we will talk about $MG(J)$ in the language of free ideals and their σ-completions. Here is a sample of results.

Theorem 1. *The following statements are equivalent for a free ideal J on a set S:*

(a) $J=\langle J\rangle$

(b) *TWO has a winning 1-tactic in $MG(J)$.*

Let J be a free ideal on S and let X be a subset of S which is not in J. Then J_X denotes the collection of sets in J which are subsets of X, and we call J_X the *relativization of J to X.*

Theorem 2. *Let $k>1$ be an integer and let J be a free ideal on a set S such that $J\neq\langle J\rangle$ and the cofinality of $(\langle J\rangle,\subset)$ is $\aleph_1$. The following statements are equivalent:*

(a) *TWO has a winning k-tactic in $MG(J)$*

(b) *For each X in $\langle J\rangle$ but not in J, TWO has a winning k-tactic in $MG(J_X)$.*

For distinct elements f and g of ${}^{\omega}\omega$, we put $f\ll g$ if there is an n in ω such that $f(m)<g(m)$ whenever m is an integer larger than n. $({}^{\omega}\omega,\ll)$ is a partially ordered set. We denote the cardinality of the continuum by $\mathbf{c}$. We need the following hypothesis in the statement of our next result. We call it "the embedding hypothesis" and denote it by EH.

EH: for every partially ordered set $(P,<)$ of cardinality at the most $\mathbf{c}$, there is an order preserving function from $(P,<)$ into $({}^{\omega}\omega,\ll)$.

Theorem 8. *Let S be a set of cardinality at the most* $\mathbf{c}$.

(a) *Assume that $2^{<\mathbf{c}}=\mathbf{c}$ and that EH holds. If J is a free ideal on*

S and the cofinality of $(\langle J\rangle, \subset)$ is at most $\aleph_1$, then player TWO has a winning 3-tactic in $MG(J)$.

(b) *If there is a real-valued measurable cardinal κ less than or equal to the cardinality of S, then there is for each cardinal λ below κ a free ideal J on S such that the cofinality of $(\langle J\rangle, \subset)$ is at least λ, and TWO does not have a winning k-tactic in $MG(J)$ for any positive integer k.*

Part 1 is organized as follows. Theorems 1 and 2 are proven in the first two sections. In section 3 we consider free ideals J with S in $\langle J\rangle$ and find necessary (but not sufficient) and sufficient (but not necessary) conditions for the existence of a winning k-tactic for TWO. Theorem 8 is proven and discussed in section 4. In section 5 we make some remarks about free ideals for which there would be winning $k+1$-tactics but not winning k-tactics. Section 6 contains some closing remarks about Theorems 2 and 8.

1. 1-Tactics.

The situation for 1-tactics in $MG(J)$ is particularly nice.

Theorem 1. *The following statements are equivalent for a free ideal J on a set S.*

(a) *Player TWO has a winning 1-tactic in $MG(J)$.*

(b) *$J = \langle J\rangle$.*

Proof. Statement (b) clearly implies statement (a). Assume that (b) is false, and pick a set X in $\langle J\rangle$ which is not in J. Write $X = A\cup B$ where A and B are pairwise disjoint sets of the same cardinality. Without loss of generality, B is not in J. Suppose now that F is a 1-tactic of TWO.

Define $\Phi : \wp(A)\to B$ so that $\Phi(Y)$ is in $B\setminus F(B\cup Y)$ for each subset Y of A. By Corollary 8 of [**12**], $(\wp(A), \subset)\to (|A|)^1_B$ (which means that for each function which assigns points of B to subsets of A, there will be a collection of subsets of A, linearly ordered by $\subset$ in order type $|A|$, such that the same point of B got assigned to each set in this well-ordered chain). Pick an increasing ω-chain $X_1\subset X_2\subset \dots\subset X_n\subset\dots\subset A$ and a point b in B so that for each positive integer n, $\Phi(X_n) = b$. For each positive integer n, put $M_n = B \cup X_n$ and $N_n = F(B_n)$.

The sequence $(M_1, N_1, \dots, M_n, N_n, \dots)$ then is a play of $MG(J)$ during which TWO used the 1-tactic F and yet it is lost by TWO since b is in each M_n but in no N_n. Since F was arbitrary, TWO does not have a winning 1-tactic in $MG(J)$. □

2. *Proof of Theorem* 2. Let k be a positive integer. If TWO has a winning k-tactic in $MG(J)$, then TWO necessarily has a winning k-tactic in $MG(J_X)$ for each subset X of S which is not in J. This is, in particular, true if X is in $\langle J\rangle$ but not in J. Theorem 2 states that under certain additional assumptions on $\langle J\rangle$, this necessary condition is also sufficient.

Theorem 2. *Let $k > 1$ be an integer and let J be a free ideal on S for which $J \neq \langle J\rangle$ and the cofinality of $\langle J\rangle$ is $\aleph_1$. The following statements are equivalent*:

(a) *TWO has a winning k-tactic in $MG(J)$.*

(b) *For each X in $\langle J\rangle$ but not in J, TWO has a winning k-tactic in $MG(J_X)$.*

Proof. We have to show that (b) implies (a). So, assume that (b) is true. By the cofinality assumption on $\langle J\rangle$, fix a family $\{S_\alpha : \alpha < \omega_1\}$ such that if $\alpha < \beta < \omega_1$, then $S_\alpha \subset S_\beta$ are both in $\langle J\rangle$ and not in J, and for each B in $\langle J\rangle$ there is an α with B a subset of S_α. Fix, furthermore, for each $\beta < \omega_1$ a winning k-tactic F_β of TWO in $MG(J_{S_\beta})$ and let G be a winning perfect information strategy of TWO in $MG(J)$.

Let $(<_n : n \in \omega)$ be a sequence of binary relations on ω_1 such that

(a) $T_n = (\omega_1, <_n)$ is a tree of height at most $n+2$ for each n in ω,

(b) for $m < n$ in ω, $<_m$ is contained in $<_n$, and

(c) for $\alpha < \beta$ in ω_1, there is an n in ω with $\alpha <_n \beta$.

(The reader could consult [**5**, pp. 85,86] for a proof that these exist.) For $\alpha < \beta < \omega_1$, we let $\phi(\alpha, \beta)$ be the smallest n in ω so that $\alpha <_n \beta$. For B in $\langle J\rangle$ we let $\alpha(B)$ be the smallest β below ω_1 such that B is a subset of S_β. We are now ready to define a k-tactic, F, for TWO. So let $(X_1, \dots, X_k)$ be given with each X_i in $\langle J\rangle$ and let $\alpha_i = \alpha(X_i)$ for

$1 \leq i \leq k$.

Case 1. $\varnothing \neq X_1 \subset \cdots \subset X_k$.

(i) $\alpha_{k-1} < \alpha_k$. Put $T = \{\beta < \omega_1 : \phi(\beta, \alpha_k) \leq \phi(\alpha_{k-1}, \alpha_k)\}$ and define $F(X_1, \ldots, X_k)$ to be the union of the finite (by (a) and (b)) collection $\{G(S_{\delta_1}, \ldots, S_{\delta_r}) : \delta_1 < \cdots < \delta_r$ and $\{\delta_1, \ldots, \delta_r\}$ is a subset of $T\}$.

(ii) $\alpha_{k-1} = \alpha_k$. Then put $F(X_1, \ldots, X_k) = F_{\alpha_k}(X_1, \ldots, X_k)$.

Case 2. In all other cases put $F(X_1, \ldots, X_k) = \varnothing$.

This defines a k-tactic F for TWO and we now show that F is winning for TWO. So let $(M_1, N_1, \ldots, M_n, N_n, \ldots)$ be a play of $MG(J)$ during which TWO used the k-tactic F. For each positive integer n, let $\alpha_n = \alpha(M_n)$. Since $M_1 \subset \ldots \subset M_n \subset \ldots, \alpha_n \leq \alpha_{n+1}$ for each positive integer n.

Possibility 1. $\{\alpha_n : n$ a positive integer$\}$ is finite.

Then choose the smallest $n \geq k$ such that $\alpha_m = \alpha_n$ for all integers m bigger than n. We evidently have that

$$N_n \text{ contains } F_{\alpha_n}(\varnothing, \ldots, \varnothing, M_n)$$
$$\vdots$$
$$\vdots$$
$$N_{n+j+k} \text{ contains } F_{\alpha_n}(M_{n+j+1}, \ldots, M_{n+j+k}) \text{ for all } j \text{ in } \omega.$$

But $(M_n, F_{\alpha_n}(\varnothing, \ldots, \varnothing, M_n), \ldots, M_{n+j+k}, F_{\alpha_n}(M_{n+j+1}, \ldots, M_{n+j+k}), \ldots)$ is a play of $MG(J_{S_{\alpha_n}})$ during which TWO used the winning k-tactic F_{α_n} and hence is won by TWO. Thus, the play of $MG(J)$ which is under consideration is won by TWO.

Possibility 2. $\{\alpha_n : n$ a positive integer$\}$ is infinite.

Let $\alpha_{i_1} < \alpha_{i_2} < \alpha_{i_3} < \ldots$ be a strictly increasing infinite subsequence. For $j \geq 2$, let $m_j = \phi(\alpha_{j-1}, \alpha_j)$; thus, $m_j = 0$ when $\alpha_{j-1} = \alpha_j$. The sequence $m_2, m_3, \ldots$ must be unbounded since there are no infinite chains in $(\omega_1, <_n)$. Let r be a positive integer. Choose n so that

$\alpha_1 \leq_n \alpha_2 \leq_n \cdots \leq_n \alpha_{i_r}$. Choose the least j such that $m_j > n$ and put $m = m_j$.

Then $\alpha_1 \leq_m \alpha_2 \leq_m \cdots \leq_m \alpha_{j-1<m}\alpha_j$ and $i_r < j$, so we have

$$\alpha_{i_1} <_m \alpha_{i_2} <_m \alpha_{i_3} <_m \cdots <_m \alpha_{i_r} <_m \alpha_j.$$

Because of the way F was defined in Case 1, it follows that $G(S_{\alpha_{i_1}}, \dots, S_{\alpha_{i_r}})$ is contained in N_j. We have shown that for each positive integer r, $G(S_{\alpha_{i_1}}, \dots, S_{\alpha_{i_r}})$ is a subset of $\cup_{n=1}^{\infty} N_n$. Since G is a winning perfect information strategy for TWO in $MG(J)$ it follows that TWO has won the play of $MG(J)$ under consideration.

This covers all possible plays of $MG(J)$ in which TWO has used the k-tactic F. Thus, F is a winning k-tactic for TWO in $MG(J)$. □

This theorem suggests that we study k-tactics for those free ideals J on S for which S is in $\langle J\rangle$, i.e., the cofinality of $(\langle J\rangle, \subset)$ is 1. We also should study k-tactics for those free ideals J on S for which the cofinality of $(\langle J\rangle, \subset)$ is at least $\aleph_2$.

3. Free ideals J with cofinality of $(\langle J\rangle, \subset)$ equal to 1.

The following cardinal functions are useful in our discussions.

Definition. Let J be a free ideal on S with $J \neq \langle J\rangle$.

(a) $\mu(J)$ is the smallest cardinality of a set which is in $\langle J\rangle$ but not in J. We call $\mu(J)$ the *minimality number of J.*

(b) $d(J)$ is the smallest cardinal number, κ, for which J is a union of a family $\{J_\alpha : \alpha < \kappa\}$ where each J_α is an ideal with $\langle J_\alpha\rangle \neq \langle J\rangle$. We call $d(J)$ the *decomposition number of J.*

(c) $dir(J)$ is the smallest cardinal number, κ, for which J is a union of a family $\{J_\alpha : \alpha < \kappa\}$ which is up-directed by $\subset$ and where each J_α is an ideal with $\langle J_\alpha\rangle \neq \langle J\rangle$. We call $dir(J)$ the *directedness number of J.*

The following relationships among these cardinals are easily verified:

(1) $\aleph_0 \leq d(J) \leq \mu(J)$

(2) $d(J) \leq dir(J)$

(3) $dir(J) \leq \min\{\text{cofinality}(J_X, \subset) : X \text{ a subset of } S \text{ not in } J\} \leq 2^{\mu(J)}$.

There are examples showing that the strict inequalities $d(J) < dir(J)$, $dir(J) < cof(J, \subset) < 2^{\mu(J)}$ and $d(J) < \mu(J)$ are possible.

In this section we use a partition relation which is studied in [**10**]. This partition relation was invented to discuss k-tactics in $MG(J)$. We recall the definition for the convenience of the reader.

Definition. Let $(P, <)$ be a partially ordered set, let k be a positive integer and let κ be an infinite cardinal number. The symbol

$$(P, <) \to (\omega\text{-path})^n_{\kappa/<\omega}$$

abbreviates the statement: for every partition of $[P]^n$ into disjoint classes $\{K_\alpha\}_{\alpha<\kappa}$, there is an increasing ω-sequence $p_1 < p_2 < \cdots < p_j < \dots$ in P for which the members of the infinite set $\{\{p_{j+1}, \dots, p_{j+n}\} : j < \omega\}$ belong to finitely many K_α's.

The symbol $(P, <) \not\to (\omega\text{-path})^n_{\kappa/<\omega}$ abbreviates the negation of this statement.

Proposition 3. *Let $k > 1$ be an integer and let J be a free ideal on S with S in $\langle J\rangle$. If $(\wp(S), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$, then player TWO has a winning k-tactic in $MG(J)$. If, in addition, $k > 2$, TWO has a winning 3-tactic in $MG(J)$.*

Proof. Since S is in $\langle J\rangle$, we can write $S = \cup_{n<\omega} S_n$ where for each n in ω, $S_n \subset S_{n+1}$ and S_n is in J. We also recall from [**10**, Proposition 36] that if $(\wp(S), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$ for some integer $k > 2$, then we have that $(\wp(S), \subset) \not\to (\omega\text{-path})^3_{\omega/<\omega}$. Thus, k in this proof will be 2 or 3. In either case, we proceed as follows. Let $[\wp(S)]^k = \cup_{n<\omega} K_n$ be a partition witnessing that $(\wp(S), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$.

We define a k-tactic, F, of TWO as follows. Let $(X_1, \dots, X_k)$ be a k-tuple from $\langle J\rangle$. If $X_1 \subset \dots \subset X_k$, we let $F(X_1, \dots, X_k) = S_n$ where $n < \omega$ is minimal with $\{X_1, \dots, X_k\}$ in K_n. Otherwise, we set $F(X_1, \dots, X_k) = \varnothing$. It follows from the properties of the given partition that F is a winning k-tactic for TWO in $MG(J)$. □

In the example below, we show that the ω-path hypothesis in Proposition 3 is sufficient but not necessary.

Example 1. The ideals $J(\kappa)$. Let κ be an infinite cardinal number, and let S be the set of finite subsets of κ (i.e., $S = [\kappa]^{<\aleph_0}$). For each α in κ we let Y_α be the collection of those Z in S with α not in Z. The family $\{Y_\alpha : \alpha \text{ in } \kappa\}$ has the following properties:

(i) $S \backslash Y_\alpha$ has cardinality κ for each α in κ,

(ii) $\cup\{Y_\alpha : \alpha \text{ in } F\} \neq S$ for each finite subset F of κ and

(iii) $\cup\{Y_\alpha : \alpha \text{ in } F\} = S$ for each infinite subset F of κ.

Let $J(\kappa)$ be the smallest hereditary family of subsets of S which includes $\{Y_\alpha : \alpha \text{ in } \kappa\}$ and which is also closed under finite unions. Then $J(\kappa)$ is a free ideal on S and

Claim. *Player TWO has a winning* 2*-tactic in* $MG(J(\kappa))$*.*

Proof. For A and B in $\langle J(\kappa)\rangle$ with A a proper subset of B, pick $z_{(A,B)}$ from $B \backslash A$. Define $F(A,B)$ as the union of $\{Y_\alpha : \alpha \text{ is in } z_{(A,B)}\}$. Since this union is finite, $F(A,B)$ is in $J(\kappa)$ for each such A and B. It follows from property (iii) that F is a winning 2-tactic for TWO in $MG(J(\kappa))$. $\square$

However, for $\kappa > \mathbf{c}$ $(\wp(S), \subset) \to (\omega\text{-path})^k_{\omega/<\omega}$ for every positive integer k [**10**, Proposition 1 and Corollary 10].

Note that Proposition 3 implies that if J is a free ideal on ω, then player TWO has a winning 2-tactic in $MG(J)$. Using this fact in conjunction with Theorem 2, we get that

Corollary 4. *If J is a free ideal on ω_1, each element of which is at most countable, then player TWO has a winning* 2*-tactic in* $MG(J)$*.*

If the free ideal J of Proposition 3 has additional structure, the condition in that proposition is also necessary.

Proposition 5. *Let $k > 1$ be an integer. For a free ideal J on S for which $\aleph_0 = dir(J)$ and $\langle J \rangle = \wp(S)$, the following statements are equivalent*:

(a) $(\wp(S), \subset) \not\to (\omega\text{-}path)^k_{\omega/<\omega}$,
(b) *TWO has a winning k-tactic in $MG(J)$.*

If, in addition, $k > 2$, the following statement is also equivalent to (b):

(c) *TWO has a winning* 3*-tactic in $MG(J)$.*

Proof. (a) $\Rightarrow$ (b) follows from Proposition 3.

(b) $\Rightarrow$ (a). Write $J = \cup_{n=1}^{\infty} J_n$ where each J_n is a free ideal, $\langle J_n \rangle \neq \wp(S)$ and J_n is contained in J_{n+1}. Write $S = A \cup B$ where A and B are pairwise disjoint sets of equal cardinality. We may assume without loss of generality that A is not in $\langle J_n \rangle$ for each positive integer n. Let F be a winning k-tactic for TWO in $MG(J)$. We define a partition $[\wp(B)]^k = \cup_{n=1}^{\infty} K_n$ as follows. Let $\{X_1, \dots , X_k\}$ be a k-tuple of subsets of B. If $X_1 \subset \dots \subset X_k$, we put $\{X_1, \dots , X_k\}$ in K_m where m is minimal with $F(A \cup X_1, \dots , A \cup X_k)$ in J_m. Otherwise, we put $\{X_1, \dots , X_k\}$ in K_1.

Since F is a winning k-tactic for TWO in $MG(J)$, it follows that this partition witnesses that $(\wp(B), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$ and since S and B have the same cardinality, that $(\wp(S), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$. The proof of (b) $\Rightarrow$ (a) is complete.

The equivalence of (c) with the other statements when $k > 2$ follows from the fact that then, if $(\wp(S), \subset) \not\to (\omega\text{-path})^k_{\omega/<\omega}$ we also have that $(\wp(S), \subset) \not\to (\omega\text{-path})^3_{\omega/<\omega}$ [**10**, Proposition 36]. □

In general, we have the following necessary condition for the existence of a winning k-tactic in $MG(J)$.

Proposition 6. *Let $k > 1$ be an integer and let J be a free ideal on S with S in $\langle J \rangle$. If player TWO has a winning k-tactic in $MG(J)$, then $(\wp(S), \subset) \not\to (\omega\text{-}path)^3_{dir(J)/<\omega}$.*

Proof. Assume that $(\wp(S),\subset)\not\to(\omega\text{-path})^3_{dir(J)/<\omega}$. By [**10**, Propositions 1 and 36] it then follows that $(\wp(S),\subset)\to(\omega\text{-path})^k_{dir(J)/<\omega}$. Let F be a k-tactic of TWO in $MG(J)$ and let $J=\cup_{\alpha<dir(J)}J_\alpha$ be a decomposition of J as in the definition of $dir(J)$. By the up-directedness of the family $\{J_\alpha:\alpha<dir(J)\}$ we write $S=A\cup B$ where A and B are pairwise disjoint sets of equal cardinality and A is not in $\langle J_\alpha\rangle$ for each α below $dir(J)$. We define a partition $[\wp(B)]^k=\cup_{\alpha<dir(J)}K_\alpha$ as follows.

Let $\{X_1,\dots,X_k\}$ be a k-tuple of subsets of B. If $X_1\subset\cdots\subset X_k$, we put $\{X_1,\cdots,X_k\}$ in K_α where α is minimal with $F(A\cup X_1,\dots,A\cup X_k)$ in J_α. Otherwise, we put $\{X_1,\dots,X_k\}$ in K_0.

$(\wp(B),\subset)\to(\omega\text{-path})^k_{dir(J)/<\omega}$ since S and B have the same cardinality. Thus, pick a finite subset H of $dir(J)$ and an increasing sequence $X_1\subset\dots\subset X_k\subset\dots\subset X_m\subset\dots\subset B$ such that each element of the infinite set $\{\{X_1,\dots,X_k\},\{X_2,\dots,X_{k+1}\},\dots,\{X_{j+1},\dots,X_{j+k}\},\dots\}$ is in K_α for some α in H. Put $M_n=B\cup X_n$ for each positive integer n. By our partition we get that $F(M_{j+1},\dots,M_{j+k})$ is in J_α for some α in H. Since the family $\{J_\alpha:\alpha<dir(J)\}$ is up directed, we find a β below $dir(J)$ such that each response of player TWO (using k-tactic F) in the play of the game where ONE consecutively moves $M_1,M_2,M_3,\dots$, is in J_β. Since the union of TWO's moves is in $\langle J_\beta\rangle$, A, a set covered by ONE, is not covered by TWO. Thus, TWO loses this play of $MG(J)$, and F could not have been a winning k-tactic. The proposition follows by contraposition. □

In Example 2 we will illustrate that this necessary condition is not sufficient.

Corollary 7. *Let $k>1$ be an integer and let J be a free ideal on S with S in $\langle J\rangle$. If player TWO has a winning k-tactic in $MG(J)$, then the cardinality of S is at the most $2^{dir(J)}$ and the cardinality of S is less than $2^{dir(J)}$ if $k=2$.*

Proof. It follows from [**10**, Proposition 1 and Corollary 10] that if the cardinality of S is bigger than $2^{dir(J)}$, then $(\wp(S),\subset)\to(\omega\text{-path})^k_{dir(J)/<\omega}$ for every positive integer k. Now apply Proposition

6. When $k = 2$ [**10**, Corollary 28] and Proposition 6 similarly imply that the cardinality of S is below $2^{dir(J)}$. $\square$

4. The proof of Theorem 8. It is a well-known fact that both the hypotheses $2^{<\mathbf{c}} = \mathbf{c}$ and EH are consequences of CH, the continuum hypothesis. Laver [**7**] showed that the theory "$ZFC + \neg CH + EH + 2^{<\mathbf{c}} = \mathbf{c}$" is consistent relative to the consistency of ZFC. Thus, the assumption that both EH and $2^{<\mathbf{c}} = \mathbf{c}$ are true is weaker than the assumption that CH is true.

Theorem 8. *Let S be a set of size at the most $\mathbf{c}$.*

(a) *Assume that $2^{<\mathbf{c}} = \mathbf{c}$ and that* EH *holds. If J is a free ideal on S and the cofinality of $(\langle J\rangle, \subset)$ is at most $\aleph_1$, then player TWO has a winning 3-tactic in $MG(J)$.*

(b) *If there is a real-valued measurable cardinal κ less than or equal to the cardinality of S, then there is for each infinite cardinal λ below κ a free ideal J on S such that the cofinality of $(\langle J\rangle, \subset)$ is at least λ and TWO does not have a winning k-tactic in $MG(J)$ for any positive integer k.*

Proof. (a) As noted in [**10**, discussion after Corollary 31], it follows from the assumption that $2^{<\mathbf{c}} = \mathbf{c}$ and that EH holds that the partition relation $(\wp(\mathbf{c}), \subset) \not\to (\omega\text{-path})^3_{\omega/<\omega}$ holds. Thus, for each X in $\langle J\rangle$ player TWO has a winning 3-tactic in $MG(J_X)$ (by Proposition 3). Since, furthermore, the cofinality of $(\langle J\rangle, \subset)$ is at most $\aleph_1$, it follows from Theorem 2 that TWO has a winning 3-tactic in $MG(J)$. This completes the proof of (a).

(b) Let κ be a real-valued measurable cardinal less than or equal to the continuum. Let S be a set of cardinality κ and write $S = A \cup B$ where A and B are pairwise disjoint and the cardinality of A is λ^+. Let J be the collection of subsets of S which have finite intersection with A. Then J Is a free ideal on S and $\langle J\rangle$ is the collection of subsets of S which intersect A in a countable set. The cofinality of $(\langle J\rangle, \subset)$ is at least λ^+. We show that TWO does not have a winning k-tactic in $MG(J)$ for every positive integer k.

Let C be a countably infinite subset of A and enumerate C bijectively as $\{c_1, c_2, \dots, c_n, \dots\}$. For each positive integer n, we let J_n be those sets in J whose intersection with C is contained in $\{c_1, c_2, \dots, c_n\}$. The collection $\{J_n : n \text{ a positive integer}\}$ witnesses that $dir(J_{B\cup C}) = \aleph_0$. By [**10**, Proposition 19], $(\wp(B \cup C), \subset) \to (\omega\text{-path})^k_{\omega/<\omega}$ for every positive integer k. By Proposition 5, player TWO does not have a winning k-tactic in $MG(J_{B\cup C})$ for any positive integer k. Thus TWO does not have a winning k-tactic in $MG(J)$ for any positive integer k. This completes the proof of the theorem. □

We now give some criteria for the nonexistence of winning k-tactics of TWO in $MG(J)$ and use these to discuss the optimality of the conclusion in Theorem 8(a) and to discuss some points about Proposition 6.

Proposition 9. *Let J be a free ideal on S. If there is a set of cardinality $2^{\mu(J)}$ in $\langle J\rangle$, then player TWO does not have a winning 2-tactic in $MG(J)$.*

Proof. Pick a set A in $\langle J\rangle$ of size $\mu(J)$ and which is not in J. Let V be a set in $\langle J\rangle$ of size $2^{\mu(J)}$, and which is disjoint from A. It follows from [**10**, Corollary 28] that $(\wp(V), \subset) \to (\omega)^2_{\mu(J)}$.

Let F be a 2-tactic of TWO, and define a partition of $[\wp(V)]^2 = \cup\{K_x : x \text{ in } A\}$ as follows. Let $\{B, C\}$ be in $[\wp(V)]^2$, and let a_0 be some element of A which we fix in advance. If $\{B, C\}$ is linearly ordered by set inclusion we may assume that B is a proper subset of C and we pick an x in $A\backslash(F(\varnothing, A\cup B) \cup F(A\cup B, A\cup C))$ and put $\{B, C\}$ in K_x. Otherwise, we put $\{B, C\}$ in K_{a_0}.

By $(\wp(V), \subset) \to (\omega)^2_{\mu(J)}$ we get a point x in A and a sequence $C_1 \subset C_2 \subset \dots \subset C_m \subset C_{m+1} \subset \dots \subset V$ such that $\{C_m, C_n\}$ is in K_x for m and n distinct integers. Put $M_n = A \cup C_n$ for each positive integer n. Put $N_1 = F(\varnothing, M_1)$ and $N_{k+1} = F(M_k, M_{k+1})$ for each positive integer k. Then $(M_1, N_1, \dots, M_n, N_n, \dots)$ is a play of $MG(J)$ in which TWO used the 2-tactic F and x is in no N_k but is in each M_k. Thus, this play is lost by TWO. This completes the proof. □

Example 2. $J = \{N \subset \mathbf{R} : N \text{ is nowhere dense}\}$. For this example, we have the following corollary.

Corollary 10.

(a) *TWO does not have a winning* 2*-tactic in* $MG(J)$.

(b) *CH implies that TWO has a winning* 3*-tactic in* $MG(J)$.

Proof. (a) Note that $\mu(J) = \aleph_0$ and that there are meager sets of size continuum, and apply Proposition 9.

(b) If CH holds, the hypotheses of Theorem 8(a) are all satisfied and the result follows. □

Letting A be a countable everywhere dense subset of $\mathbf{R}$ and B a meager subset of $\mathbf{R}$ of size continuum and letting $J' = J_{A\cup B}$ it follows similarly from Proposition 9 that TWO does not have a winning 2-tactic in $MG(J')$. It can be shown that $dir(J')$ is uncountable. Thus, if CH is true, we get that $(\wp(\mathbf{c}), \subset) \not\to (\omega\text{-path})^2_{dir(J')/<\omega}$. This shows that the necessary condition of Proposition 6 is not sufficient.

Let J be a free ideal on a set J. The following Proposition shows that the cardinality bound in Proposition 9 is the break point for the existence of winning k-tactics of TWO in $MG(J)$.

Proposition 11. *Let J be a free ideal on S. If there is a subset of S of cardinality bigger than $2^{\mu(J)}$ with every subset of size at most $2^{\mu(J)}$ in $\langle J\rangle$, then TWO does not have a winning k-tactic in $MG(J)$ for any positive integer k.*

Proof. Pick disjoint subsets A and V of S with A in $\langle J\rangle \backslash J$ and the cardinality of A is $\mu(J)$, the cardinality of V is bigger than $2^{\mu J)}$ and every subset of V of size at most $2^{\mu(J)}$ in $\langle J\rangle$. Also pick a family $\{C_\alpha : \alpha < (2^{\mu(J)})^+\}$ with $A \subset C_\alpha \subset C_\beta \subset (A \cup V)$ for $\alpha < \beta < (2^{\mu(J)})^+$ and C_α in $\langle J\rangle$ for $\alpha < (2^{\mu(J)})^+$.

Let k be a positive integer and let F be a given k-tactic for TWO. We may assume without loss of generality that $F(X_1, \dots, X_k) = \varnothing$ whenever $X_1, \dots, X_k$ are in $\langle J\rangle$ and $X_1 = \varnothing$.

Write $(2^{\mu(J)})^+ = \cup\{T_\alpha : \alpha < (2^{\mu(J)})^+\}$ with $\{T_\alpha : \alpha < (2^{\mu(J)})^+\}$ a disjoint collection of intervals, each of length k. Enumerate $T_\alpha = \{\alpha_1, \dots, \alpha_k\}$ in increasing order for each such α.

We define a partition $[(2^{\mu(J)})^+]^2 = \cup_{x\in A} K_x$ as follows. Let $\alpha < \beta < (2^{\mu(J)})^+$ be given. Pick x in

$$A \backslash (F(C_{\alpha_1}, \dots, C_{\alpha_k}) \cup F(C_{\alpha_2}, \dots, C_{\alpha_k}, C_{\beta_1}) \cup \dots \cup F(C_{\beta_1}, \dots, C_{\beta_k}))$$

and put $\{\alpha, \beta\}$ in K_x.

By the Erdös-Rado theorem, we pick an x in A and an increasing ω-sequence

$$\alpha_1 < \alpha_2 < \dots < \alpha_m < \dots < (2^{\mu(J)})^+$$

which is monochromatic of class x for this partition and enumerate $\cup_{n=1}^{\infty} T_{\alpha_n} = \{\gamma_1, \gamma_2, \dots, \gamma_m, \dots\}$ in increasing order.

Put $M_m = C_{\gamma_m}$ for each positive integer m, and consider the play $(M_1, N_1, \dots, M_m, N_m, \dots)$ of $MG(J)$ where TWO now has used F. By the properties of our partition x is in each M_m but in no N_m, so TWO has lost this play of $MG(J)$. Since k was arbitrary and F was an arbitrary k-tactic, the proof is complete. □

5. Winning $(k+1)$-tactics but no winning k-tactics? In the preceding paragraphs, we saw examples of (1) a free ideal J for which TWO has a winning 1-tactic in $MG(J)$, (2) a free ideal J for which TWO has a winning 2-tactic but not a winning 1-tactic in $MG(J)$, and (3) a free ideal J for which TWO does not have a winning 2-tactic in $MG(J)$, but if CH holds, then TWO has a winning 3-tactic in $MG(J)$.

At present, this is the only information we have about

Problem 1. Is there for every positive integer k a free ideal J_k on a set S_k such that

(1) TWO has a winning $(k+1)$-tactic in $MG(J_k)$, but

(2) TWO does not have a winning k-tactic in $MG(J_k)$?

We now discuss some specific examples in this context and we prove a theorem that rules out some ways of trying to construct examples that will solve the problem positively. To this end, we introduce a cardinal function for partially ordered sets.

Definition. Let $(P,<)$ be an up-directed partially ordered set with no largest element. Then c.n.$(P,<)$ is the minimal cardinality that a subset Q of P with the property that there is no p in P so that $q \leq p$ for each q in Q, can have. We call c.n.$(P,<)$ the *completeness number* of $(P,<)$.

The equivalence of the following statements for an up-directed partially ordered set which has no largest element is easily verified:

(a) $(P,<)$ has a cofinal chain,

(b) the cofinality of $(P,<)$ is equal to c.n.$(P,<)$,

(c) $(P,<)$ has a cofinal chain of order type equal to the cofinality of $(P,<)$.

Proposition 12. *Let k be a positive integer and let J be a free ideal on S with $J \neq \langle J\rangle$ and with $1 <$ cofinality$(\langle J\rangle,\subset) =$ c.n.$(\langle J\rangle,\subset)$. Assume that TWO has a winning k-tactic in $MG(J_X)$ for each X in $\langle J\rangle\backslash J$ Then the following statements are equivalent*:

(a) *TWO has a winning k-tactic in $MG(J)$.*

(b) *For some positive integer n, TWO has a winning n-tactic in $MG(J)$.*

Proof. Let J be as in the hypothesis. Only (b) $\Rightarrow$ (a) needs proof. This is proven by induction on n. If $n \leq k$, there is nothing to prove. So assume that $n > k$, say $n = k+j+1$ for some integer j, and that (b) $\Rightarrow$ (a) is true for $k+j$. Assume furthermore that TWO has a winning $k+j+1$-tactic in $MG(J)$. We will show that TWO has a winning $k+j$-tactic in $MG(J)$, which will complete the proof.

Let $\kappa =$ c.n.$(\langle J\rangle,\subset)$, and let $\{C_\alpha : \alpha < \kappa\}$ be a $\subset$-cofinal chain of order type κ in $\langle J\rangle\backslash J$. Write $\kappa = \cup_{\alpha<\kappa} T_\alpha$ where $\{T_\alpha : \alpha < \kappa\}$ is a sequence of intervals, each of length $k+j$. We enumerate each $T_\alpha = \{\alpha_1,\dots,\alpha_{k+j}\}$ in increasing order. For B in $\langle J\rangle$, we let $\beta(B)$ denote the smallest α in κ with $B \subset C_{\alpha_1}$. For each α in κ pick a winning k-tactic, say F_α, for TWO in $MG(J_{C_{\alpha_1}})$. We may assume without loss of generality that $F_\alpha(X_1,\dots,X_k) = \varnothing$ whenever $X_1,\dots,X_k$ are in $\langle J_{C_{\alpha_1}}\rangle$ and $X_1 = \varnothing$. Let F be a winning $k+j+1$-tactic for TWO in $MG(J)$. We now define a $k+j$-tactic for TWO.

Let $X_1 \subset \cdots \subset X_{k+j}$ in $\langle J \rangle$ be given, and put $\beta_m = \beta(X_m)$ for $1 \leq m \leq k+j$.

Case 1. $\delta = \beta_{k+j-1} < \beta_{k+j} = \nu$. Then we let $G(X_1, \dots, X_{k+j})$ be the union of sets of the form $F(Y_1, \dots, Y_{k+j+1})$ where $\{Y_1, \dots, Y_{k+j+1}\}$ is an increasing $k+j+1$-tuple from the collection $\{C_{\gamma_i} : 1 \leq i \leq k+j$ and $\gamma = \delta$ or $\gamma = \nu\}$.

Case 2. $\delta = \beta_{k+j-1} = \beta_{k+j}$. Then we let $G(X_1, \dots, X_{k+j})$ be the union of sets of the form $F_\delta(Y_1, \dots, Y_k)$ where $\{Y_1, \dots, Y_k\}$ is a subset of $\{X_1, \dots, X_{k+j}\}$.

This defines a $k+j$-tactic G for TWO in $MG(J)$. We show that G is winning for TWO. So let $(M_1, N_1, \dots, M_t, N_t, \dots)$ be a play of $MG(J)$ during which TWO used the $k+j$-tactic G. Let β^t denote $\beta(M_t)$ for every positive integer t. Note that if $t < s$, then $\beta^t \leq \beta^s$. There are two possibilities to be considered.

Possibility 1. $\{\beta^t : t$ a positive integer$\}$ is finite.

Let β be the maximum of this set and let $m > k+j$ be minimal such that for each $t \geq m$, $\beta^t = \beta$. Then N_{r+k} contains $F_\beta(M_{r+1}, \dots, M_{r+k})$ for each integer $r \geq m$. By the choice of F_β it follows that the given play of $MG(J)$ is won by TWO.

Possibility 2. $\{\beta^t : t$ a positive integer$\}$ is infinite.

Let $\{\gamma^1, \gamma^2, \dots\}$ enumerate $\{\beta^t : t$ a positive integer$\}$ in increasing order. For each positive integer n there then is a positive integer $m \geq n$ for which N_m contains the set of finite unions of sets of the form $F(Y_1, \dots, Y_{k+j+1})$ where $\{Y_1, \dots, Y_{k+j+1}\}$ is an increasing $k+j+1$-tuple from the collection $\{C_{\delta_i} : 1 \leq i \leq k+j$ and $\delta = \gamma^n$ or $\delta = \gamma^{n+1}\}$. By the choice of F, it then follows that the given play of $MG(J)$ is won by TWO.

The proof of the Proposition is complete. □

Before we apply this information to our examples, we prove another useful fact that will rule out some ideals as examples that will solve Problem 1 for $k = 3$.

Proposition 13. *Let J be a free ideal on S with the property that for each X in $\langle J \rangle$ there is a Y in $\langle J \rangle$ which is disjoint from X and has the same cardinality as X. If TWO has a winning k-tactic in $MG(J)$ for some positive integer k, then TWO has a winning 3-tactic in $MG(J_A)$ for each A in $\langle J \rangle$.*

Proof. Let J be as in the hypothesis and let F be a winning k-tactic of TWO in $MG(J)$ for some positive integer k. Only the case $k > 3$ requires proof. Thus, assume that $k > 3$ and put $m = k - 2$.

Observation. Let Γ be a collection of sets in $\langle J \rangle$ with the property that if A and B are in Γ and $A \subset B$, then there is a C in Γ with $A \subset C \subset B$. Consider that modification of $MG(J)$ where ONE is restricted to picking sets from Γ. Then TWO has a winning 3-tactic in this modified game.

Proof of the Observation. For each pair A and B in Γ with $A \subset B$, pick sets $A \subset C_1(A,B) \subset \cdots \subset C_m(A,B) \subset B$ with $C_i(A,B)$ in Γ for $1 \leq i \leq m$. We define a 3-tactic G as follows. Let $X_1 \subset X_2 \subset X_3$ in Γ be given.

We let $G(X_1, X_2, X_3)$ be the union of sets of the form $F(Y_1, \dots, Y_k)$ where $\{Y_1, \dots, Y_k\}$ is a $\subset$-increasing k-tuple form

$$\{X_1, X_2, X_3\} \cup \{C_i(X_r, X_j) : 1 \leq i \leq m,\ 1 \leq r < j \leq 3\}.$$

That G is as required now follows from the fact that F is a winning k-tactic for TWO in $MG(J)$. The proof of the observation is complete. □

Note that this observation also applies to $MG(J_X)$ for any subset X of S. Now let A be a set in $\langle J \rangle$ which is not in J. We may assume (by Proposition 3) that A is uncountable. Pick a set Y in $\langle J \rangle$ which is disjoint from A and has the same cardinality as A. Write $Y = \cup_{x \in A} S_x$

where $\{S_x : x \in A\}$ is a pairwise disjoint collection of infinite sets. Pick for each x in A a collection $\mathcal{R}_x$ of subsets of S_x with the following properties:

(a) $\varnothing$ and S_x are both in $\mathcal{R}_x$,

(b) $\mathcal{R}_x$ is totally ordered by $\subset$, and

(c) whenever $C \subset D$ are in $\mathcal{R}_x$, there is an E in $\mathcal{R}_x$ with $C \subset E \subset D$.

Let Γ be the family of those subsets X of $A \cup Y$ for which $X \cap S_x$ is in $\mathcal{R}_x$ for each x in A and with $X \cap S_x$ not empty for each x in $X \cap A$. Then Γ has the properties required by the observation above. Let G be a winning 3-tactic for TWO in the modified version of $MG(J_{A\cup Y})$ where ONE is required to pick his moves from Γ. We now use G to define a winning 3-tactic of TWO in $MG(J_A)$.

For X a subset of A put $X^* = X \cup (\cup_{x\in X} S_x)$. Note that, for each such X, X^* is in Γ and that if $U \subset V$ are subsets of A, then $U^* \subset V^*$.

For $U \subset V \subset W$ subsets of A, put $H(U,V,W) = G(U^*,V^*,W^*) \cap A$. Then H is a winning 3-tactic for TWO in $MG(J_A)$, for let $(M_1, N_1, \dots, M_t, N_t, \dots)$ be a play of $MG(J_A)$ during which TWO used the 3-tactic H. Let t be a positive integer and let x be a point in M_t. We show that there is a positive integer s with x in N_s.

First note that $M_1^* \subset M_2^* \subset \cdots \subset M_t^* \subset \cdots$ is a sequence of consecutive moves by ONE in the game where ONE is restricted to moving from Γ and that x is a point in M_t^*. Since G is a winning 3-tactic of TWO in this game, we find a positive integer s with x in $G(M_{s-2}^*, M_{s-1}^*, M_s^*)$. But then x is in $G(M_{s-2}^*, M_{s-1}^*, M_s^*) \cap A$, which is N_s. The proof is complete. □

Combining Proposition 12 and Proposition 13, we obtain

Corollary 14. *Let $k > 2$ be a positive integer. Let J be a free ideal on S with $J \neq \langle J \rangle$ and with $1 <$ cofinality$(\langle J \rangle, \subset) =$ c.n.$(\langle J \rangle, \subset)$. Assume that for each X in $\langle J \rangle$ there is a Y in $\langle J \rangle$ which is disjoint from X and has the same cardinality as X. Then the following statements are equivalent*:

(a) *TWO has a winning k-tactic in $MG(J)$.*

(b) *TWO has a winning 3-tactic in $MG(J)$.*

Example 2 (Continued). By Corollary 10 player TWO does not have a winning 2-tactic in $MG(J)$, but if CH holds TWO has a winning 3-tactic in $MG(J)$. We used the following three consequences of CH in this demonstration.

(i) EH holds.

(ii) $2^{<\mathbf{c}} = \mathbf{c}$.

(iii) The cofinality of $\langle J \rangle$ is $\aleph_1$.

The next few problems all relate to Example 2.

Problem 2. Is the theory "$ZFC + EH + 2^{<\mathbf{c}} = \mathbf{c} + \mathbf{c} > \aleph_1 +$ the cofinality of $\langle J \rangle$ is $\aleph_1$" consistent relative to ZFC?

As Laver points out in [**7**], the theory "$ZFC + EH + MA + \mathbf{c} > \aleph_1$" is consistent relative to ZFC. In such a model, one has that the cofinality of $\langle J \rangle$ is equal to c.n.$(\langle J \rangle, \subset)$. In light of Proposition 12, we have

Problem 3. Consider a special case of Laver's models in [**7**] in which $EH + MA + \mathbf{c} = \aleph_2$ holds. Does player TWO have a winning 3-tactic in $MG(J)$ in this model?

On the other hand, consider the models obtained by starting with models of CH and iteratively with countable support adding $\aleph_2$ Mathias reals. In such models $\mathbf{R}$ is a union of $\aleph_1$ meager sets, so c.n.$(\langle J \rangle, \subset) = \aleph_1$. Moreover, every subset of $\mathbf{R}$ of cardinality at most $\aleph_1$ is meager, whence the cofinality of $\langle J \rangle$ is $\aleph_2$. The reader interested in details could consult Section 6 of Miller's paper [**8**]. We henceforth call a model obtained in this way a *Mathias reals model*.

Problem 4. Consider a Mathias reals model. Does player TWO have a winning k-tactic in $MG(J)$ for some positive integer k?

A variety of other problems suggest themselves for Example 2. We mention the following one before moving on.

Problem 5. Is it possible that there is no positive integer k for which player TWO has a winning k-tactic in $MG(J)$?

Also note that J satisfies the condition of Proposition 13.

Example 3. Let κ and λ be cardinal numbers with $\aleph_0 = \mathrm{cof}(\lambda) \leq \lambda \leq \kappa$. We consider the free ideals $J = [\kappa]^{<\lambda}$. In the special case when $\lambda = \aleph_0$, we call $MG(J)$ the *countable-finite game*.

Proposition 5 tells us that if $\kappa = \lambda$ and if player TWO has a winning k-tactic in $MG(J)$ for some positive integer k, then TWO has a winning 3-tactic in $MG(J)$. If $\kappa = \lambda^+$ and TWO has a winning k-tactic in $MG(J)$, Propositions 5 and 12 tell us that TWO has a winning 3-tactic in $MG(J)$. What is the situation when κ is larger than λ^+? The next proposition shows that this class of examples cannot solve Problem 1.

Proposition 15. *Let k be a positive integer and let $J = [\kappa]^{<\lambda}$ where κ and λ are cardinal numbers with $\aleph_0 = cof(\lambda) \leq \lambda \leq \kappa$. If TWO has a winning k-tactic in $MG(J)$, then TWO has a winning 3-tactic in $MG(J)$.*

Proof. Let κ, λ and k be as in the hypotheses. Let F be a winning k-tactic of TWO in $MG(J)$. We may assume that $k > 3$; put $m = k-2$.

Observation. Let Γ be a collection of sets in $\langle J \rangle$ with the property that if A and B are in Γ and $A \subset B$, then there is a C in Γ with $A \subset C \subset B$. Consider that modification of $MG(J)$ where ONE is restricted to picking sets from Γ. Then TWO has a winning 3-tactic in this modified game.

Proof of the Observation. For each pair A and B in Γ with $A \subset B$, pick sets $A \subset C_1(A,B) \subset \cdots \subset C_m(A,B) \subset B$ with $C_i(A,B)$ in Γ for $1 \leq i \leq m$. We define a 3-tactic G as follows. Let $X_1 \subset X_2 \subset X_3$ in Γ be given.

We let $G(X_1, X_2, X_3)$ be the union of sets of the form $F(Y_1, \ldots, Y_k)$ where $\{Y_1, \ldots, Y_k\}$ is a $\subset$-increasing k-tuple from

$$\{X_1, X_2, X_3\} \cup \{C_i(X_r, X_j) : 1 \leq i \leq m,\ 1 \leq r < j \leq 3\}.$$

That G is as required now follows from the fact that F is a winning k-tactic for TWO in $MG(J)$. The proof of the observation is complete. $\square$

Now write $\kappa = \cup_{\alpha<\kappa} S_\alpha$ where $\{S_\alpha : \alpha < \kappa\}$ is a pairwise disjoint collection of countably infinite sets. For each $\alpha < \kappa$ pick a collection $\mathcal{R}_\alpha$ of subsets of S_α with the following properties:

(a) $\varnothing$ and S_α are both in $\mathcal{R}_\alpha$,

(b) $\mathcal{R}_\alpha$ is totally ordered by $\subset$, and

(c) whenever $A \subset B$ are in $\mathcal{R}_\alpha$, there is a C in $\mathcal{R}_\alpha$ with $A \subset C \subset B$.

Let Γ be the set of those subsets X of κ for which $X \cap S_\alpha$ is in $\mathcal{R}_\alpha$ for each $\alpha < \kappa$ and for which X is in $\langle J \rangle$. Then Γ has the properties required by the Observation above. So let G be a winning 3-tactic for TWO as in the Observation.

We now use G to define a winning 3-tactic for TWO in $MG(J)$. Note that for X in $\langle J \rangle$, $X^* = \cup_{\alpha \in X} S_\alpha$ is also in $\langle J \rangle$ and that if $X \subset Y$, then $X^* \subset Y^*$. Note also that if X is in J, then $\{\alpha < \kappa : X \cap S_\alpha \neq \varnothing\}$ is in J.

Let $A \subset B \subset C \in \langle J \rangle$ be given and define $H(A, B, C)$ to be the set of these α in κ for which $G(A^*, B^*, C^*) \cap S_\alpha$ is not the empty set. By the foregoing remarks, H is a 3-tactic for TWO in $MG(J)$. We show that H is winning for TWO.

Consider a play $(M_1, N_1, \dots, M_t, N_t, \dots)$ of $MG(J)$ during which TWO has used the 3-tactic H. Let t be a positive integer and let α be an element of M_t. Then S_α is a subset of M_t^*. We show that α is in N_m for some positive integer m.

By our earlier remarks $M_1^* \subset M_2^* \subset \cdots \subset M_t^* \subset \cdots$ is a sequence of consecutive moves by ONE in the game where ONE is restricted to moving from Γ and hence $\cup_{n=1}^{\infty} M_n^*$ is contained in the union of the responses by TWO, using G. Pick the smallest k for which $G(M_k^*, M_{k+1}^*, M_{k+2}^*) \cap S_\alpha$ is not the empty set. Then α is in $H(M_k, M_{k+1}, M_{k+2})$ and we are done. □

We saw in Corollary 4 that TWO has a winning 2-tactic in $MG([\aleph_1]^{<\aleph_0})$. Beyond this, our knowledge about pairs κ and λ for which TWO has a winning 3-tactic in $MG([\kappa]^{<\lambda})$ is quite unsatisfactory.

Problem 6. Does TWO have a winning 3-tactic in $MG([\aleph_2]^{<\aleph_0})$?

At this stage, even consistency results would be interesting. It is also not clear that "3-" is the optimal conclusion in Proposition 15.

Problem 7. Is there (consistently) a J as in Proposition 15 for which TWO does not have a winning 2-tactic in $MG(J)$ but does have a winning 3-tactic in $MG(J)$?

Laver's model in [**7**] with $\mathbf{c} > \aleph_{\omega+1}$ and $J = [\aleph_{\omega+2}]^{<\aleph_\omega}$ may shed some light on problem 7.

6. Remarks about the cofinality condition in theorems 2 and 8. In both Theorem 2 and Theorem 8 the hypothesis that the cofinality of $\langle J \rangle$ is at the most $\aleph_1$ was sufficient to make various general conclusions about the existence of winning k-tactics. In this paragraph, we give an example which shows that this cofinality hypothesis is not necessary to have winning k-tactics.

New ideas are needed to deal with free ideals J for which the confinality of $\langle J \rangle$ is at least $\aleph_2$. The ideals of problems 3, 4 and 6 are test cases which will probably shed some light on the combinatorics that will be needed to handle cofinalities above $\aleph_1$.

Example 1 (continued and expanded).

Claim. *For each infinite cardinal number κ there is a free ideal J on a set S such that*

(i) $J \neq \langle J \rangle$,

(ii) *the cofinality of $\langle J \rangle$ is at least κ and*

(iii) *TWO has a winning 2-tactic in $MG(J)$.*

Proof. Let κ be a given infinite cardinal number. Take a set S of cardinality κ^{++} and write $S = A \cup B$ where A and B are pairwise disjoint and the cardinality of A is κ. Now define J so that $J_A = J(\kappa)$ and so that for each X in J the cardinality of $B \cap X$ is at the most κ. Since A is in $\langle J \rangle$ but not in J, (i) is clear. It is also evident that the cofinality of $\langle J \rangle$ is κ^{++} and thus (ii) also holds. We now prove (iii).

Let $\{Y_\alpha : \alpha < \kappa\}$ be a collection of subsets of A which are in J and has the property that $\cup_{\alpha\in F} Y_\alpha = A$ for every infinite subset F of κ. Also let F_A be a winning 2-tactic for TWO in $MG(J_A)$. We assume (without loss of generality) that $F_A(X,T) = \varnothing$ whenever $X = \varnothing$. By Proposition 15 of [**10**] we fix a partition $[[B]^{\leq\kappa}]^2 = \cup_{\alpha<\kappa} K_\alpha$ which witnesses that

$$([B]^{\leq\kappa}, \subset) \not\to (\omega\text{-path})^2_{\kappa/<\omega}.$$

Define a 2-tactic G of TWO as follows. Let U and V in $\langle J\rangle$ be given with $U \subset V$. Let $U' = U\cap A$, $V' = V\cap A$, $U'' = U\cap B$ and $V'' = V\cap B$.

Case 1. If $U'' \subset V''$, pick α in κ with $\{U'', V''\}$ in K_α and put $G(U,V) = Y_\alpha \cup V''$.

Case 2. Otherwise, let $G(U,V) = F_A(U',V') \cup V''$.

G is a winning 2-tactic for TWO, for consider a play $(M_1, N_1, \dots, M_k, N_k, \dots)$ of $MG(J)$ in which TWO used G. For each positive integer k, put $B_k = B\cap M_k$ and $C_k = A\cap M_k$. If there is some positive integer k such that $B_m = B_k$ for all $m \geq k$, then $C_{m+1} \supset C_m$ for all $m \geq k$, and F_A then guarantees that the given play is a win for TWO. Otherwise, the choice of the partition $[[B]^{\leq\kappa}]^2 = \cup_{\alpha<\kappa} K_\alpha$ and the properties of the family $\{Y_\alpha : \alpha < \kappa\}$ guarantees that this play is a win for TWO. □

Part 2—The game $SMG(J)$. Let J be a free ideal on a set S and let $\langle J\rangle$ be its σ-completion. For the remainder of this article, when we consider such a free ideal, we will tacitly assume that $J \neq \langle J\rangle$ and that S is not in $\langle J\rangle$. We make these two assumptions to avoid technicalities or trivialities.

$SMG(J)$, which we call "the strongly monotonic meager-nowhere dense game on J," is played as follows. Player ONE starts the game by picking an M_1 from $\langle J\rangle$, and TWO responds by picking an N_1 from J. In the second inning, ONE picks an M_2 which contains $M_1 \cup N_1$ from $\langle J\rangle$ and TWO responds by picking an N_2 from J, and so on. The players construct an infinite sequence $(M, N_1, \dots, M_k, N_k, \dots)$ where for each positive integer k

(i) M_k denotes the set ONE picked from $\langle J\rangle$ during the kth inning,

(ii) N_k denotes the set TWO picked from J during the kth inning, and

(iii) M_{k+1} contains $M_k \cup N_k$.

Such a sequence is a *play of* $SMG(J)$ and TWO wins this play if $\cup_{n=1}^{\infty} M_n = \cup_{n=1}^{\infty} N_n$. Note that TWO has a winning perfect information strategy $SMG(J)$. Furthermore, if TWO has a winning k-tactic in $MG(J)$, then TWO hs a winning k-tactic in $SMG(J)$. In section 1 of this part, we show that the converse fails badly. We give an example of a free ideal J on a set S for which TWO does not have a winning k-tactic in $MG(J)$ for any positive integer k, and yet TWO has a winning 1-tactic in $SMG(J)$.

In section 2 we start our discussion of winning k-tactics for TWO in $SMG(J)$ and we prove

Theorem 18. *If J is a free ideal on a set S and if the cofinality of $(\langle J\rangle, \subset)$ is $\aleph_1$, then TWO has a winning 2-tactic in $SMG(J)$.*

In sections 3 and 4 we discuss various points about k-tactics for $k > 1$. In section 5, we give an application of $SMG(J)$ to the game $BM(\mathbf{R}, \tau)$. Let J be the collection of nowhere dense subsets of $\mathbf{R}$. We prove

Theorem 22. *Let k be a positive integer. If TWO has a winning k-tactic in $SMG(J)$, then TWO has a winning k-tactic in $BM(\mathbf{R}, \tau)$.*

1. 1-tactics in $SMG(J)$. We have the following necessary condition for the existence of winning 1-tactics for TWO in $SMG(J)$.

Proposition 16. *Let J be a free ideal on S and let C be a cofinal subset of J. Then* (a) $\Rightarrow$ (b), *where* (a) *TWO has a winning 1-tactic in $SMG(J)$.* (b) *For each T in $\langle J\rangle$, but not in J, there is a family G_T in C such that*

(i) *the cardinality of G_T is at least* $\text{c.n.}(\langle J\rangle, \subset)$ *and*

(ii) *for every infinite subcollection H of G_T, T is a subset of $\cup H$.*

Proof. Let κ denote c.n.$(\langle J\rangle, \subset)$ and suppose, contrary to the claim of the Proposition, that C is a cofinal subset of J for which there is some set T in $\langle J\rangle$ but not in J such that for each collection G of cardinality at least κ of sets from C, there is an infinite subcollection G' of G whose union does not cover T. Fix such a set T.

Let F be a 1-tactic for TWO. Construct a chain $\{D_\alpha : \alpha < \kappa\}$ in $\langle J\rangle$ which has the following properties:

(i) for $\alpha < \kappa$, T is a proper subset of D_α,

(ii) for $\alpha < \beta < \kappa$, $D_\alpha \cup F(D_\alpha)$ is a proper subset of D_β and

(iii) $\cup_{\alpha<\kappa} D_\alpha$ is not in $\langle J\rangle$.

For each $\alpha < \kappa$ pick a C_α from C which is different from any C_β with $\beta < \alpha$ which had already been picked, so that $F(D_\alpha)$ is a subset of C_α. Then $\{C_\alpha : \alpha < \kappa\}$ is a family of κ many sets from C. By our choice of C and T there is an infinite sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ in κ for which T is not a subset of $\cup_{n=1}^{\infty} C_{\alpha_n}$. Put $M_n = D_{\alpha_n}$ and $N_n = F(D_{\alpha_n})$ for each positive integer n. Then $(M_1, N_1, \ldots, M_k, N_k, \ldots)$ is a play of $SMG(J)$ during which TWO used the 1-tactic F and lost. We proved the contrapositive of the Proposition. □

We now introduce one more cardinal function for free ideals before applying this Proposition.

Definition. Let λ be an infinite cardinal number and let J be a free ideal on a set S. Then $K(J, \lambda)$ is the minimal cardinality of a subcollection G of J which has the property that each set in J of cardinality less than λ is a subset of some set in G.

It is clear that if $\lambda < \kappa$ are infinite cardinals, then $K(J, \lambda) \leq K(J, \kappa)$ and that the cofinality of J is an upper bound for these cardinal numbers.

Corollary 17. *Let J be a free ideal on S. If any of conditions* (a), (b) *or* (c) *applies to J, TWO does not have a winning* 1*-tactic in $SMG(J)$.*

(a) $d(J) <$ c.n.$(\langle J\rangle, \subset)$

(b) *for some infinite cardinal $\rho <$* c.n.$(\langle J\rangle, \subset)$ *there is an X in $\langle J\rangle$*

for which $\rho^+ < K(J_X, \rho^+)$

(c) $\mathrm{cof}\,(\langle J\rangle, \subset) < K(J, \mathrm{c.n.}(\langle J\rangle, \subset))$.

Proof. (a) Let κ denote $d(J)$ and let $J = \cup_{\alpha<\kappa} J_\alpha$ be a decomposition of J witnessing this. By assumption (a), we pick a set T in $\langle J\rangle$ with T not in $\langle J_\alpha\rangle$ for each $\alpha < \kappa$. Note that if C is any collection of cardinality bigger than κ of sets in J, then there is an α in κ for which the set $\{D \in G : D \text{ is in } J_\alpha\}$ is infinite. T, together with the fact that the cofinality of $(J, \subset)$ is no smaller than $\mathrm{c.n.}(\langle J\rangle, \subset)$, witnesses that the necessary condition of Proposition 16 fails.

(b) Pick ρ and X as in the hypothesis, and consider any family G of cardinality at least ρ^+ of sets in J. Pick by hypothesis (b) a set Y in J which is a subset of X, has cardinality at the most ρ and which is not a subset of any set in G. Then there is an infinite subcollection G' of G and a point y in Y such that y is in no set in G'. Thus, X is not a subset of $\cup G'$. Thus X witnesses that the necessary condition of Proposition 16 fails.

(c) Let H be a cofinal subset of $\langle J\rangle$ of minimal cardinality. By (c) we pick an X in $\langle J\rangle$ for which $K(J_X, \mathrm{c.n.}(\langle J\rangle, \subset))$ is bigger than cof $(\langle J\rangle, \subset)$. Then we have the inequality $\mathrm{c.n.}(\langle J\rangle, \subset) < K(J_X, \mathrm{c.n.}(\langle J\rangle, \subset))$. If $\mathrm{c.n.}(\langle J\rangle, \subset)$ is the successor of an infinite cardinal number ρ, then ρ and X satisfy hypothesis (b) and we are done. Otherwise, $\mathrm{c.n.}(\langle J\rangle, \subset)$ is an uncountable limit cardinal and there is an infinite cardinal number ρ below $\mathrm{c.n.}(\langle J\rangle, \subset)$ for which $\rho^+ < K(J_X, \rho^+)$ and (b) applies again. □

Example 2 (continued—see Part 1, Section 4). Since $\mu(J) < \mathrm{c.n.}(\langle J\rangle, \subset)$ and $d(J) \leq \mu(J)$ for this example, Corollary 17(a) implies that TWO does not have a winning 1-tactic in $SMG(J)$. Note also that if CH holds, then neither (b) nor (c) applies to this example.

Example 3 (continued—see Part 1, Section 5). Now we conclude from the formulae $d(J) = dir(J) < \mathrm{c.n.}(\langle J\rangle, \subset)$ and Corollary 17(a) that TWO does not have a winning 1-tactic in $SMG(J)$. For the particular example $J = [\aleph_\omega]^{<\aleph_0}$ neither (b) nor (c) of Corollary 17 applies.

Next we show that there are nontrivial free ideals J for which TWO has a winning 1-tactic in $SMG(J)$.

Example 4. Let κ and λ be infinite cardinal numbers and let S and T be disjoint sets of cardinality κ and λ, respectively. Write $S = \cup_{\alpha<\kappa} R_\alpha$ where $\{R_\alpha : \alpha < \kappa\}$ is a pairwise disjoint collection of countably infinite sets. The underlying set for the ideal J we are about to define is $S \cup T$, which we denote by E.

Let X be a subset of E. We put X in J if the supremum of the set $\{\gamma < \kappa : X \cap R_\gamma \text{ is infinite}\}$ is below κ and if the cardinality of $X \cap T$ is less than λ. Then J is a free ideal on E having the following readily verifiable properties:

(a) E is in $\langle J \rangle$ if and only if λ has countable cofinality.

We assume henceforth that the cofinality of λ is uncountable.

(b) S is in $\langle J \rangle$ but not in J.

(c) $\mu(J)$ is equal to the cofinality of κ.

(d) c.n.$(\langle J \rangle, \subset)$ is equal to the cofinality of λ.

Claim. *The following statements are equivalent*:

(i) *TWO has a winning* 1*-tactic in* $SMG(J)$.

(ii) *Both the cofinality of* κ *and of* λ *are equal to* $\aleph_1$.

Proof. (ii) $\Rightarrow$ (i). Let $\{T_\alpha : \alpha < \omega_1\}$ be a collection of subsets of T which has the following properties:

(1) for $\alpha < \beta < \omega_1$, T_α is a proper subset of T_β.

(2) the cardinality of each T_α is less than λ and

(3) $T = \cup\{T_\alpha : \alpha < \omega_1\}$.

For each B in $\langle J \rangle$ we let $\alpha(B)$ denote the smallest ordinal ν below ω_1 for which T_ν is not contained in B.

Let $\{C_\alpha : \alpha < \omega_1\}$ be a collection of subsets of κ with the properties:

(a) $C_\alpha = \{a\}$ if κ is ω_1,

(b) for $\alpha < \beta < \omega_1$ each ordinal in C_α is less than each in C_β and

(c) if κ is not ω_1, then for $\alpha < \beta < \omega_1$ the cardinality of C_α is less than that of C_β.

For δ in κ, let $\{r_\delta(n) : n < \omega\}$ be a bijective enumeration of R_δ. We also fix a collection of functions $\{f_\alpha : \alpha < \omega_1\}$, each of which has domain ω_1 and values in ω and such that

(*)

$$\text{if } \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots < \omega_1, \text{ then } \{f_{\alpha_n}(\delta) : n \text{ a positive integer}\}$$

is infinite for each δ which is above each α_n.

(Proof that these exist: for each $\gamma < \omega_1$, pick an injection h_γ from γ to ω. For α less than ω_1, define f_α so that $f_\alpha(\delta) = h_\delta(\alpha)$ if δ is bigger than α, and $f_\alpha(\delta) = 0$ otherwise.)

We now define a 1-tactic for TWO. Let B be a set in $\langle J\rangle$ and let β be the ordinal $\alpha(B)$. We define $F(B)$ to be the set

$$(\cup\{R_\delta : \delta \text{ is in } C_\gamma \text{ and } \gamma \leq \beta\})$$
$$\cup\,(\{r_\sigma(n) : n \leq f_\beta(\gamma),\ \sigma \text{ is in } C_\gamma \text{ and } \gamma \text{ in } \omega_1\}) \cup (B \cap T) \cup C_\beta.$$

Then $F(B)$ is in J and F is a 1-tactic for TWO. We claim that F is a winning 1-tactic for TWO. For let $(M_1, N_1, \dots, M_k, N_k, \dots)$ be a play of $SMG(J)$ during which TWO used F. For each positive integer k, we let β_k denote the ordinal $\alpha(M_k)$.

Observation 1. $\beta_k < \beta_{k+1}$ for each positive integer k.

(This is because C_{β_k} is a subset of N_k and N_k of M_{k+1} for each k.)

Observation 2. $\cup_{k=1}^{\infty}(T \cap M_k)$ is a subset of $\cup_{k=1}^{\infty} N_k$.

(This is because of the third term in the union of four that constitutes $F(B)$.)

We will be done if we show that S is a subset of $\cup_{k=1}^{\infty} N_k$. Let δ denote the supremum of the set $\{\beta_k : k \text{ a positive integer}\}$. From the first term in the definition of F, it follows that S will be a subset of $\cup_{k=1}^{\infty} N_k$ if we can show that R_γ is a subset of $\cup_{k=1}^{\infty} N_k$ for each γ not in C_τ for some τ less than δ. Consider such a γ. Then γ is in C_τ for some τ bigger than or equal to δ and thus $\{r_\gamma(n) : n \leq f_{\beta_k}(\tau) \text{ and } k \text{ is a positive integer}\}$ is a subset of $\cup_{k=1}^{\infty} N_k$. It follows from Observation 1, the choice of τ

and the property $(*)$ of the family $\{f_\alpha : \alpha < \omega_1\}$ that R_γ is a subset of $\cup_{k=1}^{\infty} N_k$ and we are done.

(i) $\Rightarrow$ (ii). Suppose that at least one of κ or λ has cofinality different from $\aleph_1$. We may assume that the cofinality of λ is uncountable.

Case 1. $\mathrm{cof}(\kappa) < \mathrm{cof}(\lambda)$. *Then* $\mu(J) <$ c.n.$(\langle J\rangle, \subset)$ *and since* $d(J)$ *is at most* $\mu(J)$ *Corollary 17(a) implies that TWO does not have a winning* 1*-tactic.*

Case 2. $\mathrm{cof}(\kappa) \geq \mathrm{cof}(\lambda)$. *Then the cofinality of* κ *is larger than* $\aleph_1$. *Since* $K(J_S, \aleph_1)$ *is larger than* $\aleph_1$ *and* c.n.$(\langle J\rangle, \subset)$ *is larger than* $\aleph_0$*,it follows from Corollary 17(b) that TWO does not have a winning* 1*-tactic.*

The proof is complete. □

If we now take κ and λ both of cofinality $\aleph_1$ and λ larger than 2^κ in this example, then TWO has a winning 1-tactic in $SMG(J)$ and by Proposition 11 TWO does not have a winning k-tactic in $MG(J)$ for all positive integers k.

2. k-tactics in $SMG(J)$ for k bigger than 1.

Theorem 18. *If* J *is a free ideal on* S *for which the cofinality of* $(\langle J\rangle, \subset)$ *is* $\aleph_1$, *then TWO has a winning* 2*-tactic in* $SMG(J)$.

Proof. Without loss of generality, J is not σ-complete. Choose a family $\{C_\alpha : \alpha < \omega_1\}$ of sets in $\langle J\rangle$ which are not in J such that

(i) for $\alpha < \beta < \omega_1$, C_α is a proper subset of C_β and

(ii) for each B in $\langle J\rangle$ there is an α less than ω_1 with B a subset of C_α.

Let G be a winning perfect information strategy of TWO in $MG(J)$ and pick for each α less than ω_1 a set D_α in J which is not a subset of C_α. For each B in $\langle J\rangle$, let $\beta(B)$ denote the least ordinal α less than ω_1 for which B is a subset of C_α.

Let $(<_n: n < \omega)$ be a sequence of binary relations on ω_1 such that

(a) $T_n = (\omega_1, <_n)$ is a tree of height at most $n+2$ for each n in ω,

(b) for $m < n$ in ω, $<_m$ is contained in $<_n$ and

(c) for $\alpha < \beta$ in ω_1, there is an n in ω for which $\alpha <_n \beta$.

We now define a 2-tactic F for TWO. So let B and C be sets in $\langle J \rangle$ with B a proper subset of C and let α denote $\beta(B)$ and $\gamma, \beta(C)$. Then α is less than or equal to γ.

Case 1: α is less than γ. Pick the smallest m in ω with $\alpha <_m \gamma$ and put $T = \{\delta < \omega_1 : \delta = \gamma \text{ or } \delta <_m \gamma\}$. By (a) T is a finite set. Define $F(B, C)$ to be the set $D_\gamma \cup (\cup\{F(C_{\alpha_1}, \dots, C_{\alpha_n}) : \{\alpha_1, \dots, \alpha_n\}$ is an n-element subset of T for an $n \leq m\}$.

Case 2: α is equal to γ. Then define $F(B, C)$ to be the set D_γ.

F is a winning 2-tactic for TWO, for let $(M_1, N_1, \dots, M_k, N_k, \dots)$ be a play of $SMG(J)$ during which TWO used F. For each positive integer n we let α_n denote $\beta(M_n)$ and we let m_n denote the smallest integer m for which $\alpha_n <_m \alpha_{n+1}$. E_n denotes C_{α_n} and S_n denotes D_{α_n} for each positive integer n.

Observation 1. α_n is less than α_{n-1} for each positive integer n. (This is because S_n is a subset of M_{n+1}.)

Observation 2. The function f defined so that for each positive integer k $f(k)$ is the smallest t with $k \leq m_t$, has the property that if $k \leq s$ are positive integers, then

(i) $f(k)$ is less than or equal to $f(s)$,

(ii) $m_{f(k)} \leq m_{f(s)}$,

(iii) $\lim_{n\to\infty} f(n) = \infty$ and

(iv) for each $j \leq f(k)$, $\alpha_j <_{m_{f(k)}} \alpha_{(f(k)+1)}$.

It follows that $N_{f(k)}$ contains the set

$$G(E_1) \cup \cdots \cup G(E_1, \dots, E_m) \cup \cdots \cup G(E_1, \dots, E_{f(k)})$$

for each positive integer k. This, together with the fact that G is a winning perfect information strategy of TWO in $MG(J)$, implies that the given play of $SMG(J)$ is won by TWO. $\square$

Example 2 (Continued). It is consistent with ZFC (even with $ZFC + \neg CH$) that the cofinality of $(\langle J \rangle, \subset)$ is $\aleph_1$. Consequently, it is consistent with ZFC that TWO has a winning 2-tactic in $SMG(J)$. In our applications of Corollary 17 we noted that TWO does not have a winning 1-tactic in $SMG(J)$.

Example 3 (Continued). For $J = [\omega_1]^{<\aleph_0}$ the cofinality of $(\langle J \rangle, \subset)$ is $\aleph_1$. By Theorem 18, TWO has a winning 2-tactic in $SMG(J)$. In Corollary 4, we have already noted the better result that TWO has a winning 2-tactic in $MG(J)$.

Problem 8. Is there some positive integer k for which TWO has a winning k-tactic in $SMG([\aleph_2]^{<\aleph_0})$?

3. Winning $(k+1)$-tactics but no winning k-tactics? We have given examples of

(1) a free ideal J for which TWO has a winning 1-tactic in $SMG(J)$

(2) a free ideal J for which TWO has a winning 2-tactic in $SMG(J)$ but no winning 1-tactic in $SMG(J)$.

These settle the case $k = 1$ of the following open problem.

Problem 9. Is there for each positive integer k a free ideal J_k such that TWO has a winning $(k+1)$-tactic in $SMG(J_k)$ but not a winning k-tactic in $SMG(J_k)$?

The following theorem rules out some ideals as candidates for the case $k > 1$ in Problem 9.

Theorem 19. *Let J be a free ideal on a set S. If the cofinality of $(\langle J \rangle, \subset)$ is equal to the completeness number of $(\langle J \rangle, \subset)$, the following statements are equivalent:*

(1) *TWO has a winning* 2*-tactic in* $SMG(J)$.

(2) *For some positive integer* k *TWO has a winning* k*-tactic in* $SMG(J)$.

Proof. Only (2) $\Rightarrow$ (1) requires proof. This implication is proven by induction on k. It is clearly true for $k \leq 2$. Let $k \geq 2$ be given and suppose that it is known that if TWO has a winning k-tactic in $SMG(J)$, then TWO has a winning 2-tactic in $SMG(J)$ for J as in the hypothesis of the theorem. Let J be as in the hypothesis of the theorem and assume that TWO has a winning $(k+1)$-tactic in $SMG(J)$. Let F be such a winning $(k+1)$-tactic for TWO. Let κ denote the cofinality of $(\langle J\rangle, \subset)$. By the induction hypothesis, it suffices to show that TWO has a winning k-tactic in $SMG(J)$.

Choose a family $\{C_\alpha : \alpha < \kappa\}$ from $\langle J\rangle$ such that

(1) for $\alpha < \beta < \kappa$, C_α is a proper subset of C_β,

(2) for each B in $\langle J\rangle$, there is an α less than κ with B a subset of C_α, and

(3) for $\alpha_1 < \cdots < \alpha_{k+1} < \beta < \kappa$, $F(C_{\alpha_1}, \ldots, C_{\alpha_{k+1}})$ is a subset of C_β.

Write $\kappa = \cup_{\alpha<\kappa} T_\alpha$ where for $\alpha < \kappa$, $T_\alpha = \{x_\alpha, y_\alpha\}$ is a set of two consecutive ordinals listed in increasing order and $\{T_\alpha : \alpha < \kappa\}$ is a pairwise disjoint family. Put $D_\alpha = C_{x_\alpha}$ for $\alpha < \kappa$. Then $\{D_\alpha : \alpha < \kappa\}$ still has properties (1), (2) and (3) above.

For each $\alpha < \kappa$, pick a z_α in S which is not in C_{y_α} and for each B in $\langle J\rangle$, let $\alpha(B)$ denote the smallest ordinal γ with B a subset of D_γ. We now define a k-tactic G for TWO.

Let $\{B_1, \ldots, B_k\}$ be sets from $\langle J\rangle$ with B_j a subset of B_s for $j \leq s \leq k$. Let α_i denote the ordinal $\alpha(B_i)$ for $1 \leq i \leq k$.

Case 1. $\alpha_1 < \cdots < \alpha_k$. Let E be the set consisting of the terms in the chain $D_{\alpha_1} \subset C_{y_{\alpha_1}} \subset \cdots \subset D_{\alpha_k} \subset C_{y_{\alpha_k}}$ and we define $G(B_1, \ldots, B_k)$ by the set

$$(\cup\{F(X_1, \ldots, X_{k+1}) : \{X_1, \ldots, X_{k+1}\} \text{ an increasing chain from } E\}) \cup \{z_{\alpha_k}\}.$$

Case 2. In all other cases, put $G(B_1, \dots, B_k) = \{z_{\alpha_k}\}$.

Suppose that $(M_1, N_1, \dots, M_t, N_t, \dots)$ is a play of $SMG(J)$ during which TWO used the k-tactic G. For each positive integer j, let α_j denote the ordinal $\alpha(M_j)$, let u_j denote x_{α_j}, let v_j denote y_{α_j} and let w_j denote z_{α_j}. We also let D_j denote C_{u_j} and H_j denote C_{v_j}.

Observation 1. For $1 \leq j < k$, $N_j = \{w_j\}$.

Observation 2. For $k \leq j$, N_j is defined by Case 1.

Observation 3. For each positive integer j, $\alpha_j < \alpha_{j+1}$. (This is guaranteed by the set $\{w_j\}$ in the definition of G.

Observation 4. $M_j \subset D_j \subset H_j \subset D_{j+1}$.

Using these observations and the fact that F is a winning $(k+1)$-tactic for TWO in $SMG(J)$, it follows that the given play of $SMG(J)$ is won by TWO. Thus, G is a winning k-tactic for TWO in $SMG(J)$. □

The hypothesis that the cofinality of $(\langle J \rangle, \subset)$ is $\aleph_1$ in Theorem 18 implies that $\mathrm{cof}\,(\langle J \rangle, \subset) = \mathrm{c.n.}(\langle J \rangle, \subset)$. We show in the next section that this sufficient condition for the existence of a winning 2-tactic for TWO is not necessary and also that it is not necessary to have $\mathrm{cof}\,(\langle J \rangle, \subset) = \mathrm{c.n.}(\langle J \rangle, \subset)$ in order to have a winning 2-tactic in $SMG(J)$.

4. Criteria for nonexistence of winning k-tactics, $k > 1$. Note that when considering a k-tactic F for TWO in $SMG(J)$ for some J, we may without loss of generality assume that $F(X_1, \dots, X_k) = \varnothing$ whenever $X_1 = \varnothing$. This will henceforth be our tacit assumption unless we explicitly assume something else.

Theorem 20. *Let J be a free ideal on a set S. If J satisfies any one of the following conditions, then TWO does not have a winning*

k-tactic in $SMG(J)$ for any positive integer k.

(a) $2^{d(J)} <$ c.n.$(\langle J\rangle, \subset)$.

(b) (c.n.$(\langle J\rangle, \subset), <) \to (\omega\text{-path})^2_{dir(J)/<\omega}$.

(c) *there is an infinite cardinal number λ with $2^\lambda <$ c.n.$(\langle J\rangle, \subset)$ and $(2^\lambda)^+ < K(J_X, \lambda^+)$ for some X in $\langle J\rangle$.*

Proof. (a) Let κ denote $d(J)$, let μ denote c.n.$(\langle J\rangle, \subset)$ and choose a decomposition $J = \cup_{\alpha<\kappa} J_\alpha$ where for each α less than κ, J_α is an ideal with $\langle J_\alpha\rangle \neq \langle J\rangle$. Since κ is less than μ, pick an X in $\langle J\rangle$ which is in none of the $\langle J_\alpha\rangle$. Let k be a positive integer and let F be a k-tactic of TWO. Construct a chain $\{C_\alpha : \alpha < \mu\}$ in $\langle J\rangle$ such that if $\alpha_1 < \cdots < \alpha_k < \beta < \mu$, then $X \cup F(C_{\alpha_1}, \ldots, C_{\alpha_k}) \cup C_{\alpha_k}$ is a subset of C_β. Write $\mu = \cup_{\alpha<\mu} T_\alpha$ where for $\alpha < \beta < \mu$,

(a) T_α has k elements and

(b) each element of T_α is less than each element of T_β. Enumerate each T_α in increasing order as $\{\alpha_1, \ldots, \alpha_k\}$.

Define a partition $[\mu]^2 = \cup_{\gamma<\kappa} S_\gamma$ as follows. We put $\{\alpha, \beta\}$ in S_γ if γ is minimal with $F(C_{\alpha_1}, \ldots, C_{\alpha_k}) \cup F(C_{\alpha_2}, \ldots, C_{\alpha_k}, C_{\beta_1}) \cup \cdots \cup F(C_{\beta_1}, \ldots, C_{\beta k})$ in J_γ.

Since 2^κ is less than μ, we get (by the Erdös-Rado theorem) a γ in κ and a subset A of μ of order type at least $\kappa^+ + 1$ which is homogeneous of class S_γ. Pick an increasing sequence $\alpha_1 < \cdots < \alpha_t < \cdots$ from A and enumerate $\cup_{n=1}^\infty T_{\alpha_n} = \{\beta_m : m \text{ is a positive integer}\}$ in increasing order.

For each positive integer t, we put $M_t = C_{\beta_t}$. Consider the corresponding play $(M_1, N_1, \ldots, M_t, N_t, \ldots)$ of $SMG(J)$ during which TWO used the k-tactic F. By our construction X is a subset of M_t while N_t is in J_γ for each positive integer t. Hence, TWO lost this play of $SMG(J)$.

(b) Let κ denote $dir(J)$, let μ denote c.n.$(\langle J\rangle, \subset)$ and choose a decomposition $J = \cup_{\alpha<\kappa} J_\alpha$ where $\{J_\alpha : \alpha < \kappa\}$ is an up directed family of ideals. Since κ is less than μ, pick an X in $\langle J\rangle$ which is in none of the $\langle J_\alpha\rangle$. Let k be a positive integer and let F be a k-tactic of TWO. Construct a chain $\{C_\alpha : \alpha < \mu\}$ in $\langle J\rangle$ such that if

$\alpha_1 < \cdots < \alpha_k < \beta < \mu$, then $X \cup F(C_{\alpha_1}, \ldots, C_{\alpha_k}) \cup C_{\alpha_k}$ is a subset of C_β.

We define a partition $[\mu]^k = \cup_{\gamma<\kappa} S_\gamma$ by putting $\{\alpha_1, \ldots, \alpha_k\}$ in S_γ where γ is minimal with $F(C_{\alpha_1}, \ldots, C_{\alpha_k})$ in J_γ.

Since $(\mu, <) \to (\omega\text{-path})^2_{\kappa/<\omega}$ we also have that $(\mu, <) \to (\omega\text{-path})^k_{\kappa/<\omega}$ [**10**, Corollary 10]. So pick a finite subset H of κ and an increasing sequence $\alpha_1 < \cdots < \alpha_t < \cdots$ from μ such that H is the set of those γ for which some $\{\alpha_{j+1}, \ldots, \alpha_{j+k}\}$ for a, j in ω is in S_γ.

Since $\{J_\alpha : \alpha < \kappa\}$ is an up directed family, pick a δ in κ such that J_γ is a subset of J_δ for each γ in H. For each positive integer t, put $M_t = C_{\alpha_t}$. Consider the corresponding play $(M_1, N_1, \ldots, M_t, N_t, \ldots)$ of $SMG(J)$ during which TWO used the k-tactic F. By our construction, X is a subset of M_t while N_t is in J_δ for each positive integer t. Hence, TWO lost this play of $SMG(J)$.

(c) Pick an X in $\langle J\rangle$ and an infinite cardinal number λ as in (c). Let μ denote c.n.$(\langle J\rangle, \subset)$ and let κ denote $(2^\lambda)^+$. Let k be a positive integer and let F be a k-tactic for TWO in $SMG(J)$. Construct a chain $\{C_\alpha : \alpha < \mu\}$ in $\langle J\rangle$ such that if $\alpha_1 < \cdots < \alpha_k < \beta < \mu$, then $X \cup F(C_{\alpha_1}, \ldots, C_{\alpha_k}) \cup C_{\alpha_k}$ is a subset of C_β.

The family G consisting of finite unions from the collection $\{F(C_{\alpha_1}, \ldots, C_{\alpha_k}) : \alpha_1 < \cdots < \alpha_k < \kappa\}$ of sets in J has cardinality at the most κ. Pick a subset Y of X which has cardinality at the most λ and which is not a subset of any set in G. Write $\kappa = \cup_{\alpha<\mu} T_\alpha$ where for $\alpha < \beta < \kappa$,

(a) T_α has k elements and

(b) each element of T_α is less than each element of T_β.

Enumerate each T_α in increasing order as $\{\alpha_1, \ldots, \alpha_k\}$ and define a partition $[\kappa]^2 = \cup_{y\in Y} S_y$ so that $\{\alpha, \beta\}$ is in S_y for some y in Y which is not in $F(C_{\alpha_1}, \ldots, C_{\alpha_k}) \cup F(C_{\alpha_2}, \ldots, C_{\alpha_k}, C_{\beta_1}) \cup \cdots \cup F(C_{\beta_1}, \ldots, C_{\beta_k})$. On account of the size of κ we pick (by the Erdös-Rado theorem) a y in Y and a subset A of κ of order type at least $\lambda^+ + 1$ which is homogeneous of class S_y. Pick an increasing sequence $\alpha_1 < \cdots < \alpha_t < \cdots$ from A and enumerate $\cup_{n=1}^\infty T_{\alpha_n} = \{\beta_m : m \text{ is a positive integer}\}$ in increasing order.

For each positive integer t, we put $M_t = C_{\beta_t}$. Consider the corresponding play $(M_1, N_1, \dots, M_t, N_t, \dots)$ of $SMG(J)$ during which TWO used the k-tactic F. By our construction, Y is a subset of M_t while y is not in N_t for each positive integer t. Hence, TWO lost this play of $SMG(J)$. □

Example 1 (continued and expanded). Let κ be an infinite cardinal number and let S be a set of cardinality λ where $\lambda \geq \kappa$. Write $S = A \cup B$ where A and B are pairwise disjoint sets with cardinalities respectively κ and λ. Define J on S so that J_A is $J(\kappa)$ and J_B is a σ-complete free ideal. Since the cofinality of $J(\kappa)$ is κ, $dir(J) \leq \kappa$.

Claim. *The following statements are equivalent.*

(a) *TWO has a winning* 2*-tactic in* $SMG(J)$*.*

(b) *There is a positive integer* k *for which TWO has a winning* k*-tactic in* $SMG(J)$*.*

(c) c.n.$(\langle J\rangle, \subset) \not\to (\omega\text{-path})^2_{\kappa\backslash<\omega}$.

Proof. Only the implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) require proof.

(b) $\Rightarrow$ (c). If (c) fails, it follows from the fact that $dir(J) \leq \kappa$ and Theorem 20(b) that TWO does not have a winning k-tactic in $SMG(J)$ for any positive integer k. (b) $\Rightarrow$ (c) is the contrapositive of this.

(c) $\Rightarrow$ (a). Let λ denote c.n.$(\langle J\rangle, \subset)$ and pick a chain $\{C_\alpha : \alpha < \lambda\}$ in $\langle J\rangle$ such that $A \subset C_\alpha \subset C_\beta$ for $\alpha < \beta < \lambda$ and so that $\cup_{\alpha<\lambda} C_\alpha$ is not in $\langle J\rangle$. Let $\{Y_\beta : \beta < \kappa\}$ be a family of sets in J_A with the property that $A = \cup_{\alpha\in F} Y_\alpha$ for each infinite subset F of κ. Let $[\lambda]^2 = \cup_{\alpha<\kappa} S_\alpha$ be a partition witnessing that $\lambda \not\to (\omega\text{-path})^2_{\kappa\backslash<\omega}$. For each T in $\langle J\rangle$, we let $\alpha(T)$ be the least γ in λ with $(C_\gamma \backslash A) \cap T$ nonempty if such exists, or else let $\alpha(T) = 0$.

We now define a 2-tactic for TWO. So let M and M' be given sets in $\langle J\rangle$ with M a subset of M'. Let $\alpha = \alpha(M)$ and $\alpha' = \alpha(M')$. Then evidently, $\alpha \leq \alpha'$.

Case 1. $\alpha = \alpha'$. Then put $F(M, M') = C_{\alpha'}$.

Case 2. $\alpha < \alpha'$. Then put $F(M, M') = Y_\gamma \cup [(M' \cup C_{\alpha'}) \cap B]$, where γ is minimal with $\{\alpha, \alpha'\}$ in S_γ.

This defines a 2-tactic F of TWO and it follows easily that whenever $(M_1, N_1, \dots, M_t, N_t, \dots)$ is a play of $SMG(J)$ during which TWO used F, then A is a subset of $\cup_{j=1}^{\infty} N_j$ and consequently that this play is won by TWO. The proof is complete. □

Thus, if we let κ be an infinite cardinal number, if λ is at least κ^{++} and if J_B is $[B]^{\leq\kappa}$, then c.n.$(\langle J\rangle, \subset) \not\to (\omega\text{-path})^2_{\kappa/<\omega}$. It follows that TWO has a winning 2-tactic in $SMG(J)$ despite the facts that the cofinality of $\langle J\rangle$ is different from the completeness number of $\langle J\rangle$ and larger than $\aleph_1$. It follows that the sufficient hypotheses of Theorems 18 and 19 are not necessary for the existence of winning 2-tactics in $SMG(J)$.

Example 2 (continued). This example is of particular interest when the cofinality of $\langle J\rangle$ is larger than $\aleph_1$. Solutions to the following two problems may shed some light on this.

Problem 10. Consider models of $ZFC + MA + \neg CH$. Is there such a model in which TWO has a winning 2-tactic in $SMG(J)$? (By Theorem 19, it suffices to ask for the existence of winning 2-tactics.)

Problem 11. Consider a Mathias reals model (see Part 1, Section 5). Does TWO have a winning k-tactic in $SMG(J)$ for some positive integer k in such a model?

Example 3 (continued). We have cardinal numbers λ and κ with $\aleph_0 \leq \lambda < \kappa$ and $J = [\kappa]^{<\lambda}$. For these ideals, we know that $d(J) = dir(J) = \aleph_0$ and that c.n.$(\langle J\rangle, \subset) = \lambda^+$. If λ is bigger than the continuum, Theorem 20 implies that TWO does not have a winning k-tactic in $SMG(J)$ for any positive integer k. If λ is less than the continuum, Theorem 20 (a) and (c) do not apply anymore. Yet Theorem 20 (b) might still apply since it could be that there is a real-valued measurable cardinal below λ, in which case [**10**, Proposition 19] shows that Theorem 20 (b) applies.

We noted earlier that TWO has a winning 2-tactic in $SMG([\aleph_1]^{<\aleph_0})$. As far as we know, this is the present state of knowledge concerning k-tactics in $SMG(J)$ for this class of ideals.

Problem 12. Is there some positive integer k such that TWO has a winning k-tactic in $SMG([\aleph_2]^{<\aleph_0})$?

(At this stage, even consistency results would be illuminating.)

Problem 13. Consider models of ZFC in which $\aleph_\omega$ is less than the continuum. Is it possible that TWO has

(i) a winning 2-tactic in $SMG(([\aleph_{\omega+1}]^{<\aleph_\omega})$?

(ii) a winning k-tactic in $SMG(([\aleph_{\omega+2}]^{<\aleph_\omega})$ for some positive integer k?

5. An application to Banach-Mazur games. In the introduction to this article we briefly introduced the Banach-Mazur game and Debs' example $(\mathbf{R}, \tau)$ of a space for which TWO does not have a winning 1-tactic in $BM(\mathbf{R}, \tau)$. Let J be the ideal of nowhere dense subsets of $\mathbf{R}$ (in the usual topology). It is further known that if the cofinality of $\langle J \rangle$ is $\aleph_1$, then TWO has a winning 2-tactic in $BM(\mathbf{R}, \tau)$. In this section we show the relevance of the game $SMG(J)$ to questions concerning k-tactics in $BM(\mathbf{R}, \tau)$.

Recall that a family $\mathcal{R}$ of subsets of a topological space (X, τ) is a π-base for the topology τ if $\mathcal{R}$ consists of nonempty elements of τ, and each nonempty element of τ contains an element of $\mathcal{R}$. Now for a π-base $\mathcal{R}$ of a topological space (X, τ), the following statements are equivalent:

(a) TWO has a winning k-tactic in $BM(X, \tau)$

(b) TWO has a winning k-tactic in the Banach-Mazur game on (X, τ) where the players are restricted to picking their open sets from the π-base $\mathcal{R}$.

The family $\mathcal{R} = \{I \backslash M : I$ is an open interval of $\mathbf{R}$ and M is a meager set with $M = M + \mathbf{Q}\}$ is a π-base for the space $(\mathbf{R}, \tau)$. The reader should remember our convention that "open," "meager," "nowhere dense," "interior," "closure," and "dense" in the present context are to be understood in the sense of the usual topology, σ, of $\mathbf{R}$ unless it is further qualified by τ.

For A in $\mathcal{R}$, $I(A)$ denotes the interior of the closure of A and $M(A)$ denotes the set $(I(A)\backslash A)+\mathbf{Q}$. The crucial properties of $\mathcal{R}$ are recorded in the next lemma. The proofs are routine and at some point rely on the fact that if X is a subset of $\mathbf{R}$ with $X = X+\mathbf{Q}$, then $X = (X\cap I)+\mathbf{Q}$ for each nonempty open interval I.

Lemma 21. *Let A and A' be in $\mathcal{R}$.*

(a) *If $A = I\backslash X$ where I is an open interval and X is meager and $X = X+\mathbf{Q}$, then $I(A) = I$ and $M(A) = X$.*

(b) *The following statements are equivalent*:

(i) $A \subseteq A'$

(ii) $I(A) \subseteq I(A')$ and $M(A') \subseteq M(A)$.

And now a final remark before proving the main result of this section; when considering a winning k-tactic F of TWO in $SMG(J)$, we may (and will) assume that F has the following two properties:

(i) for all $X_1, \dots, X_k$ in $\langle J\rangle$, $F(X_1, \dots, X_k)\backslash X_k \neq \varnothing$, and

(ii) for each ω-sequence $X_1 \subseteq \cdots \subseteq X_t \subseteq \dots$ of sets in $\langle J\rangle$ with the property that there is a positive integer m such that whenever $j > m$ and $s < j-k$, then $F(X_{s+1}, \dots, X_{s+k}) \subseteq X_j$, it follows that $\cup_{n=1}^{\infty} X_n = \cup_{n=0}^{\infty} F(X_{n+1}, \dots, X_{n+k})$. (Intuitively speaking, this says that F is then also a winning k-tactic for TWO in those plays where ONE plays a weakly monotonically and eventually obeys the rules of $SMG(J)$).

We are now ready to prove

Theorem 22. *Let $k > 1$ be an integer. If TWO has a winning k-tactic in $SMG(J)$, then TWO has a winning k-tactic in $BM(\mathbf{R}, \tau)$.*

Proof. We consider the Banach-Mazur game where both players pick their open sets from the π-base $\mathcal{R}$ defined above. Let $k > 1$ be an integer and let F be a winning k-tactic of TWO in $SMG(J)$ which has the properties stated above. Let G be a winning tactic of TWO in $BM(\mathbf{R},\sigma)$.

Define a k-tactic, H, of TWO in $BM(\mathbf{R},\tau)$ as follows. Let $D_1 \supseteq \cdots \supseteq D_k$ be sets in $\mathcal{R}$. Put $I_j = I(D_j)$ and $M_j = M(D_j)$ for $1 \leq j \leq k$. By Lemma 21 we know that $I_1 \supseteq \cdots \supseteq I_k$ and $M_1 \subseteq \cdots \subseteq M_k$. Pick a nonempty open interval $V(I_1,\dots,I_k) \subseteq G(I_k)$ which is disjoint from $F(M_1,\dots,M_k)$ and put

$$H(D_1,\dots,D_k) = V(I_1,\dots,I_k)\setminus((F(M_1,\dots,M_k)\cup M_k)+\mathbf{Q}).$$

We show that each play $(E_1,N_1,\dots,E_t,N_t,\dots)$ of $BM(\mathbf{R},\tau)$ for which each E_t and each N_t is in $\mathcal{R}$ and with $N_1 = H(E_1,\dots,E_1)$, $N_2 = H(E_1,\dots,E_1,E_2),\dots,$ $N_{j+k} = H(E_{j+1},\dots,E_{j+k})$ for j in ω is won by TWO. This will show that TWO has a winning k-tactic in $BM(\mathbf{R},\tau)$.

Consider such a play and put $M_t = M(E_t)$ and $I_t = I(E_t)$ for each positive integer t. So $E_t = I_t\setminus M_t$ for each positive integer t. Furthermore, let $W_1 = F(M_1,\dots,M_1)$, $W_2 = H(M_1,\dots,M_1,M_2),\dots,$ $W_{j+k} = H(M_{j+1},\dots,M_{j+k})$ for j in ω. An inductive computation, using the properties of F, shows that $M_t \subseteq M_{t+1}$ for each positive integer t and that if t is bigger than k, then in fact $M_t\cup W_t \subseteq M_{t+1}$. Consequently, $\cup_{n=1}^{\infty}M_n = \cup_{n=1}^{\infty}W_n$. Since $(I_1,G(I_1),\dots,I_t,G(I_t),\dots)$ is a play of $BM(\mathbf{R},\sigma)$ during which TWO used the winning 1-tactic G, $\cap_{n=1}^{\infty}I_n$ is nonempty. An inductive computation shows that for each positive integer m, I_{m+1} and W_m are disjoint, and thus $\cap_{n=1}^{\infty}I_n$ and $\cup_{n=1}^{\infty}W_n$ are pairwise disjoint. Consequently, $\cap_{n=1}^{\infty}I_n$ is a subset of $\cap_{n=1}^{\infty}E_n$ whence this latter intersection is nonempty and TWO has won the play under consideration. The proof is complete. □

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