ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

NO CONTINUUM IN E^2 HAS THE TMP; II. TRIODIC CONTINUA

L.D. LOVELAND

ABSTRACT. A subset X of the Euclidean plane E^2 has the triple midset property (TMP) if, for every line segment $[x, y]$ in E^2 such that $\{x, y\} \subset X$, the perpendicular bisector of $[x, y]$ meets X in exactly three points. Resolving the planar aspect of a more general question, the main theorem shows that no compact, connected, nondegenerate subset of E^2 can possess this triple midset property.

1. Introduction. Let (X, ρ) be a metric space, and let x and y be two points of X. The *midset* $M(x, y)$ of x and y is the set of all points m of X such that $\rho(x, m) = \rho(y, m)$. If each of its midsets consists of two points, the metric space X is said to have the *double midset property* (DMP); for example, a circle in the Euclidean plane E^2 has the DMP. It has been conjectured that a continuum with the DMP must be homeomorphic to a simple closed curve, a conjecture which has been confirmed for continua lying in E^2 [**3**]. A metric space in which every midset consists of three points is said to have the *triple midset property* (TMP), but no example is available of a continuum with the TMP. In this paper I show that no such continuum exists in E^2, the space where one might first look for examples.

Although no examples have been found of continua with the TMP, it follows from a theorem of Bagemihl and Erdős [**1**] that there exists a subset of E^2 with the property that its intersection with every line consists of three points. Such a three-point set has the TMP. Mazurkiewicz [**6**] had previously demonstrated the existence of a subset E^2 that meets every line in exactly two points.

Midsets have also been called bisectors [**2**] or equidistant sets [**8, 9**], but, for subsets of Euclidean spaces, it is helpful to distinguish between

Received by the editors on November 25, 1989, and in revised form on February 2, 1990.
1991 AMS *Mathematics Subject Classification.* Primary 54F50, 54F15, Secondary 54F65, 51K05, 51M05.
Key words and phrases. Arc, bisector, continuum, Euclidean plane, equidistant set, midset, simple closed curve, triod, triple midset property.

bisectors and midsets. If a and b are two points of a subset X of E^2, the *bisector* $B(a, b)$ of a and b is the straight line that bisects, and is perpendicular to, the line segment joining a and b, while the *midset* $M(a, b)$ is the intersection of $B(a, b)$ with X.

A *continuum* is a compact, connected metric space containing more than one point, and a *planar continuum* is a continuum that lies in the Euclidean plane E^2 and inherits its metric topology. Also, $E^2 - B(a, b)$ has two components; the one containing the point a is called the *a-side* of $B(a, b)$. The standard Euclidean metric ρ is used for E^2. An *arc* is a continuum homeomorphic to a closed interval $[0, 1]$ on the real line, while a *triod* is any homeomorphic image of the union of the closed intervals $[(-1, 0), (1, 0)]$ and $[(0, 0), (0, 1)]$ in E^2. The image v of $(0, 0)$ in a triod T is called the *vertex* of T while the closures of the components of $T - \{v\}$ are called the *legs* of T. A *simple closed curve* is a homeomorphic image of a circle in E^2.

Lemma 1.1. *A continuum with the* TMP *is locally connected, path connected, and locally path connected.*

An indirect proof of Lemma 1.1 is easily obtained using [**10**, 28D, p. 209] (see [**5**, Lemma 2]). For more details relating to Lemma 1.2, see [**5**, Lemma 3].

Lemma 1.2 [**7**, Theorem 75, p. 218]. *A locally connected continuum is either an arc, a simple closed curve, or it contains a triod.*

The proof of the result in the title breaks into three parts by virtue of Lemma 1.2. The first two of the three parts follow from [**4**]. Theorem 2.1 of [**4**] shows that no simple closed curve in E^2 can have the TMP, and Theorem 3.1 of [**4**] shows that no arc in E^2 can have the TMP. Theorem 2.1 of this paper completes the result because it states that no continuum containing a triod can have the TMP. The summary theorem is given as Theorem 2.2. However, the general question [**5**, Question 4] about the existence of a continuum with the TMP remains open.

An arc or a simple closed curve A is said to *cross a line* L in E^2 at a point m if there are arcs A' and A'' in A such that $A' \cap A'' = \{m\}$

and A' and A'' lie on opposite sides of L. The arc A is said to *bounce off* L at a point m if there is a subarc A' of A such that m lies in the interior of A', $A' \cap L = \{m\}$, and $A' - \{m\}$ lies in a single component of $E^2 - L$. Also A is said to *bounce off a bisector $B(a,b)$ at m to the a-side of $B(a,b)$* if $A' - \{m\}$ lies on the a-side of $B(a,b)$. An arc A is said to *hang to the side* S of a line L at a point $v \in L$ if v is an endpoint of A and there exists a neighborhood V of v such that $(A - \{v\}) \cap V \subset S$.

Frequently used in the sequel, Lemma 1.3 says, roughly, that if $X \subset E^2$ and some bisector B of X has one bounce point and two crossing points of X, then, under certain limit point conditions, there must be a bisector near B that intersects X at least four times. Thus, X cannot have the TMP. A slightly different version of Lemma 1.3 appears as Lemma 2.1 of [**4**]. The proof outlined there establishes Lemma 1.3.3 below, and it is easily modified to prove both Lemma 1.3.1 and 1.3.2.

Lemma 1.3. *If $X \subset E^2$, X has the* TMP, *C is a circle centered at v, $\mathcal{V}$ is a component of $E^2 - C$, a and b are points of $C \cap X$, and X contains three disjoint arcs, two that cross $B(a,b)$ and one, say A, that bounces off $B(a,b)$ at the point v, then:*

1. *a and b cannot both be limit points of $\mathcal{V} \cap X$,*
2. *a cannot be a limit point of both $X \cap \operatorname{Int} C$ and $X \cap \operatorname{Ext} C$, and*
3. *If, in addition, A bounces off $B(a,b)$ to the a-side of $B(a,b)$ at v, then a cannot be a limit point of $X \cap \operatorname{Ext} C$ and b cannot be a limit point of $X \cap \operatorname{Int} C$.*

Open questions and conjectures relating to the TMP, and to sets whose midsets consist of n points, are stated at the end of Section 3 of [**4**] and in [**5**].

2. No triodic continuum in E^2 has the TMP. From Lemmas 1.1 and 1.2, a continuum with the triple midset property is either an arc, a simple closed curve, or it must contain a triod. Theorem 2.1 rules out the latter for plane continua.

Theorem 2.1. *If a continuum in E^2 has the* TMP, *then it cannot contain a triod.*

Proof. Suppose X is a continuum in E^2 such that X contains a triod Υ and X has the TMP. Let the vertex of Υ be v, and let W_i, for $i = 1, 2, 3$, denote its legs. For each $r > 0$ let $C(r)$ denote the circle of radius r which has its center at v, and choose $u > 0$ such that $C(u) \cap W_i \neq \varnothing$, for each i. Then Υ contains a triod T with vertex v and legs L_i such that, for $i = 1, 2, 3$, $L_i \subset W_i$ and L_i has an endpoint e_i such that $L_i \cap C(u) = \{e_i\}$. For $p \in E^2$, let $C(p) = C(\rho(v, p))$. If x and y are two points of X, then a point c of $M(x, y) - \{v\}$ is called a *bad point* of $B(x, y)$ if no arc in X crosses $B(x, y)$ at c. The proof is broken into a sequence of 13 enumerated assertions, the last of which gives the contradiction.

(1). For $v' \in X$, define $\beta(v')$ to be the collection of all bisectors $B(x, y)$ such that $\{x, y\} \subset X$, $v' \in B(x, y)$, and some point of $M(x, y)$ is a bad point of $B(x, y)$. Then $\beta(v')$ is countable.

Proof of (1). Suppose there exists $v' \in X$ such that $\beta(v')$ is uncountable. Let $\beta(v') = \beta$. For each $B \in \beta$, there exists an interval $I(B) \subset B$ such that the midpoint c of $I(B)$ is a bad point of B. The collection $\beta' = \{I(B) : B \in \beta\}$ is an uncountable collection of pairwise disjoint intervals, each of which lies on a line through v', so there must exist a sequence $\{I_i\}$ of intervals in β' which converges to an interval I_0 in β' in such a way that I_{2n} and I_{2n+1} lie on opposite sides of the bisector B_0 containing I_0, for each n. Let c_i be the midpoint of I_i for $i = 0, 1, 2, \ldots$, and note that $\{c_i\}$ converges to c_0. Let N be a neighborhood of c_0 such that N contains no endpoints of any interval I_i and $N \cap X$ is path connected (see Lemma 1.1). Let A be a path in $N \cap X$ joining two points, c_{2n} and c_{2n+1}. Then A contains an arc that crosses B_0 at c_0, contradicting the fact that c_0 is a bad point of B. It follows that β' and β are countable sets, and (1) follows.

(2). If, for some $r > 0$, $C(r) \cap X$ contains an arc and c is a point of $C(r) \cap X$, then c cannot be a limit point of both $X \cap \operatorname{Int} C(r)$ and $X \cap \operatorname{Ext} C(r)$.

Proof of (2). Suppose there exists $r > 0$, an arc $A \subset C(r) \cap X$, and a point $c \in X \cap C(r)$ such that c is a limit point of both $X \cap \operatorname{Int} C(r)$ and

$X \cap \mathrm{Ext}\, C(r)$. Let $m \in \mathrm{Int}\, A$ such that $m \neq c$. Since X must contain an arc that bounces off $B(m, c)$ at v and X has the TMP, it follows from Lemma 1.3.2 that $B(m, c)$ contains a bad point p. Then X contains an arc E which hangs to the, say, c-side of $B(m, c)$ at p. Choose a point $m_1 \in A$ such that E crosses $B(m_1, c)$, and, using Lemma 1.3 again, let p_1 be a bad point of $B(m_1, c)$. Then an arc E_1 exists in $X - E$ such that $p_1 \in E_1$. Choose a point $m_2 \in A$ such that E and E_1 each cross $M(m_2, c)$. But this contradicts Lemma 1.3.2 because $B(m_2, c)$ must contain, in addition to these two crossing points, the point v where an arc in X bounces off $B(m_2, c)$. This proves (2).

(3). If, for some $r > 0$, $C(r) \cap X$ contains an arc A, then the endpoints of A cannot both be limit points of either $X \cap \mathrm{Int}\, C$ or of $X \cap \mathrm{Ext}\, C$.

Proof of (3). The proof is much the same as for (2).

(4). If $r \in (0, u)$, then each component of $C(r) \cap X$ is a point.

Proof of (4). Suppose there exists an $r \in (0, u)$ such that $C(r) \cap X$ contains an arc. It follows from (2) that, for each $i \in \{1, 2, 3\}$, there exists an arc θ_i such that $\theta_i \subset L_i \cap C(r)$, and from (3) that one endpoint a_i of θ_i is a limit point of $L_i \cap \mathrm{Int}\, C(r)$ and the other endpoint b_i of θ_i is a limit point of $L_i \cap \mathrm{Ext}\, C(r)$. If each θ_i is given an orientation from a_i to b_i, then some two of them, say θ_1 and θ_2, must have the same orientation, and with no loss in generality it may be assumed that $a_1 b_1 a_2 b_2$ is the clockwise orientation on $C(r)$. Let ω be the clockwise rotation about v such that $\omega(a_1) = a_2$, and choose an arc Ψ in $\omega(\theta_1) \cap \theta_2$ such that $a_2 \in \Psi$. For each $x \in \Psi$, $B(a_1, x)$ separates a_1 from a_2 and there exists an arc in T that bounces off $B(a_1, x)$ at v. For $i \in \{1, 2\}$, define $\Psi_i = \{x \in \Psi :$ there is an arc $A_x \subset T$ such that A_x bounces off $B(a_1, x)$ at v to the a_i-side of $B(a_1, x)\}$.

Suppose Ψ_2 is dense in some open subset Φ of Ψ, and let $x \in \Phi$. From Lemma 1.3.3 there is an arc P in X such that P hangs off $B(a_1, x)$ to one side at a point $p \neq v$. Choose $y \in \Phi \cap \Psi_2$ such that P crosses $B(a_1, y)$ and, using Lemma 1.3.3 again, let P' and Q be arcs in X such that Q hangs off $B(a_1, y)$ at a point q where $q \neq v$, $P' \subset P$, P' crosses $B(a_1, y)$, and $P' \cap Q = \varnothing$. Choose $z \in \Phi \cap \Psi_2$ such that both P' and Q

cross $B(a_1, z)$. Since $z \in \Psi_2$, there is an arc A_z in X that bounces off $B(a_1, z)$ at v to the a_2-side of $B(a_1, z)$. But this contradicts Lemma 1.3.3 since a_1 is a limit point of $X \cap \operatorname{Int} C$. It follows that Ψ_2 is not dense in any open subset of Ψ.

Since $\Psi = \Psi_1 \cup \Psi_2$, Ψ_1 must be dense in Ψ. From the definition of Ψ, for each $x \in \Psi_1$, the point $\omega^{-1}(x) = x' \in \theta_1$ has the property $B(a_1, x) = B(x', a_2)$. This means that there is an open subset Φ' of θ_1 and a dense subset Ψ_1' of Φ' such that, for $x' \in \Psi_1'$, there is an arc $A_{x'}$ in T that bounces off $B(x', a_2)$ at v to the a_1-side of $B(x', a_2)$. Then, because a_2 is a limit point of $X \cap \operatorname{Int} C$, a contradiction to Lemma 1.3.3 is obtained just as in the previous paragraph, and (4) follows.

An arc A is said to *span an open annulus* U if $\operatorname{Int} A \subset U$ and the endpoints of A lie in different components of BdU. An *annulus at a point* v' is the open annulus between two circles centered at v'. If $v' \in X$ and $r > 0$, let $C(r, v')$ denote the circle at v' with radius r.

(5). If X contains a triod T' with vertex v', U is an annulus at v', u' is a number such that $C(u', v')$ intersects each leg of T', and there exist arcs A_1, A_2, A_3 in $X \cap \operatorname{Int} C(u', v')$, each spanning U, such that, for every $x \in A_1$, points $y \in A_2$ and $z \in A_3$ exist with the following properties:

(a) $\rho(v', x) = \rho(v', y) = \rho(v', z)$ and

(b) $\{B(x, y), B(y, z), B(x, z)\} \subset \beta(v')$,

then there exists an annulus U' in U at v' and three arcs X_1, X_2, X_3 such that, for each i, $X_i \subset A_i$, X_i spans U', and $X_i \cup \{v'\}$ lies in a straight line.

Proof of (5). Since $\beta(v')$ is countable by (1), the collection β' of all ordered triples of elements of $\beta(v')$ is also countable. Let T_i, $i = 1, 2, 3, \ldots$, denote the elements of β', and, for each i, define M_i to be the set of all points $x \in A_1$ such that there exist $y \in A_2$ and $z \in A_3$ with $(B(x, y), B(y, z), B(x, z)) = T_i$. By hypothesis, $A_1 = \cup M_i$, and it is not difficult to prove that each M_i is closed. A Baire category theorem [**10**, 25, p. 185] shows the existence of an integer n and an arc X_1' in A_1 such that $X_1' \subset M_n$. Since v' is the vertex of a triod in X and $C(u', v')$ intersects each leg of T', (4) can be applied at the vertex v' to see that X_1' cannot lie in a circle at v'. This means that there exists an open annulus U' in U at v' and a subarc X_1 of X_1' such that X_1 spans U'. Let $(B_1, B_2, B_3) = T_n$, and let R_i, for $i \in \{1, 2, 3\}$, be the reflection of E^2 in B_i. If the composition R_2R_1 is denoted by R, a rotation about v', then $R_3R_2R_1(x) = R_3R(x)$ for $x \in X_1$. However, the reflection R_3 changes the orientation of three noncollinear points, so the set $X_1 \cup \{v'\}$ must be collinear. Let $X_2 = R_1(X_1)$ and $X_3 = R_2(X_2)$ to complete (5).

(6). If v' is the vertex of a triod T' in X such that two legs of T' lie in $C(u) \cup \operatorname{Int} C(u)$, then $v = v'$.

Proof of (6). Suppose X contains a triod T' as in the statement of (6) such that $v' \neq v$. There exist open annulus U at v' and three disjoint arcs A_1, A_2, A_3 in distinct legs of T' that span U such that $A_1 \cup A_2 \subset C(u) \cup \operatorname{Int} C(u)$. Applying (4) at the point v', one can also insist that, for each circle C' centered at v' and lying in U, each component of each $A_i \cap C'$ is a point. Let $x \in \operatorname{Int} A_1$, and let C' be the circle at v' with radius $\rho(v', x)$. Then x must be a limit point of $\mathcal{V} \cap X$ where $\mathcal{V}$ is either $\operatorname{Int} C'$ or $\operatorname{Ext} C'$. Since each component of $A_i \cap C'$ is a point, there must exist points $y \in A_2 \cap C'$ and $z \in A_3 \cap C'$ such that each is a limit point of $X \cap \mathcal{V}$. By Lemma 1.3.1, each of $B(x, y)$, $B(x, z)$, $B(y, z)$ has a bad point, so the hypothesis of (5) is satisfied. By the conclusion of (5) it may be assumed that each arc A_i lies in a line through v'. One of A_1 or A_2, say A_1, must fail to lie on the line through v and v'. Using (4), Lemma 1.3.1, and the fact that the segment A_1 must have its interior in $\operatorname{Int} C(u)$, choose an annulus U' at v and three disjoint arcs A_1', A_2', A_3' in X spanning U' such that $A_1' \subset A_1$ and conditions (a) and (b) of (5) are satisfied relative to the point v. Then, from (5), there must exist a subarc X_1 of A_1' such that $X_1 \cup \{v\}$ lies on a line. But $X_1 \subset A_1$ and the segment A_1 does not lie on a line through v. This contradiction establishes (6).

(7). If L is an arc in $X \cap (C(u) \cup \operatorname{Int} C(u))$ such that v is an endpoint of L and $0 < t < u$, then $C(t) \cap L$ cannot contain three points.

Proof of (7). Suppose there exists $t \in (0, u)$ such that $C(t) \cap L$ contains three points. From (4) each component of $L \cap C(t)$ is a point, so an annulus U exists at v such that L contains three disjoint arcs A_1, A_2, A_3 each spanning U and $U \subset \operatorname{Int} C(u)$. In the order on L with v as the first point, assume $A_1 < A_2 < A_3$. From (4) and Lemma 1.3.1, as used in the proof of (6), one sees that (5) applies to these arcs, and, using (5), it may be assumed that each of the sets $A_i \cup \{v\}$ lies in a straight line. Let r and s be such that $BdU = C(r) \cup C(s)$ with $r < s$. For $i \in \{1, 2, 3\}$, let a_i be the point of $A_i \cap C(s)$, let $B_i = B(a_i, a_{i+1})$ (mod 3), and let p be the point where L crosses B_3. Impose a rectangular coordinate system such that v is the origin, B_3 is the x-axis, and p has a positive x-coordinate. For convenience, assume also that A_1 lies above the x-axis, and, using (6), assume $(L_2 \cup L_3) \cap L = \{v\}$.

Let W and W^* be the two open sectors of E^2 at v defined by the two rays from v through A_1 and A_3 such that $p \in W$. Then $A_2 \subset W$ because if $A_2 \subset W^*$ and $p \in W$, then L would cross either B_1, B_2, or B_3 twice, contrary to Lemma 1.3.2. For similar reasons, no B_j, $j \in \{1, 2, 3\}$, can separate $A_1 \cup A_3$ from A_2. Then each B_j intersects both W and W^*. From this it can be deduced that $B_1 \cup B_2$ separates $A_1 \cup A_3$ from $B_3 - \{v\}$. Let $B_3 \cap X = \{p, q, v\}$, let P be an arc in $L - \{v\}$ that crosses B_3 at p, and observe that, from Lemma 1.3.2, no arc in X can cross B_3 at q. To see that no arc in X can bounce off B_3 at q, suppose G is such an arc. Because A_1 and A_3 are segments, there is a circle C at q and points $a \in C \cap A_1$ and $b \in C \cap A_3$ such that a and b are limit points of $A_1 \cap \operatorname{Int} C$ and $A_3 \cap \operatorname{Int} C$, respectively. But this contradicts Lemma 1.3.1 because X crosses $B(a, b)$ at v, P crosses $B(a, b)$, and G bounces off $B(a, b)$ at q. Therefore, q must be an endpoint of each arc in X that contains it.

Suppose $q \in L$. Since $B_1 \cup B_2$ separates $A_1 \cup A_3$ from $B_3 - \{v\}$ and $q \in B_3 - \{v\}$, L must cross either B_1 or B_2 twice. But this contradicts Lemma 1.3, so $q \notin L$. For the same reason q cannot belong to any arc in $X - \{v\}$ that contains L. Let L' be an arc in X from v to q, and note that $L' \cap L = \{v\}$ from (6). With no loss in generality, assume $L_3 \not\subset L'$ since L' cannot contain both L_2 and L_3. Then $L \cap L' = \{v\}$, $L \cap L_3 = \{v\}$, and $L' \cap (L_3 - \{e_3\}) = \{v\}$.

For points x and y on the x-axis, let "$x < y$" refer to the usual order of their x-coordinates; therefore, $v < p$. Suppose $p < q$. Then an arc in L from A_1 to A_3 separates q from v in the closure of W. By Lemma 1.3, L' cannot cross either B_1 or B_2, so L' must intersect $L - \{v\}$. But this contradicts the previous paragraph, and $q < p$. Let σ denote the degree measure of the angle BdW, and let K be the circle of radius $\rho(q, a_1)$ centered at q.

Suppose $q \in \operatorname{Int} C(u)$. Then $q \notin L_i$ for $i \in \{1, 2, 3\}$ because the endpoints of the L_i's lie in $C(u)$. Suppose further that $L_2 \subset L'$, as could happen if L' goes through e_2 before getting to q. Choose u' such that $0 < u' < u$ and $C(u')$ separates q from e_2. Then there exists a component G of $L' - C(u')$ with endpoints x and y in $C(u')$. By Lemma 1.3.1, $B(x, y)$ cannot be crossed by L since it is already crossed by G. This means that $B(x, y)$ cannot intersect both W and W^*. For the same reason, $B(x, y) \not\subset W \cup \{v\}$, so $B(x, y) \subset W^* \cup \{v\}$. Then $\sigma < 90°$, $q < v$ because L' cannot cross B_1 or B_2, and $A_1 \cup A_3 - \{a_1, a_3\} \subset \operatorname{Int} K$. Suppose $L' - \{q, v\}$ lies on the A_1-side of B_3, and choose a point a_3' near a_3 and in A_3 such that $B(a_1, a_3')$ intersects P near p, intersects L' near q, and intersects both L and L' near v. This is possible since a_3' lies in both $\operatorname{Int} C(\rho(v, a_3))$ and $\operatorname{Int} K$. But this contradicts the TMP, so $L' - \{q, v\}$ lies on the A_3-side of B_3. Since L crosses B_1, L' cannot cross it by Lemma 1.3.1, so $L' - \{v\}$ lies on the q-side of B_1. But then $B(x, y)$ is trapped between the x-axis and B_1, which means that $B(x, y)$ intersects W. However, $B(x, y) \subset W^* \cup \{v\}$, so this contradiction shows that $L_2 \not\subset L'$.

Then $L \cup L' \cup L_2 \cup L_3$ contains four arcs whose pairwise intersections are either $\{v\}$ or $\{v, e_3\}$. If any three of the four hang off B_3 to the same side at v, then, for x and y carefully chosen in A_1 and A_3, respectively, $B(x, y)$ would intersect all three of these arcs near v and would also intersect P near p. This would contradict the TMP so, of the four arcs, two hang to each side of B_3 at v. But this results in the same contradiction to the TMP because points $x \in A_1$ and $y \in A_3$ can be chosen so that $B(x, y)$ is close enough to B_3, with $v \notin B(x, y)$, that $B(x, y)$ intersects L' near q and L near p as well as intersecting two of the four distinct arcs in X hanging off B_3 at v. The supposition is that $q \in \operatorname{Int} C(u)$ led to this contradiction; therefore, $q \in C(u) \cup \operatorname{Ext} C(u)$.

Since $p \in \operatorname{Int} L \subset C(u) \cup \operatorname{Int} C(u)$, $q < p$, and $q \in C(u) \cup \operatorname{Ext} C(u)$, it follows that $q < v < p$ and $\rho(v, p) \le u \le \rho(v, q)$. Also, since L cannot cross B_3 except at p and $A_1 \subset L$, L must hang off B_3 at v to the A_1-side of B_3.

Suppose $\sigma \leq 90^\circ$. Then, since $q < v$, it follows that $A_1 \cup A_3 - \{a_1, a_3\} \subset \operatorname{Int} K$, and the argument given in the fifth paragraph of this proof of (7) shows the existence of a point a_3' in A_3 such that $B(a_1, a_3')$ intersects X four times if $L' - \{q, v\}$ lies on the A_1-side of B_3. It follows that $L' - \{q, v\}$ lies on the A_3-side of B_3. The same argument, but using a point a_1' near a_1 and in A_1, shows that $B(a_1', a_3)$ would intersect X four times if L_3 hangs below the x-axis at v. Because neither L_3 nor L' can cross either B_1 or B_3, and because $e_3 \notin L$ by (6), L_3 and L' must each hang off B_1 at v to the A_1-side of B_1 at v. But L must also hang off B_1 at v to the A_1-side of B_1, so there exists a point $a_2' \in \operatorname{Int} A_2$, near a_2, such that $B(a_1, a_2')$ intersects each of L, L', and L_3 near v. However, $B(a_1, a_2')$ also intersects L at a point between A_1 and A_2. Since this contradicts the TMP, $\sigma > 90^\circ$.

Since $\sigma > 90^\circ$, A_1 and A_3 lie in the second and third quadrants, respectively. Also, because B_1 intersects both W and W^* and $A_2 \subset W$, A_2 lies either in the first or fourth quadrants. Using this together with $\sigma > 90^\circ$ and $q < v$, choose a vertical line H that separates $\{q\} \cup A_1 \cup A_3$ from $\{v\} \cup A_2$. Then H intersects L three times, once between v and A_1, again between A_1 and A_2, and a third time between A_2 and A_3. Since H also intersects L', $H \cap X$ contains four points. The object in Case 1 below is to show that H can be chosen close enough to the y-axis that it is a bisector for some two points of X, which contradicts the TMP. Let R denote the reflection of E^3 in H.

Case 1. Assume $\rho(p, v) < \rho(q, v)$. In this case it is clear that H can be chosen such that $R(v) < v < p < R(q)$, which means that $R(L')$ must intersect L at a point x. Then $H = B(x, R^{-1}(x))$ and the contradiction follows.

Case 2. Assume $\rho(p, v) = \rho(q, v)$; that is, $\{p, q\} \subset C(u)$. In this case the y-axis is $B(p, q)$ and L crosses $B(p, q)$ twice, once between A_1 and A_2 and again between A_2 and A_3. By the TMP neither L nor L' can intersect $B(p, q)$ except at v and the two points of $L \cap B(p, q)$. This means $L' \cup L$ bounces off $B(p, q)$ to the q-side at v. By (2) there can be no arc in $X \cap C(u)$ with p in its interior; therefore, since $L \subset C(u) \cup \operatorname{Int} C(u)$, p must be a limit point of $X \cap \operatorname{Int} C$. But this contradicts Lemma 1.3.3, and (7) follows.

(8). For $i \in \{1,2,3\}$ and $t \in (0,u]$, $C(t) \cap L_i$ is a point.

Proof of (8). Suppose (8) is false. Then there exists $i \in \{1,2,3\}$ and $t \in (0,u)$ such that $L_i \cap C(t)$ contains two points. But since $C(t)$ separates the endpoints of L_i, there must exist t' near t such that $L_i \cap C(t')$ contains three points. This contradicts (7) and (8) follows.

(9). For $i \in \{1,2,3\}$, L_i is a straight line segment.

Proof of (9). From Lemma 1.3 and (8), it follows that each bisector $B(x,y)$, for $\{x,y\} \subset C(r) \cap T$, lies in $\beta(v)$. From (8), the collection $F_{12} = \{B(x,y) : x \in L_1, y \in L_2, \{x,y\} \subset C(r), \text{ and } 0 < r \leq u\}$ is a continuous family of lines in $\beta(v)$. If F_{12} contains two distinct bisectors, then it contains uncountably many between them. Therefore, since $\beta(v)$ is countable by (1), F_{12} consists of a single line B_1. The analogous sets F_{23} and F_{13} similarly consist of single lines B_2 and B_3, respectively. As in the last part of the proof of (5), this means that L_1 transposes to itself under the composition of a rotation about v and a reflection in B_3. Unless L_1 is a line segment, this is a contradiction, and, similarly, L_2 and L_3 must be segments.

The only restriction on u in order for the corresponding triod T to have its three legs on straight lines, as in (9), was that $C(u)$ meet each leg of Υ. Let $\Gamma = \{u : C(u) \text{ meets all three legs of } \Upsilon\}$. Because X is compact, Γ has a least upper bound μ, and since Υ is compact, it follows from (9) that $\mu \in \Gamma$. By enlarging Υ if possible, it may be assumed that for $t > \mu$, there is no triod T' in X having vertex v such that $\Upsilon \subset T'$ and all three legs of T' meet $C(t)$. In the sequel, let $C = C(\mu)$, let T be this maximal straight-legged triod, and, for each i, let L_i denote a leg of T. For $i \in \{1,2,3\}$, let θ_i denote the component of $X \cap C$ containing the endpoint e_i of L_i.

(10). The components θ_1, θ_2, and θ_3 of $X \cap C$ are pairwise disjoint.

Proof of (10). Suppose, for example, that θ_1 and θ_2 intersect. Then $\theta_2 = \theta_1$. With no loss in generality assume that $L_2 \cap L_3$ bounces off $B(e_1,e_2)$ to the e_2-side of $B(e_1,e_2)$ at v, and, using (9), choose $x \in \theta_2$ near enough to e_2 that $L_2 \cup L_3$ bounces off $B(e_1,x)$ at v to the x-side of $B(e_1, x)$. Let $M(e_1, x) = \{v, p_1, p_2\}$ where $p_1 \in \theta_2$. From Lemma 1.3.3, there must exist an arc P such that $p_2 \in P$ and P hangs to one side of $B(e_1, x)$ at p_2. Choose a point $y \in \theta_2$ near x such that P crosses $B(e_1, y)$. But θ_2 also crosses $B(e_1, y)$ and $v \in B(e_1, y)$. This contradicts Lemma 1.3.3, and (10) follows.

In view of (10) and Lemma 1.1, it would violate the maximality of T and μ for all three θ_i's to contain limit points of $X \cap \operatorname{Ext} C$, so let L_1 be a leg of T such that no point of θ_1 is a limit point of $X \cap \operatorname{Ext} C$. Using (6), let e_1 and e_1' be the endpoints of θ_1 with $e_1 = e_1'$ if $\theta_1 = \{e_1\}$.

(11). The endpoint e_1' of $L_1 \cup \theta_1$ is not a limit point of $X - (L_1 \cup \theta_1)$.

Proof of (11). Suppose (11) is false. Then e_1' must be a limit point of $(X - L_1) \cap \operatorname{Int} C$, and there is an arc A in $X \cap \operatorname{Int} C$ such that A has endpoints e_1' and x, where $x \in \operatorname{Int} C$, and $A \not\subset L_1 \cup \theta_1$. By (6), $A \cap (L_1 \cup \theta_1) = \{e_1'\}$, so $L_1 \cup \theta_1 \cup A$ is an arc. Choose r such that $\rho(v, x) < r < \mu$. Since $C(r)$ intersects L_1, it follows from (7) that $C(r) \cap A$ consists of one point. Then, for $a \in (A - \{e_1', x\}) \cap C(r)$, a is a limit point of both $A \cap \operatorname{Int} C(a)$ and $A \cap \operatorname{Ext} C(a)$, and the proof of (9) applies, where A replaces L_1, to show that $A \cup \{v\}$ lies in a straight line. From this, $e_1 \neq e_1'$. Since an arc in $L_1 \cup L_2 \cup L_3$ bounces off $B(e_1, e_1')$ at v, the proof of (10) applies here to show that no component of $C \cap X$ can contain both e_1 and e_1'. This contradiction establishes (11).

To establish the contradiction in (13), it seems necessary to strengthen (6) as in (12) below.

(12). If T' is a triod in X and v' is its vertex, then $v = v'$.

Proof of (12). Suppose T' is a triod in X such that the vertex v' of T' is not v. For $r > 0$ let $K(r)$ denote the circle of radius r centered at v'. By (9) there must be two legs, say L_1 and L_2, of T and an open annulus V at v' such that $v \in BdV$ and, for every r such that $K(r) \subset V$, $K(r)$ intersects both L_1 and L_2. Let $\{r_i\}$ converge to $\rho(v, v')$ such that $K(r_i) = K_i \subset V$ and, for each i, let $\{a_i\} = L_1 \cap K_i$, $\{b_i\} = L_2 \cap K_i$, and let $B(a_i, b_i) = \{x_i, y_i, v'\}$ where $x_i \in L_1 \cup L_2$. Note that $\{B(a_i, b_i)\}$ converges to the line M through v and v' and that $\{x_i\}$ converges to v. Let S and S' be the two sides of M, and assume $L_2 \cup L_3 - \{v\} \subset S$.

Suppose $L_1 - \{v\} \subset S$. Choose two points a_i and b_i close enough to v that $B(a_i, b_i)$ separates $\{e_1, e_2, e_3\}$ from $\{v\}$. Then, in contradiction to the TMP, $M(a_i, b_i)$ contains four points, one in each L_i and the point v'. Thus, $L_1 - \{v\} \subset S'$.

Suppose L_1 and L_2 are symmetric about the line M. Then, for each i, $B(a_i, b_i) = M$, $x_i = v$ and $y_i = y$ where $M \cap X = \{v, v', y\}$. Fix points a and b in $\operatorname{Int} L_1$ and $\operatorname{Int} L_2$, respectively, such that $B(a, b) = M$, and let $K = K(\rho(v', a))$. If all three legs of T' hang to the same side of M at v', then a bisector B near M is easily found such that B intersects all three legs of T' and B intersects at least one leg of T. Since this contradicts the TMP, an arc in T' must cross M at v'. Let A be an arc in X hanging off M at y, and let Y be the circle at y with radius $\rho(y, a)$. Adjust the points a and b, if necessary, so that the lines through L_1 and L_2 are not tangent to either circle K or Y, which means that L_1 and L_2 each cross all three circles K, Y, and $C(\rho(v, a))$ at a and b, respectively. In the subsegments (v, a) and (v, b) of L_1 and L_2 choose points a' and b' near a and b, respectively. There are three cases depending on the order of y, v and v' on M, but the orders yvv' and $yv'v$ are really the same since the previous assertions apply to v' as well as to v. Let $I = (\operatorname{Int} K) \cap (\operatorname{Int} Y)$, $E = (\operatorname{Ext} K) \cap (\operatorname{Ext} Y)$, and note that, with the order yvv' eliminated, either $v \in I$ or $v \in E$. Assume $v \in I$. Then $\{a', b'\} \subset I$, and all three of v, v', and y lie on the same side of both $B(a', b)$ and $B(a, b')$. But this contradicts the TMP because one of these two bisectors must intersect $T \cup A \cup T'$ four times. Therefore, $v \in E$, which means that $\{a', b'\} \subset E$. But then $B(a', b) \cup B(a, b')$ separates $\{y, v'\}$ from $\{v\}$, and there is no way to arrange the seven arcs in $T \cup A \cup T'$ without one of $B(a', b)$ or $B(a, b')$ intersecting four of them. Since this contradicts the TMP, L_1 and L_2 are not symmetric about M.

Since L_1 and L_2 are not symmetric about M and each of $\{a_i\}$ and $\{b_i\}$ converge to v, it follows from (9) that $\{B(a_i, b_i)\}$ is a sequence of distinct lines converging to M. If infinitely many x_i belong to L_2, then, for i sufficiently large, both L_2 and L_3 would cross $B(a_i, b_i)$, contrary to Lemma 1.3. Then, for convenience, assume that $x_i \in L_1$ for each i. Let $y \in M$ be a limit point of $\{y_i\}$. For each i there exists an arc A_i in X joining y_i to y. For i sufficiently large, it follows from (6) that A_i misses $L_1 - \{v\}$. Then A_i and L_1 are two distinct arcs each of which must cross $B(a_j, b_j)$ for some j. If $K' = K(\rho(v', a_j))$, then a_j and

b_j are limit points of both $X \cap \operatorname{Int} K'$ and $X \cap \operatorname{Ext} K'$, contradicting Lemma 1.3. (12) follows.

(13). X cannot have the TMP.

Proof of (13). Let $\mathcal{L} = L_2 \cup L_3$. In the first of two similar cases, suppose that $e_1' \notin B(e_2, e_3)$, and let p and q be points of $\mathcal{L}$ such that $M(p,q) = \{x, y, e_1'\}$, where x lies between p and q in $\operatorname{Int} \mathcal{L}$. Then $x \neq v$. Assume p and q are named in such a manner that, for p' near p and between p and q on $\mathcal{L}$, $B(p', q) \cap \theta_1 = \varnothing$. Let $\{p_i\}$ converge to p on $\mathcal{L}$ such that, for each i, p_i lies between p and q on $\mathcal{L}$, $B(p_i, q) = \{x_i, y_i, z_i\}$, and $\{x_i\}$ converges to x on $\mathcal{L}$. From (6), (9), (11), and $x \neq v$, it follows that neither x nor e_1' lies in the limiting set of $\{y_i, z_i\}$. Then $\{y_i, z_i\}$ converges to y, and, from Lemma 1.1 and (6), $y \notin L_1 \cup \theta_1$. Using (12), let α be an arc in $X - (\mathcal{L} \cup L_1)$ such that $\alpha \cap B(p,q) = \{y\}$, $\{y_i, z_i\} \subset \alpha$ for all but finitely many i, and $\alpha - \{y\}$ lies in one side S of $B(p,q)$. Choose a sequence $\{q_i\}$ of points between p and q in $\mathcal{L}$ that converges to q such that, for each i, $M(p, q_i) = \{x_i', y_i', z_i'\}$, $\{x_i'\}$ converges to x, and $\{z_i'\}$ converges to e_1'. Since $x \neq v$, it follows from (11) that $\{y_i'\}$ converges to y. Clearly, $y_i' \notin S$. Using (12) again, let α' be an arc in X such that $\alpha' \cap B(p,q) = \{y\}$, α' contains all but finitely many y_i', and $\alpha' \subset E^2 - S$. Then α cannot bounce off $B(p,q)$ at y because, by (12), $\alpha \cup \alpha'$ cannot contain a triod. The other case being similar, it may be assumed that S is the q-side of $B(p,q)$. Let d be the endpoint of α such that $d \neq y$, and choose n such that $B(q, p_n)$ separates y from d. Since $B(q, p_n) \cap \mathcal{L} = \{x_n\}$, the TMP ensures that $\alpha \cap B(q, p_n) = \{y_n, z_n\}$. Thus, α must cross $B(q, p_n)$ at one point, say y_n, and must bounce off $B(q, p_n)$ at the other point z_n. Choose a point w in $\mathcal{L}$ near enough to, and on the appropriate side of, p_n that $B(q, w)$ intersects α twice near z_n and again near y_n. Since $B(q, w)$ also must intersect $\mathcal{L}$, this contradicts the TMP.

This leaves the case where $e_1' \in B(e_2, e_3)$. However, L_1 cannot lie in $B(e_2, e_3)$, so $e_1 \notin B(e_2, e_3)$. In this case points p and q exist in $\mathcal{L}$ such that $M(p,q) = \{e_1, x, y\}$, $x \in \mathcal{L}$, and p and q are named such that, for p' near p and between p and q on $\mathcal{L}$, $B(p', q) \cap (L_1 \cup \theta_1) = \varnothing$. Under this new definition of x, $x \neq v$. It follows from (6), and the fact that no point of θ_1 is a limit point of $X \cap \operatorname{Ext} C$, that e_1 is not a limit point of $X - (L_1 \cup \theta_1)$. From these two facts and the technique of the previous paragraph, a contradiction to the TMP is obtained. Because (13) follows, Theorem 2.1 is established.

Theorem 2.2. *No continuum in E^2 can have the triple midset property.*

Proof of Theorem 2.2. From Theorems 2.1 and 3.1 of [**4**], neither an arc nor a simple closed curve in E^2 can have the TMP, and from Theorem 2.1 no continuum in E^2 that contains a triod can have the TMP. Therefore, Theorem 2.2 follows from Lemma 1.2.

Note added in proof. Theorem 2.2 was recently proved in more general form by the author and S.M. Loveland.

REFERENCES

1. F. Bagemihl and P. Erdős, *Intersections of prescribed power, type, or measure*, Fund. Math. **41** (1954), 57–67.

2. A.D. Berard, Jr. and W. Nitka, *A new definition of the circle by use of bisectors*, Fund. Math. **85** (1974), 49–55.

3. L.D. Loveland, *The double midset conjecture for continua in the plane*, Top. Applications **40** (1991), 117–129.

4. ———, *No continuum in E^2 has the TMP*; I. *Arcs and spheres*, Proc. Amer. Math. Soc. **110** (1990), 1119–1128.

5. L.D. Loveland and S.G. Wayment, *Characterizing a curve with the double midset property*, Amer. Math. Monthly **81** (1974), 1003–1006.

6. S. Mazurkiewicz, *Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs*, C.R. Sc. et Letters de Varsovie **7** (1914), 382–383.

7. R.L. Moore, *Foundations of point set theory*, revised ed., Amer. Math. Soc. Colloq. Publ. **13**, 1962.

8. Sam B. Nadler, Jr., *An embedding theorem for certain spaces with an equidistant property*, Proc. Amer. Math. Soc. **59** (1976), 179–183.

9. J.B. Wilker, *Equidistant sets and their connectivity properties*, Proc. Amer. Math. Soc. **47** (1975), 446–452.

10. Stephen Willard, *General topology*, Addison-Wesley, Reading, MA, 1970.

Department of Mathematics, Utah State University, Logan, UT 84322–3900.