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$\ell^{p,\infty}$ HAS A COMPLEMENTED SUBSPACE ISOMORPHIC TO ℓ^2

DENNY H. LEUNG

ABSTRACT. We prove the assertion claimed in the title.

The weak L^p spaces play an important role in interpolation theory as well as harmonic analysis. In this paper, we concentrate on their geometry as Banach spaces. More precisely, we show that the sequential weak L^p spaces have complemented subspaces isomorphic to ℓ^2. We remark that the corresponding assertion for non-atomic weak L^p spaces is also true. For, in this case, it is easy to see that the Rademacher functions span a complemented subspace isomorphic to ℓ^2. The idea of the proof in the atomic case is to create a similar situation in the $\ell^{p,\infty}$ spaces.

We start by recalling some standard definitions. Let (Ω, Σ, μ) be an arbitrary measure space. For $1 < p < \infty$, the weak L^p space $L(p, \infty, \mu)$ is the space of all Σ-measurable functions f such that $\{\omega : |f(\omega)| > 0\}$ is σ-finite and $||f|| \equiv \sup_B \int_B |f|\, d\mu/\mu(B)^{1-1/p} < \infty$, where the supremum is taken over all measurable sets B with $0 < \mu(B) < \infty$. In case $(\Omega, \Sigma, \mu) = [0, 1]$ endowed with Lebesgue measure or $\mathbf{N}$ with the counting measure, we use the notation $L^{p,\infty}[0, 1]$ and $\ell^{p,\infty}$, respectively. For an real valued function f defined on (Ω, Σ, μ), we denote by f^* the decreasing rearrangement of $|f|$ [**1**], similarly for (a_n^*), where (a_n) is a sequence of real numbers. It is well known that $L^{p,\infty}[0, 1]$ is naturally isomorphic to the dual of $L^{q,1}[0, 1]$, $1/p + 1/q = 1$, where $L^{q,1}[0, 1]$ denotes the space of all measurable functions f such that $||f|| = \int_0^1 t^{-1/p} f^*(t)\, dt < \infty$. Furthermore, $L(p, \infty, \mu)$ satisfies an upper p-estimate [**2**]. For further details on the weak L^p spaces, we refer to [**2, 1**].

Fix $1 < p < \infty$, let (e_i) denote the coordinate unit vectors in $\ell^{p,\infty}$, and let $F = [e_i]$.

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Proposition 1. *Let* $G = \{(a_n) \in \ell^{p,\infty} : \exists\, j$ *such that* $\sum_{n=k(2^j)+1}^{(k+1)2^j} a_n = 0,\ k = 0, 1, 2, \dots\}$ *and define* $f : \operatorname{span}\{G, (n^{-1/p}), e_1\} \to \mathbf{R}$ *by*

$$f(g + \alpha(n^{-1/p}) + \beta e_1) = \alpha \quad \text{for all} \quad g \in G \text{ and } \alpha, \beta \in \mathbf{R},$$

then f *is bounded with respect to the norm of* $\ell^{p,\infty}$.

Proof. Let $g = (a_n) \in G$ and suppose $||g + \alpha(n^{-1/p}) + \beta e_1|| \le 1$. Choose j such that $\sum_{n=k(2^j)+1}^{(k+1)2^j} a_n = 0$ for all $k \ge 0$. For any $k \in \mathbf{N}$, consider the set $B = \{n : 2^j < n \le k2^j\}$. By definition of the norm in $\ell^{p,\infty}$, we obtain

$$\begin{aligned}
&\sum_{n=2^j+1}^{k(2^j)} |a_n + \alpha n^{-1/p}| \le ((k-1)2^j)^{1-1/p} \\
&\Rightarrow \left| \sum_{n=2^j+1}^{k(2^j)} (a_n + \alpha n^{-1/p}) \right| \le ((k-1)2^j)^{1-1/p} \\
&\Rightarrow \left| \sum_{n=2^j+1}^{k(2^j)} \alpha n^{-1/p} \right| \le ((k-1)2^j)^{1-1/p} \\
&\Rightarrow \left| \int_{2^j+1}^{k(2^j)+1} \alpha t^{-1/p}\, dt \right| \le ((k-1)2^j)^{1-1/p}.
\end{aligned}$$

Taking the limit as $k \to \infty$ in the last inequality, we see that $|\alpha| \le 1 - 1/p$. This shows that f is bounded, as claimed. $\square$

By the Hahn-Banach theorem, there is a norm preserving extension $x' \in (\ell^{p,\infty})'$ of f. We claim that $x' \in F^\circ$. Indeed, let (a_n) be a finitely nonzero sequence, then $(a_n) - (\sum a_n)e_1 \in G$, hence

$$\left\langle (a_n) - \left(\sum a_n\right) e_1, x' \right\rangle = f\left((a_n) - \left(\sum a_n\right) e_1\right) = 0.$$

Therefore,

$$\langle (a_n), x' \rangle = \left(\sum a_n\right) \langle e_1, x' \rangle = \left(\sum a_n\right) f(e_1) = 0.$$

By continuity, we obtain $x' \in F^\circ$, as claimed.

This x' will play the role of the identically 1 function in the scheme of the Rademacher functions. The tree structure involved will be given by the "dyadic" decomposition of $\mathbf{N}$: $A_1 = \mathbf{N}$, $A_2 = \{1, 3, 5, \dots\}$, $A_3 = \{2, 4, 6, \dots\}$, etc. At this point, we need some additional notation:

(1) Let (a_n) be a real sequence and let $A \subset \mathbf{N}$, then $(a_n)\chi_A$ denotes the sequence whose n-th coordinate is a_n if $n \in A$ and 0 otherwise;

(2) For $y' \in (\ell^{p,\infty})'$ and $A \subset \mathbf{N}$, $y'\chi_A \in (\ell^{p,\infty})'$ is given by $\langle x, y'\chi_A\rangle = \langle x\chi_A, y'\rangle$ for all $x \in \ell^{p,\infty}$;

(3) $S : \ell^{p,\infty} \to \ell^{p,\infty}$ will denote the shift operator $S(a_n) = (a_{n+1})$;

(4) For $n \geq 1$ and $0 \leq j < 2^n$, we let $N(n, j) =$ (first element of $A_{2^n+j}) - 1$. With respect to (1) and (2) and the sets A_n defined above, we will write χ_n for χ_{A_n}. A simple fact which will be used repeatedly is that for any $x \in \ell^{p,\infty}$, $x\chi_{2^n+j} - S^{N(n,j)}(x\chi_{2^n+j}) \in G$ and hence $\langle x\chi_{2^n+j}, x'\rangle = \langle S^{N(n,j)}(x\chi_{2^n+j}), x'\rangle$. Mimicking the definition of the Rademacher functions, we now define a sequence $(x'_n) \subset (\ell^{p,\infty})'$ by $x'_n = \sum_{k=2^{n-1}}^{2^n-1} (-1)^k x'\chi_k$. In the following propositions, we show that (x'_n) is equivalent to the canonical ℓ^2 basis.

Proposition 2. *For all n and $0 \leq j < 2^n$, $||x'\chi_{2^n}|| = ||x'\chi_{2n+j}||$.*

Proof. For $x \in \ell^{p,\infty}$,

$$\begin{aligned}\langle x, x'\chi_{2^n+j}\rangle &= \langle x\chi_{2^n+j}, x'\rangle = \langle S^{N(n,j)}(x\chi_{2^n+j}), x'\rangle \\ &= \langle S^{N(n,j)}(x\chi_{2^n+j}), x'\chi_{2^n}\rangle.\end{aligned}$$

Taking the supremum over all x with norm ≤ 1, we have $||x'\chi_{2^n+j}|| \leq ||x'\chi_{2^n}||$ since $||S^{N(n,j)}(x\chi_{2^n+j})|| \leq ||x||$. The reverse inequality follows by symmetry. □

Proposition 3. *For any finitely nonzero sequence (a_n), we have $||\sum a_n x'_n|| \preceq ||(a_n)||_2$, where $\preceq$ means $\leq$ up to a fixed constant and $||\cdot||_2$ denotes the ℓ_2 norm.*

Proof. Since $(\ell^{p,\infty})'$ satisfies a lower q-estimate, $1/p+1/q=1$, we have

$$\frac{1}{q} \geq ||x'|| \succeq \bigg(\sum_{j=0}^{2^n-1} ||x'\chi_{2^n+j}||^q\bigg)^{1/q} = 2^{n/q}||x'\chi_{2^n}||$$
$$\Rightarrow ||x'\chi_{2^n}|| \preceq 2^{-n/q}.$$

Let (a_n) be a finitely nonzero sequence, then there exist k and a sequence $(b_n)_{n=0}^{2^k-1}$ such that $\sum a_n x'_n = \sum_{n=0}^{2^k-1} b_n x'\chi_{2^k+n}$. But then, the same relationship exists between the Rademacher functions and the characteristic functions of the dyadic intervals. Thus, if (r_n) denotes the Rademacher functions, then we obtain $\sum a_n r_n = \sum_{n=0}^{2^k-1} b_n \chi_{[n/2^k,(n+1)/2^k)}$. Now let $x=(x_m)\in \ell^{p,\infty}$ with $||x||\leq 1$, then

$$\Big\langle x, \sum a_n x'_n\Big\rangle = \Big\langle x, \sum_{n=0}^{2^k-1} b_n x'\chi_{2^k+n}\Big\rangle = \sum_{n=0}^{2^k-1} b_n\langle x\chi_{2^k+n}, x'\rangle$$
$$= \sum_{n=0}^{2^k-1} b_n\langle S^{N(k,n)}(x\chi_{2^k+n}), x'\rangle = \langle y, x'\chi_{2^k}\rangle,$$

where we have set $y=(y_m)=\sum_{n=0}^{2^k-1} b_n S^{N(k,n)}(x\chi_{2^k+n})$. To estimate the norm of y, we must consider the sum $\sum_{m\in B}|y_m|$ for any finite $B\subset \mathbf{N}$. Suppose card $B=i$, then there are disjoint subsets $B_0,\dots,B_{2^k-1}$ of $\mathbf{N}$, each of which has cardinality i such that $\sum_{m\in B}|y_m| \leq \sum_{n=0}^{2^k-1}|b_n|(\sum_{m\in B_n}|x_m|)$. A simple computation reveals that the latter sum is $\leq \sum_{n=0}^{2^k-1} b_n^*(\sum_{m=ni+1}^{(n+1)i} x_m^*)$. Now $||x||\leq 1 \Rightarrow x_m^* \leq m^{-1/p}$ for all m; hence

$$\sum_{m\in B}|y_m| \leq \sum_{n=0}^{2^k-1} b_n^*\bigg(\sum_{m=ni+1}^{(n+1)i} m^{-1/p}\bigg)$$
$$\preceq i^{1/q}\sum_{n=0}^{2^k-1} b_n^*((n+1)^{1/q}-n^{1/q}).$$

Therefore, $||y|| \preceq \sum_{n=0}^{2^k-1} b_n^*((n+1)^{1/q} - n^{1/q})$. Consequently,

$$\begin{aligned}
\left|\left\langle x, \sum a_n x_n' \right\rangle\right| &= |\langle y, x'\chi_{2^k}\rangle| \leq ||y|| \cdot ||x'\chi_{2^k}|| \\
&\preceq 2^{-k/q} \sum_{n=0}^{2^k-1} b_n^*((n+1)^{1/q} - n^{1/q}) \\
&= \left\| \sum_{n=0}^{2^k-1} b_n \chi_{[n/2^k,(n+1)/2^k)} \right\|_{L^{q,1}[0,1]} \\
&= \left\| \sum a_n r_n \right\|_{L^{q,1}[0,1]} \\
&\preceq \left\| \sum a_n r_n \right\|_s \ \forall s > q \qquad \text{(Hölder's inequality)} \\
&\preceq ||(a_n)||_2 \qquad \text{(Khinchine's inequality)}.
\end{aligned}$$

Since this holds for all x with $||x|| \leq 1$, we obtain the desired inequality. □

Proposition 4. *For any finitely nonzero sequence* (a_n), *we have* $||\sum a_n x_n'|| \succeq ||(a_n)||_2$.

Proof. Let k and (b_n) be chosen as in the last proposition. Computing directly the L^q norm of $\sum a_n r_n$ and using (one half of) Khinchine's inequality, we get $||(a_n)||_2 \preceq 2^{-k/q}||(b_n)||_q$. On the other hand,

$$\begin{aligned}
\left\| \sum a_n x_n' \right\| = \left\| \sum_{n=0}^{2^k-1} b_n x' \chi_{2^k+n} \right\| &\succeq \left(\sum_{n=0}^{2^k-1} ||b_n x' \chi_{2^k+n}||^q \right)^{1/q} \\
&= ||x'\chi_{2^k}|| \cdot ||(b_n)||_q
\end{aligned}$$

by the lower q-estimate on $(\ell^{p,\infty})'$ and Proposition 2. To finish the proof, it remains to show that $||x'\chi_{2^k}|| \succeq 2^{-k/q}$. Let $x = (m^{-1/p})$. By direct verification, it can be shown that $x\chi_{2^k} - S^{N(k,n)}(x\chi_{2^k+n}) \in F$ for $0 \leq n < 2^k$. Therefore, since $x' \in F^\circ$, we obtain

$$\langle x\chi_{2^k}, x' \rangle = \langle S^{N(k,n)}(x\chi_{2^k+n}), x' \rangle = \langle x\chi_{2^k+n}, x' \rangle.$$

Thus,

$$\begin{aligned}1 = \langle x, x'\rangle &= \sum_{n=0}^{2^k-1} \langle x\chi_{2^k+n}, x'\rangle = 2^k\langle x\chi_{2^{k+1}-1}, x'\chi_{2^{k+1}-1}\rangle \\ &\le 2^k||x\chi_{2^{k+1}-1}||\cdot||x'\chi_{2^{k+1}-1}|| = 2^k||x\chi_{2^{k+1}-1}||\cdot||x'\chi_{2^k}|| \\ &= 2^k||((n2^k)^{-1/p})_n||\cdot||x'\chi_{2^k}|| = (2^k)^{1-1/p}||x||\cdot||x'\chi_{2^k}||.\end{aligned}$$

Hence $||x'\chi_{2^k}|| \succeq 2^{-k/q}$, as desired. □

Combining the previous propositions, we arrive at

Theorem 5. *The sequence* $(x'_n) \subset (\ell^{p,\infty})'$ *is equivalent to the* ℓ^2 *basis.*

To obtain ℓ^2 as a complemented subspace of $\ell^{p,\infty}$, it suffices to produce a sequence in $\ell^{p,\infty}$ which is (1) biorthogonal to (x'_n) and (2) equivalent to the ℓ^2 basis. A natural choice is the sequence (x_n) given by $x_n = \sum_{k=2^{n-1}}^{2^n-1}(-1)^k x\chi_k$, where $x = (m^{-1/p})$. In this case, requirement (1) is trivially true but (2) fails. However, since $x' \in F^\circ$, we may obtain a desirable sequence by "trimming" the x_n's. This is done in the following lemmas.

Lemma 6. *There is a constant* $K < \infty$ *such that for any integer* $k \ge 1$ *and any finitely nonzero sequence* (a_n), $||(\sum_{n=1}^k a_n x_n)\chi_{(2^k,\infty)}|| \le K||(a_n)||_2$.

Proof. Let $j = k-1$ and let the sequence $(b_n)_{n=0}^{2^j-1}$ be such that $\sum_{n=1}^k a_n x_n = \sum_{n=0}^{2^j-1} b_n x\chi_{2^j+n}$. Then

$$\begin{aligned}
\left\| \left(\sum_{n=1}^{k} a_n x_n \right) \chi_{(2^k,\infty)} \right\| &= \left\| \sum_{n=0}^{2^j-1} b_n x \chi_{2^j+n} \chi_{(2^k,\infty)} \right\| \\
&\preceq \left(\sum_{n=0}^{2^j-1} ||b_n x \chi_{2^j+n} \chi_{(2^k,\infty)}||^p \right)^{1/p} \qquad \text{(upper } p\text{-estimate)} \\
&\le \left(\sum_{n=0}^{2^j-1} ||b_n x \chi_{2^j} \chi_{(2^k,\infty)}||^p \right)^{1/p} \\
&= ||(b_n)||_p \cdot ||x \chi_{2^j} \chi_{(2^k,\infty)}|| \\
&= ||(b_n)||_p \cdot ||((n2^j+1)^{-1/p})_{n=2}^{\infty}|| \\
&\le ||(b_n)||_p \cdot ||((n2^j)^{-1/p})_{n=2}^{\infty}|| \\
&= 2^{-j/p} ||(b_n)||_p \cdot ||(n^{-1/p})_{n=2}^{\infty}|| \\
&\le 2^{-j/p} ||(b_n)||_p ||x|| = ||x|| \left\| \sum_{n=1}^{k} a_n r_n \right\|_{L^p[0,1]} \\
&\preceq ||(a_n)||_2, \qquad \text{(Khinchine's inequality)}
\end{aligned}$$

as desired. □

Lemma 7. *Let K be the same constant as in Lemma 6. Then there is a strictly increasing sequence of integers (k_i) such that*

(a) *$k_i > 2^i$ for all $i \ge 1$; and*

(b) *if we let $z_i = x_i \chi_{[k_i,\infty)}$, then*

$$\left\| \sum_{i=1}^{n} a_i z_i \right\| \le (3K + q + 1 - 1/n) ||(a_i)||_2$$

for all (a_i) and all n, where $1/p + 1/q = 1$.

Proof. The proof is by induction. For $n = 1$, let $k_1 = 3$. Then the assertions are trivially verified. (Note that $||x|| \le q$.) Now suppose $k_1, \dots, k_n$ has been defined ($n \ge 1$). Choose $k_{n+1} > 2^{n+1}$ such that

$$\begin{aligned}
||((k_{n+1})^{-1/p}, \dots, (k_{n+1}+m)^{-1/p}, 0, \dots)|| &> (n+1)^{-2} \\
\Rightarrow Km^{1/q} &> (3K + q + 1)(k_n^{1/q}).
\end{aligned}$$

We need to show that for all finite nonempty subsets B of $\mathbf{N}$,

$$(\operatorname{card} B)^{-1/q} \sum_{j\in B} \left| \left(\sum_{i=1}^{n+1} a_i z_i \right)(j) \right| \le (3K + q + 1 - 1/(n+1)) \cdot ||(a_i)||_2.$$

Start with such a set B; we consider two cases.

Case A. $(\operatorname{card} B)^{-1/q} \sum_{j\in B} |z_{n+1}(j)| \le (n+1)^{-2}$. In this case,

$$\begin{aligned}\sum_{j\in B} \left| \left(\sum_{i=1}^{n+1} a_i z_i \right)(j) \right| &\le \sum_{j\in B} \left| \left(\sum_{i=1}^{n} a_i z_i \right)(j) \right| + \sum_{j\in B} |a_{n+1} z_{n+1}(j)| \\ &\le (\operatorname{card} B)^{1/q} \{ (3K + q + 1 - 1/n) \cdot ||(a_i)||_2 \\ &\qquad + |a_{n+1}|(n+1)^{-2} \}\end{aligned}$$

by the inductive hypothesis. Now since $(n+1)^{-2} - 1/n \le -1/(n+1)$, the desired result follows.

Case B. $(\operatorname{card} B)^{-1/q} \sum_{j\in B} |z_{n+1}(j)| > (n+1)^{-2}$. In this case $||z_{n+1}\chi_B|| > (n+1)^{-2}$. Let $m = \operatorname{card} B$. Then

$$||((k_{n+1})^{-1/p}, \dots, (k_{n+1} + m)^{1/p}, 0, \dots)|| \ge ||z_{n+1}\chi_B|| > (n+1)^{-2}.$$

Now

$$\begin{aligned}\sum_{j\in B\cap[1,k_n)} \left| \left(\sum_{i=1}^{n+1} a_i z_i \right)(j) \right| \\ &= \sum_{j\in B\cap[1,k_n)} \left| \left(\sum_{i=1}^{n-1} a_i z_i \right)(j) \right| \\ &\le (3K+q+1-1/(n-1)) \cdot ||(a_i)||_2 \cdot (\operatorname{card}(B \cap [1,k_n)))^{1/q} \\ &\qquad \text{by induction} \\ &\le (3K+q+1) \cdot ||(a_i)||_2 \cdot k_n^{1/q} \le K||(a_i)||_2 \cdot m^{1/q} \\ &\qquad \text{by choice of } k_{n+1}.\end{aligned}$$

Finally,

$$
\begin{aligned}
\sum_{j\in B}&\left|\left(\sum_{i=1}^{n+1} a_i z_i\right)(j)\right| \\
&= \left(\sum_{j\in B\cap[1,k_n)} + \sum_{j\in B\cap[k_n,k_{n+1}]} + \sum_{j\in B\cap[k_{n+1},\infty)}\right)\left|\left(\sum_{i=1}^{n+1} a_i z_i\right)(j)\right| \\
&\le K||(a_i)||_2\cdot m^{1/q} + \sum_{j\in B\cap[k_n,k_{n+1})}\left|\left(\sum_{i=1}^{n} a_i z_i\right)(j)\right| \\
&\qquad\qquad + \sum_{j\in B\cap[k_{n+1},\infty)}\left|\left(\sum_{i=1}^{n+1} a_i z_i\right)(j)\right| \\
&\le K||(a_i)||_2\cdot m^{1/q} + \sum_{j\in B\cap[k_n,\infty)}\left|\left(\sum_{i=1}^{n} a_i x_i\right)(j)\right| \\
&\qquad\qquad + \sum_{j\in B\cap[k_{n+1},\infty)}\left|\left(\sum_{i=1}^{n+1} a_i x_i\right)(j)\right| \\
&\le K||(a_i)||_2\cdot m^{1/q} + (\operatorname{card} B)^{1/q}\left\{\left\|\left(\sum_{i=1}^{n} a_i x_i\right)\chi_{[k_n,\infty)}\right\|\right. \\
&\qquad\qquad\qquad\left. + \left\|\left(\sum_{i=1}^{n+1} a_i x_i\right)\chi_{[k_{n+1},\infty)}\right\|\right\} \\
&\le (\operatorname{card} B)^{1/q}||(a_i)||_2(K+K+K)
\end{aligned}
$$

by Lemma 6 since $k_n > 2^n$. This completes the induction. □

Theorem 5 and Lemma 7 combine to yield the following theorem.

Theorem 8. *For $1 < p < \infty$, $\ell^{p,\infty}$ has a complemented subspace isomorphic to ℓ^2.*

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Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511, Republic of Singapore