ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

PARACOMPACTNESS IN PERFECT, LOCALLY LINDELÖF SPACES

NOBUYUKI KEMOTO

Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. Under MA + ¬CH, paracompactness of certain perfect locally compact spaces is discussed in [**13**], [**7**], [**10**] and [**1**].

In this note, under no additional set theoretical assumption, we shall characterize paracompactness in perfect, locally Lindelöf spaces. As a corollary, it will be shown that $2^\omega < 2^{\omega_1}$ is equivalent to Lindelöfness of certain perfectly normal locally Lindelöf spaces.

0. Introduction. M.E. Rudin proved that, under MA + ¬CH, perfectly normal manifolds are metrizable [**13**]. Using her technique, D.S. Lane proved that, under MA + ¬CH, perfectly normal, locally compact, locally connected spaces are paracompact [**10**]. Furthermore, G. Gruenhage proved that, under MA + ¬CH, perfectly normal, locally compact, collectionwise Hausdorff spaces are paracompact [**7**]. These results are shown in Nyikos's article [**12**]. In this note, we will define the property S of topological spaces, and also define stationary collectionwise Hausdorffness (SCWH) which is a weakening of collectionwise Hausdorffness (CWH). Then we shall prove that in every perfect, locally Lindelöf space X, X is paracompact if and only if X is SCWH and has the property S. Also these applications will be studied. In particular, it will be shown that $2^\omega < 2^{\omega_1}$ is equivalent to Lindelöfness of certain perfect, locally Lindelöf spaces.

Throughout in this paper, all spaces are assumed to be regular T_1. A space is *perfect* if all closed subspaces are G_δ. For a disjoint family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of a space, an *expansion* $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of $\mathcal{F}$ is a family of subsets such that $F_\alpha \subset U_\alpha$ for every $\alpha \in \Lambda$ and $U_\alpha \cap F_\beta = 0$ if $a, \beta \in \Lambda$ with $\alpha \neq \beta$. An *open expansion*

Received by the editors on September 19, 1988, and in revised form on October 30, 1989.
AMS *Mathematics Subject Classification* (1980). 54D20, 54A35.
Key words and phrases. Paracompact, perfect, S-space, $2^\omega < 2^{\omega_1}$.

is an expansion whose elements are open. Furthermore, *separation* means pairwise disjoint expansion, and *open separation* means pairwise disjoint open separation. A subspace Y of a space is *discrete* if there is an open expansion of $\{\{y\} : y \in Y\}$. A space is *κ-collectionwise Hausdorff* (κ-CWH) if every closed discrete subspace of size $\leq \kappa$ has an open separation. A space is *collectionwise Hausdorff* (CWH) if it is κ-CWH for every infinite cardinal κ. A space is *collectionwise normal with respect to Lindelöf closed sets* if every discrete family of Lindelöf closed sets has an open separation. Let κ be a regular uncountable cardinal. A subset of κ is *stationary* if it intersects all closed (in κ with the order topology) unbounded subsets of κ. A space is *stationary κ-collectionwise Hausdorff* (κ-SCWH) if for every closed discrete subspace $\{x_\alpha : \alpha \in S\}$ indexed by a stationary subset S of κ, there is a stationary subset $S' \subset S$ of κ such that $\{x_\alpha : \alpha \in S'\}$ has an open separation. If a space is κ-SCWH for every uncountable regular cardinal, then we call it *stationary collectionwise Hausdorff* (SCWH). Note that we do not require discreteness of such open separations in these definitions. A space is *ω_1-Lindelöf* if every open cover has a subcover of size $\leq \omega_1$. A space is (*weakly*) *submetaLindelöf* if every open cover $\mathcal{U}$ has a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers (families, respectively) refining $\mathcal{U}$ such that for every $x \in X$, there is an $n < \omega$ such that $0 < |(\mathcal{U}_n)_x| \leq \omega$ (in this definition, if we replace $|(\mathcal{U}_n)_x| \leq \omega$ by $|(\mathcal{U}_n)_x| < \omega$, then we call such a space (*weakly*) *submetacompact*). Where $(\mathcal{U}_n)_x = \{U \in \mathcal{U}_n : x \in U\}$. A countable family $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of collections of subset of a space X is called an *interlacing* on X if $\cup\mathcal{V}$ is a cover of X and for each n in ω, each V in $\mathcal{V}_n$ is open in $\cup\mathcal{V}_n$. A space X is called *ultrapure* provided for each open cover $\mathcal{U}$ of X, there is an interlacing $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ on X such that for each n in ω and x in $\cup\mathcal{V}_n$ there is a countable subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $\cup(\mathcal{V}_n)_x \subset \cup\mathcal{U}'$. For other covering properties (topological notions, set theoretical notions), refer to [**3, 6, 9**], respectively. Especially, for the Martin's axiom or the S-space (hereditarily separable not hereditarily Lindelöf space) problem, [**19, 14**], respectively.

1. In perfect, locally Lindelöf spaces. In this section we give an equivalent condition in order that a perfect, locally Lindelöf space is paracompact. Note that for every locally Lindelöf space X, X is paracompact if and only if X is strongly paracompact if and only if X is the free union of Lindelöf subspaces. The next Lemma is an easy exercise, but it will be used frequently.

Lemma 1.1. *For every Lindelöf space X, X is perfect if and only if X is hereditary Lindelöf. Hence, perfect Lindelöf spaces are ccc, where a space is ccc if there is no pairwise disjoint family of uncountably many nonempty open sets.*

The technique of the proof of the following lemma is essentially due to [**2**, 1.3 and 1.7].

Lemma 1.2. *Locally ccc, SCWH spaces are collectionwise normal with respect to Lindelöf closed sets. In particular, locally ccc, SCWH spaces are CWH.*

Proof. We shall prove by induction of $c(X)$, where $c(X) = \min\{\lambda$: there does not exist a pairwise disjoint family of λ-many nonempty open sets$\}$. If $c(X) = \omega_1$ (i.e., X is ccc), then there does not exist a discrete family of uncountably many Lindelöf closed sets by ω_1-SCWH. Since in a regular space, every discrete family of countably many Lindelöf closed sets has an open separation, the first step of the induction is proved.

Assume $c(X) = \lambda > \omega_1$. Let $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ be a discrete family of nonempty Lindelöf closed sets. Then by SCWH, $\kappa < \lambda$ if κ is regular uncountable, and $\kappa \leq \lambda$ if κ is a singular cardinal. For every $\alpha < \kappa$ and $x \in F_\alpha$, take a ccc open neighborhood V_x of x such that $V_x \cap F_\beta = 0$ for all $\beta < \kappa$ with $\alpha \neq \beta$. Since F_α is Lindelöf, take a countable subset F'_α of F_α such that $F_\alpha \subset \cup\{V_x : x \in F'_\alpha\}$, call this open set U_α. Then $\{U_\alpha : \alpha < \kappa\}$ is an expansion of $\mathcal{F}$ by ccc open sets. There are two cases.

Case 1. κ is a regular cardinal. First we shall show that $S = \{\alpha < \kappa : \mathrm{cl}\,(\cup_{\beta<\alpha} U_\beta) \cap \cup_{\beta\geq\alpha} F_\beta \neq 0\}$ is not stationary in κ. Assume on the contrary that S is stationary. For every $\alpha \in S$, take an $x_\alpha \in \mathrm{cl}\,(\cup_{\beta<\alpha} U_\beta) \cap \cup_{\beta\geq\alpha} F_\beta$. For every $\alpha \in S$, let $x_\alpha \in F_{\nu(\alpha)}$. Then $\alpha \leq \nu(\alpha)$ and $\nu(\alpha) \in S$ for every $\alpha \in S$. For every $\alpha \in \kappa - S$, let

$\nu(\alpha) = 0$. Then $C = \{\alpha < \kappa : \nu''\alpha \subset \alpha\}$ is closed unbounded in κ. If α and α' are in $S \cap C$ with $\alpha < \alpha'$, then $\nu(\alpha) < \alpha' \leq \nu(\alpha')$. Thus, $Y = \{x_\alpha : \alpha \in S \cap C\}$ is a discrete closed subspace consisting of distinct points and $S \cap C$ is stationary in κ. Using κ-SCWH, take an open separation $\{V_\alpha : \alpha \in S'\}$ of $\{x_\alpha : \alpha \in S'\}$, where S' is a stationary subset of κ which is included in $S \cap C$. Of course, $x_\alpha \in V_\alpha$ for every $\alpha \in S'$. Since $x_\alpha \in \operatorname{cl}(\cup_{\beta<\alpha} U_\beta)$ for every $\alpha \in S'$, take an $f(\alpha) < \alpha$ such that $V_\alpha \cap U_{f(\alpha)} \neq 0$. Then by the pressing down lemma, there are a stationary subset $S'' \subset S'$ and an $\alpha_0 < \kappa$ such that $V_\alpha \cap U_{\alpha_0} \neq 0$ for every α of S''. Thus $\{V_\alpha \cap U_{\alpha_0} : \alpha \in S''\}$ witnesses a contradiction, since U_{α_0} is ccc. Therefore, S is not stationary.

Next, take a closed unbounded set C of κ which misses S. Enumerate $C = \{\alpha(\gamma) : \gamma < \kappa\}$ with the increasing order. For $\gamma < \kappa$, let $H_\gamma = \cup_{\beta<\alpha(\gamma+1)} U_\beta - \operatorname{cl}(\cup_{\beta<\alpha(\gamma)} U_\beta)$ and $\mathcal{F}_\gamma = \{F_\beta : \alpha(\gamma) \leq \beta < \alpha(\gamma+1)\}$. Note $\cup\mathcal{F}_\gamma \subset H_\gamma$. Since each U_β is ccc, $c(\operatorname{cl} H_\gamma) \leq \kappa (< \lambda)$ for each $\gamma < \kappa$. Using the inductive hypothesis, take an open separation $\mathcal{U}_\gamma$ of $\mathcal{F}_\gamma$ in H_γ for every $\gamma < \kappa$. Since C is closed unbounded in κ, $\cup\{\mathcal{F}_\gamma : \gamma < \kappa\} = \mathcal{F}$. Therefore, $\cup\{\mathcal{U}_\gamma : \gamma < \kappa\}$ is an open separation of $\mathcal{F}$.

Case 2. κ is a singular cardinal. Decompose κ into $\{A_\gamma : \gamma \in \Gamma\}$, where $|A_\gamma| < \kappa$ for each γ of Γ, $|\Gamma| < \kappa$ and $A_\gamma \cap A_{\gamma'} = 0$ for γ, γ' of Γ with $\gamma \neq \gamma'$. Let $\mathcal{F}_\gamma = \{F_\alpha : \alpha \in A_\gamma\}$ and $H_\gamma = \cup_{\alpha \in A_\gamma} U_\alpha$. Then it is easy to show $c(\operatorname{cl} H_\gamma) \leq (\omega \cdot |A_\gamma|)^+ < \kappa \leq \lambda$ for each γ of Γ. By the inductive hypothesis, for every γ in Γ take an open separation $\mathcal{V}_\gamma = \{V_\alpha : \alpha \in A_\gamma\}$ of $\mathcal{F}_\gamma$ in H_γ with $F_\alpha \subset V_\alpha \subset U_\alpha$ for every $\alpha \in A_\gamma$. Let $\mathcal{V} = \cup\{\mathcal{V}_\gamma : \gamma \in \Gamma\}$. Since every member of $\mathcal{V}$ is ccc, for every γ in Γ, every member of $\mathcal{V}$ meets at most countably many members of $\mathcal{V}_\gamma$. Hence, every member of $\mathcal{V}$ meets at most $\omega \cdot |\Gamma|$ members of $\mathcal{V}$. By the usual chaining argument, decompose $\mathcal{V}$ into $\{\mathcal{W}_\delta : \delta \in \Lambda\}$, where $|\mathcal{W}_\delta| \leq \omega \cdot |\Gamma|$ for every $\delta \in \Lambda$ and $(\cup\mathcal{W}_\delta) \cap (\cup\mathcal{W}_{\delta'}) = 0$ if $\delta, \delta' \in \Lambda$ with $\delta \neq \delta'$. Let $\mathcal{K}_\delta = \{F \in \mathcal{F} : F \subset \cup\mathcal{W}_\delta\}$. Since $c(\operatorname{cl} \cup \mathcal{W}_\delta) \leq (\omega \cdot |\Gamma|)^+ < \kappa \leq \lambda$, by the inductive hypothesis, take an open separation $\mathcal{U}_\delta$ of $\mathcal{K}_\delta$ in $\cup\mathcal{W}_\delta$ for each $\delta \in \Lambda$. Then $\cup\{\mathcal{U}_\delta : \delta \in \Lambda\}$ is an open separation of $\mathcal{F}$, since $\mathcal{F} = \cup\{\mathcal{K}_\delta : \delta \in \Lambda\}$. Thus the proof is complete. □

By the same argument of the proof of [**7**, Lemma 6], we can show the next lemma. Note that by 1.1 and 1.2, CWH can be replaced by SCWH in the following lemma, as well as the later results such as Theorems 1.4, 1.5, etc.

Lemma 1.3. [**7**]. *Let X be a perfect, locally Lindelöf, CWH space. Then X is the free union of ω_1-Lindelöf subspaces.*

Definition. A space has the *property S* if every right separated subspace of type ω_1 is σ-discrete (i.e., the countable union of discrete subspaces).

Then using the above definition we can characterize paracompactness in perfect, locally Lindelöf spaces as follows. First we prove the next main theorem.

Theorem 1.4. *Let X be a perfect, locally Lindelöf, ω_1-Lindelöf space. Then the following assertions are equivalent.*

1) *X is ω_1-CWH and has the property S.*

2) *X is the free union of Lindelöf subspaces.*

Proof. To show 1) $\rightarrow$ 2), we may assume that X is the union of $\{U_\alpha : \alpha < \omega_1\}$, where each U_α is an open set with the Lindelöf closure, since X is ω_1-Lindelöf and locally Lindelöf. Furthermore, assume that 1) holds.

Claim 1. *The closure of every Lindelöf open set is Lindelöf.*

Proof. Assume on the contrary that there is an open Lindelöf subset W such that $\mathrm{cl}\, W$ is not Lindelöf. Then $\mathrm{cl}\, W$ has a right separated subspace $\{x_\alpha : \alpha < \omega_1\}$, from now on we identify this set with ω_1. Since X has the property S, there is a stationary subset S of ω_1 such that S is discrete. For convenience, we shall call such an S *stationary discrete*. By perfectness of X, there is a stationary discrete closed subset S' of S (in fact, S is *σ-discrete-closed*, i.e., S is the countable union of discrete closed subspaces of X). Using the ω_1-CWH-ness of X, take an open separation of S'. Since W is ccc, this separation shows a contradiction. Thus this claim is proved. $\square$

Next by induction on $\beta < \omega_1$, we shall define an increasing sequence $\langle \alpha(\beta) : \beta < \omega_1 \rangle$ in ω_1 and an increasing sequence $\langle X_\beta : \beta < \omega_1 \rangle$ of open Lindelöf subspaces. Let $\alpha(0) = 0$ and $X_0 = U_0$. Assume that for each $\delta < \beta$, where $\beta < \omega_1$, $\alpha(\delta)$ and X_δ have been defined. When $\beta = \delta + 1$, since $\operatorname{cl} X_\delta$ is Lindelöf, take an $\alpha(\beta) > \alpha(\delta)$ with $\alpha(\beta) < \omega_1$ such that $X_\beta = \cup\{U_\alpha : \alpha \leq \alpha(\beta)\}$ includes $\operatorname{cl} X_\delta$. When β is limit, let $\alpha(\beta) = \sup\{\alpha(\delta) : \delta < \beta\}$ and $X_\beta = \cup_{\delta<\beta} X_\delta$. Then $\langle X_\beta : \beta < \omega_1 \rangle$ satisfies the following properties:

1) $\operatorname{cl} X_\alpha \subset X_\beta$ if $\alpha < \beta < \omega_1$,
2) $X_\beta = \cup_{\delta<\beta} X_\delta = \cup_{\delta<\beta} \operatorname{cl} X_\delta$ if β is limit.

Claim 2. *$S = \{\beta < \omega_1 : \partial X_\beta \neq 0\}$ is not stationary in ω_1.*

Proof. Assume on the contrary that S is stationary. Fix an $x_\beta \in \partial X_\beta$ for every $\beta \in S$. We identify $\{x_\alpha : \alpha \in S\}$ with S. Then S with the increasing enumeration is right separated by $\{X_\beta : \beta \in S\}$. Since X has the property S, there is a stationary discrete subset S_1 of S. Furthermore, using perfectness take a stationary discrete closed subspace S_2 of S_1. Let C be the set of all limit ordinals of ω_1. Note that C is closed unbounded in ω_1 (with the order topology). Using the ω_1-CWH-ness of X, take an open separation $\{V_\beta : \beta \in S_3\}$ of S_3, where $S_3 = S_2 \cap C$ is stationary. Since $x_\beta \in \partial X_\beta \subset \operatorname{cl}(\cup_{\delta<\beta} X_\delta)$ if $\beta \in S_3$, take an $f(\beta) < \beta$ such that $X_{f(\beta)} \cap V_\beta \neq 0$ for each $\beta \in S_3$. Then by the pressing down lemma, there are a stationary subset S_4 of S_3 and an α of ω_1 such that $X_\alpha \cap V_\beta \neq 0$ for each $\beta \in S_4$. This contradicts the fact that X_α is ccc. This completes the proof of the claim. □

To continue the proof, take a closed unbounded subset C of ω_1 which misses S of the above claim. Enumerate C with the increasing order, say $\{\beta(\gamma) : \gamma < \omega_1\}$. Then $F_\gamma = X_{\beta(\gamma+1)} - X_{\beta(\gamma)}$ is clopen Lindelöf for each $\gamma < \omega_1$. And since C is closed unbounded and $X = \cup_{\gamma<\omega_1} F_\gamma$, X is the free union of ω_1-many Lindelöf subspaces.

To show 2) $\rightarrow$ 1), assume that X is the free union of Lindelöf subspaces X_α's, $\alpha \in A$. Let Y be a right separated subspace of type ω_1. Then by Lemma 1.1, each X_α is hereditarily Lindelöf. So $Y \cap X_\alpha$ is countable for each α. Hence it is easy to show that Y is σ-discrete. Finally, the ω_1-CWH-ness of X is evident, since it is paracompact. □

Remark. The above argument implies a perfect Type I space (for the definition, see [**12**, 2.10]) is ω_1-CWH and have the property S if and only if it is paracompact.

Theorem 1.5. *Let X be a perfect, locally Lindelöf space. Then the following assertions were equivalent.*

1) *X is CWH and has the property S.*

2) *X is paracompact.*

Proof. By 1.3 and 1.4. □

As an easy corollary of 1.4, we can prove the following.

Corollary 1.6. *Let X be a connected* (*locally connected*), *perfect, locally Lindelöf, ω_1-Lindelöf space. Then the following assertions are equivalent.*

1) *X is ω_1-CWH and has the property S.*

2) *X is Lindelöf* (*paracompact, respectively*).

Proof. The connected case is due to 1.4. If we apply the connected case to each connected component, then we can also prove the local connected case. □

Thus we can prove the following result by 1.3 and 1.6. For the local connected case, apply the connected case to each connected component.

Corollary 1.7. *Let X be a connected* (*locally connected*), *perfect, locally Lindelöf space. Then the following assertions are equivalent.*

1) *X is CWH and has the property S.*

2) *X is Lindelöf* (*paracompact, respectively*).

In another flavor of these contexts, we can also prove the following result. Note that perfect, locally compact spaces have countable tightness.

Corollary 1.8. *Let X be a connected (locally connected), perfect, locally Lindelöf space of countable tightness. Then the following assertions are equivalent.*

1) *X is ω_1-CWH and has the property S.*

2) *X is Lindelöf (paracompact, respectively).*

Proof. As above, the locally connected case is an easy corollary of the connected case. It suffices to show 1) $\to$ 2) of the connected case. Assume 1), then as in Claim 1 of 1.4, the closure of every Lindelöf open set is Lindelöf.

Next let $X_0 \subset X$ be a Lindelöf open set. We can define an increasing sequence of Lindelöf open sets $\{X_\alpha : \alpha < \omega_1\}$ such that $\mathrm{cl}\, X_\alpha \subset X_\beta$ if $\alpha < \beta < \omega_1$ and $\cup_{\alpha<\beta} X_\alpha = \cup_{\alpha<\beta} \mathrm{cl}\, X_\alpha$ if β is limit. Because, if $\beta = \alpha + 1$, take a Lindelöf open set X_β such that $\mathrm{cl}\, X_\alpha \subset X_\beta$. If β is limit, let $X_\beta = \cup_{\alpha<\beta} X_\beta$.

Then $\cup_{\alpha<\omega_1} X_\alpha = \cup_{\alpha<\omega_1} \mathrm{cl}\, X_\alpha$ is clopen in X, since X has countable tightness. Therefore, $X = \cup_{\alpha<\omega_1} X_\alpha$ by the connectedness of X. We shall show that there is an $\alpha < \omega_1$ such that $X_\alpha = X$ (then X is Lindelöf). Assume on the contrary that $X - X_\alpha \neq 0$ for every $\alpha < \omega_1$. Take $x_\alpha \in \partial X_\alpha$ for each $\alpha < \omega_1$ using connectedness of X. Then $Y = \{x_\alpha : \alpha < \omega_1\}$ is right separated. Identify Y with ω_1. By the property S, ω_1-CWH-ness and perfectness of X, we can get a stationary discrete closed set S of ω_1 and an open separation $\{V_\alpha : \alpha \in S\}$ of S. As in the proof of claim 2 of 1.4, using the pressing down lemma, we can get a contradiction. □

Example. In 1.4, 1.5, 1.6, 1.7 and 1.8, (ω_1-)CWH-ness cannot be deleted.

Let X be the Bubble space derived from a subset of x-axis of size ω_1, see [**16**, Example F]. As is well known, X is a connected, locally connected, locally Lindelöf (but not locally compact), ω_1-Lindelöf Moore space which is neither Lindelöf nor ω_1-CWH. Since Moore implies perfect and subparacompact, it has the property S (see the next section). Note that MA + $\neg$CH implies the normality of X, but CH implies the non-normality of X.

2. Getting the property S. In this section, we study the property S. First we shall show the next result.

Theorem 2.1. *Let X be a space. Then X has the property S if and only if every locally countable subspace of X of size ω_1 is σ-discrete.*

Proof. Here a space Y is *locally countable* if every point in Y has a countable neighborhood. Assume that X has the property S and $Y = \{x_\alpha : \alpha \in \omega_1\}$ is a locally countable subspace of X. By transfinite induction on $\beta \in \omega_1$, we shall define an $\alpha(\beta) \in \omega_1$ and a countable open neighborhood V_β of $x_{\alpha(\beta)}$ in Y. Assume $\alpha(\gamma)$ and V_γ have been defined for all $\gamma < \beta$. Let $\alpha(\beta)$ be the least α in ω_1 such that $x_\alpha \notin \cup_{\gamma<\beta} V_\gamma$. Take a countable open neighborhood V_β of $x_{\alpha(\beta)}$ in Y. Since for each $\beta \in \omega_1$, $V_\beta - \cup_{\gamma<\beta} V_\gamma$ is countable, let f_β be an onto map from ω to $V_\beta - \cup_{\gamma<\beta} V_\gamma$. Then each $Y_n = \{f_\beta(n) : \beta \in \omega_1\}$ is a right separated subspace of type ω_1. Since X has the property S, each Y_n is σ-discrete. Thus Y is σ-discrete. The other direction is obvious. $\square$

Theorem 2.2. *Let X be a locally countable space. Then the following assertions are equivalent.*

1) *X is σ-discrete.*

2) *X is weakly submetacompact.*

3) *X is weakly submetaLindelöf.*

4) *X is ultrapure.*

Proof. 1) $\to$ 2). Assume that $X = \cup_{n\in\omega} X_n$ and $\mathcal{U}$ is an open cover of X, where each X_n is discrete. Let $\mathcal{U}_n$ be an open expansion of X_n, which refines $\mathcal{U}$. Then $\{\mathcal{U}_n : n \in \omega\}$ witnesses the weak submetacompactness of X.

2) $\to$ 3) $\to$ 4) is obvious.

4) $\to$ 1). Assume that X is locally countable and ultrapure. For each x in X, fix a countable open neighborhood V_x of x. By ultrapureness, take an interlacing $\{\mathcal{V}_n : n \in \omega\}$ on X such that for every n in ω and x in $\cup\mathcal{V}_n$, there is a countable subfamily $\mathcal{V}'$ of $\{V_x : x \in X\}$ with $\cup(\mathcal{V}_n)_x \subset \cup\mathcal{V}'$ (thus $\cup(\mathcal{V}_n)_x$ is countable). Put $X_n = \cup\mathcal{V}_n$ and

$\mathcal{U}_n = \{\cup(\mathcal{V}_n)_x : x \in \cup\mathcal{V}_n\}$ for each n in ω.

Claim. *For each n in ω and x in $\cup\mathcal{V}_n$, $(\mathcal{U}_n)_x$ is countable.*

Proof. Assume indirectly that $(\mathcal{U}_n)_x$ is uncountable for some n in ω and x in $\cup\mathcal{V}_n$. Then there is an uncountable subset $A \subset \cup\mathcal{V}_n$ such that x is in $\cup(\mathcal{V}_n)_y$ for each y in A. By picking V_y in $\mathcal{V}_n$ such that $x, y \in V_y$ for each y in A, we can show $A \subset \cup(\mathcal{V}_n)_x$. But, since $\cup(\mathcal{V}_n)_x$ is countable, this is a contradiction. This complete the proof of the claim. □

Since each member of $\mathcal{U}_n$ is countable and each $(\mathcal{U}_n)_x$ is countable, by the usual chaining argument, we can decompose $\mathcal{U}_n$ into $\{\mathcal{W}_{n\lambda} : \lambda \in \Lambda_n\}$ such that each $\mathcal{W}_{n\lambda}$ is countable and $\cup\mathcal{W}_{n\lambda}$'s are pairwise disjoint. Thus, $\cup\mathcal{W}_{n\lambda}$'s are a decomposition of $\cup\mathcal{V}_n$ by countable clopen sets in $\cup\mathcal{V}_n$. Then it is straightforward to show that $\cup\mathcal{V}_n$ is σ-discrete. Thus X is σ-discrete. □

In a similar way of the proof of the above theorem, we can also prove the next theorem. But Theorems 2.3, 2.5 and the equivalences of 1), 2), 3) of Theorem 2.2 are known by [**11**].

Theorem 2.3. ([**11**]). *Let X be a locally countable space. Then the following assertions are equivalent.*

1) *X is σ-discrete-closed.*
2) *X is subparacompact.*
3) *X is submetacompact.*
4) *X is submetaLindelöf.*

Next we quote the relation between 2.2 and 2.3. But the proof is not hard, so we state without proof.

Theorem 2.4. *For every space X, X is σ-discrete-closed if and only if X is perfect and σ-discrete.*

Noting that σ-discrete-closed, first countable spaces are Moore, we can show the following result by 2.3.

Theorem 2.5. ([**11**]) *Let X be a locally countable space. Then the following assertions are equivalent.*

1) *X is σ-discrete-closed and first countable.*
2) *X is subparacomapct and first countable.*
3) *X is Moore.*

By 2.1 and 2.2, if a space is hereditary ultrapure, then it has the property S. Since we can prove in a usual way that perfect, ultrapure spaces are hereditary ultrapure, we can show the following corollary using 1.5.

Corollary 2.6. *Perfect, locally Lindelöf, CWH, ultrapure spaces are paracompact.*

Remark. It is not hard to show that screenable or σ-para-Lindelöf spaces are ultrapure and SCWH (cf. 2.6). But a consistent example of a perfectly normal, locally compact, metaLindelöf space which is not paracompact is known (see [**18**]).

3. What happens in perfect, locally compact spaces under Martin's Axiom? A survey. In this section we study paracompactness of perfect, locally compact spaces. As corollaries, we shall show the Gruenhage's and Lane's (in the next section) Theorems.

It is well known that under $\mathrm{MA}(\omega_1)$, there does not exist a compact S-space of countable tightness, and there does not exist a perfect locally compact space having S-subspace, [**13, 15**]. Using essentially the same argument, Balogh showed the following.

Theorem 3.1. ([**1**])[$\mathrm{MA} + \neg\mathrm{CH}$]. *In a compact space of countable tightness, every locally countable subspace of size $< 2^\omega$ is σ-discrete (in fact, compact spaces of countable tightness have the property S under $\mathrm{MA}(\omega_1)$).*

Theorem 3.2. [MA(ω_1)]. *Perfect, locally compact spaces have the property S.*

Proof. Let X be a perfect, locally compact space and Y be a right separated subspace of type ω_1. Let ωX be the one point compactification of X. Then by the comment after 4.14 of [**12**], ωX has countable tightness. Therefore, Y is σ-discrete by 3.1. □

Using 3.2 and 1.5, we can show the next Gruenhage's result.

Theorem 3.3. ([**7**])[MA(ω_1)]. *In perfect, locally compact spaces, CWH-ness is equivalent to paracompactness.*

4. CWH-ness in normal spaces. A survey. In this section we shall show that with further set theoretical assumptions or assumptions of spaces, we can delete CWH-ness from Theorem 1.5.

F.D. Tall asked whether perfectly normal, locally compact spaces are collectionwise normal in L, see [**16**]. Note that L satisfies $V = L$. It is known that in the model adding ω_2-random reals to a model satisfying $V = L$, perfectly normal, locally compact spaces are collectionwise normal. Here we give related topics.

Theorem 4.1. [$V = L$]. *Let X be a perfectly normal, locally Lindelöf space of character $\leq 2^\omega$ $(= \omega_1)$. Then X has the property S if and only if X is paracompact.*

Proof. Since under $V = L$, normal spaces of character $\leq 2^\omega$ are CWH, by 1.5, it is evident. □

Remark. The above result implies that perfectly normal, locally compact spaces having the property S are collectionwise normal under $V = L$. Recently in [**8**], by modifying the construction of the Kunen line, Gruenhage and Daniels construct a perfectly normal, locally compact, CWH, noncollectionwise normal space under $\diamond^*$ (which holds in L). Thus such a space does not have the property S (one cannot delete the property S from 4.1).

Next we shall consider paracompactness in perfectly normal, locally compact, locally connected spaces.

Theorem 4.2. *Let X be a perfectly normal, locally compact, locally connected space. Then X has the property S if and only if X is paracompact.*

Proof. Since perfectly normal, locally compact, locally connected spaces are CWH ([**3**, 8.9]), it is evident by 1.5 (or 1.8). □

Remark. The above result implies that every perfectly normal manifold is metrizable if and only if it has the property S.

Now, as a corollary, we can get Lane's theorem using 3.2 and 4.2.

Corollary 4.3. ([**10**]) [MA(ω_1)]. *Perfectly normal, locally compact, locally connected spaces are paracompact.*

5. Equivalents of $2^\omega < 2^{\omega_1}$. In this section, we shall show that paracompactness of certain perfectly normal, locally Lindelöf spaces is equivalent to $2^\omega < 2^{\omega_1}$. To begin, we need some definitions and lemmas.

Definition. Let S be a subset of ω_1. $\Phi(S)$ denotes the following assertion.

For every $F : {}^{<\omega_1}2 \to 2$ there is a $g : \omega_1 \to 2$ such that for every $f : \omega_1 \to 2$, $\{\alpha \in S : F(f|\alpha) = g(\alpha)\}$ is stationary in ω_1, where ${}^{<\omega_1}2 = \cup_{\alpha<\omega_1} {}^{\alpha}2$ and ${}^{\alpha}2$ denotes the set of all functions from α to 2 and $f|\alpha$ denotes the restriction of f to α.

Incidentally, if $\Phi(S)$ holds, then S is stationary in ω_1. For more details, see [**5**]. The next results were proved in [**5**].

Lemma 5.1. ([**5**]). *The following assertions hold.*

1) *$I = \{S \subset \omega_1 : \Phi(S)$ does not hold$\}$ is a normal ideal on ω_1.*

2) *$2^\omega < 2^{\omega_1}$ holds if and only if $\omega_1 \notin I$.*

In a similar way of the proof of [**17**, 3.1], we can get the next result.

Lemma 5.2. *Let S be a stationary subset of ω_1 and X be a normal space of character $\leq 2^\omega$. Assume that $\Phi(S)$ holds. Then for every closed discrete subspace $\{x_\alpha : \alpha \in S\}$ of X indexed by S, there is a stationary subset S' of S such that $\{x_\alpha : \alpha \in S'\}$ has an open separation.*

Using the above results, we can get the following result.

Theorem 5.3. *The following assertions are equivalent.*

1) $2^\omega < 2^{\omega_1}$ *holds.*

2) *Every perfectly normal, locally Lindelöf, connected space of countable tightness and character $\leq 2^\omega$ which has the property S is Lindelöf.*

3) *Every perfectly normal, locally Lindelöf, connected, ω_1-Lindelöf space of character $\leq 2^\omega$ which has the property S is Lindelöf.*

Proof. To show 1) $\to$ 2), assume that X is such a space as in 2). We shall show that X is Lindelöf by modifying the proof of 1.8.

Claim. *The closure of every Lindelöf open set is Lindelöf.*

Proof. Assume on the contrary that there is an open Lindelöf W such that $\mathrm{cl}\, W$ is not Lindelöf. Then there is a right separated $Y = \{x_\alpha : \alpha < \omega_1\} \subset \mathrm{cl}\, W$. Now identify Y with ω_1. Since X has the property S, ω_1 is σ-discrete, say $\omega_1 = \cup_{n<\omega} S_n$. By 5.1, there is an $n < \omega$ such that $\Phi(S_n)$. By perfectness, S_n is the countable union of closed discrete subsets of X, say $S_n = \cup_{m<\omega} S_{nm}$. Again by 1) of 5.1 there is an $m < \omega$ such that $\Phi(S_{nm})$ holds. Then by 5.2, there is a stationary subset S' of S_{nm} such that S' has an open separation $\{V_\alpha : \alpha \in S'\}$. But $\{V_\alpha \cap W : \alpha \in S'\}$ witnesses a contradiction, since W is ccc. This completes the proof of the claim. $\square$

Next let $X_0 \subset X$ be a Lindelöf open set. We can define an increasing sequence of Lindelöf open sets $\{X_\alpha : \alpha < \omega_1\}$ such that $\mathrm{cl}\, X_\alpha \subset X_\beta$

if $\alpha < \beta < \omega_1$ and $\cup_{\alpha<\beta} X_\alpha = \cup_{\alpha<\beta} \mathrm{cl}\, X_\alpha$ if β is limit. Because, if $\beta = \alpha + 1$, take a Lindelöf open set X_β such that $\mathrm{cl}\, X_\alpha \subset X_\beta$ using the above claim. If β is limit, let $X_\beta = \cup_{\alpha<\beta} X_\beta$.

Then $\cup_{\alpha<\omega_1} X_\alpha = \cup_{\alpha<\omega_1} \mathrm{cl}\, X_\alpha$ is clopen in X, since X has countable tightness. Therefore, $X = \cup_{\alpha<\omega_1} X_\alpha$ by the connectedness of X. We shall show that there is an $\alpha < \omega_1$ such that $X_\alpha = X$ (then X is Lindelöf). Assume on the contrary that $X - X_\alpha \neq 0$ for every $\alpha < \omega_1$. Take an $x_\alpha \in \partial X_\alpha$ for each $\alpha < \omega_1$ using the connectedness of X. Then $Y = \{x_\alpha : \alpha < \omega_1\}$ is right separated. Identify Y with ω_1. In a similar way of the proof of the above claim, we can get a stationary discrete closed subset S of ω_1 and an open separation $\{V_a : \alpha \in S\}$ of S. As in the proof of claim 2 of 1.4, using the pressing down lemma, we can get a contradiction. This completes the proof of 1) $\to$ 2).

The proof of 1) $\to$ 3) is similar to the above proof.

2) $\to$ 1) and 3) $\to$ 1) can be proved simultaneously by giving a counterexample assuming $2^\omega = 2^{\omega_1}$. Assume $2^\omega = 2^{\omega_1}$. Then there is a collection of ω_1-many free ultrafilters on ω, say $\{x_\alpha : \alpha < \omega_1\}$, such that for any subset D of ω_1 there is a subset U of ω such that $U \in x_\alpha$ for any $\alpha \in D$ and $\omega - U \in x_\alpha$ for any $\alpha \in \omega_1 - D$, by [**4**].

Let R be the real line. Since R is normal and ω is closed in R (hence ω is C^*-embedded in R), $\beta\omega = \mathrm{cl}_{\beta R}\omega \subset \beta R$ is valid. Where βY denotes the Stone-Čech compactification of a Tychonoff space Y. Let X be $R \cup \{x_\alpha : \alpha < \omega_1\}$. Equip X with the subspace topology on βR. We shall show that this X is the counterexample.

Since R is connected and dense in X, X is connected.

To show normality of X, it is enough to show that the subspace $X - R$ is normalized in X. Where a subspace A is *normalized* if for every subset B of A, B and $A - B$ can be separated by disjoint open sets. Let D be a subset of ω_1 and U be the subset of ω as above. And let W be the set $\cup\{(n - 1/2, n + 1/2) : n \in U\}$, and W' be an open set of X such that $W' \cap R = W$. Then it is not hard to show that $\{x_\alpha : \alpha \in D\} \subset W'$ and $\mathrm{cl}\, W' \cap \{x_\alpha : \alpha \in \omega_1 - D\} = 0$. Hence X is normal. This argument implies $\{x_\alpha : \alpha < \omega_1\}$ is closed discrete in X. Therefore, X is not Lindelöf.

Since points of R have compact neighborhoods in X, to show the local Lindelöfness of X we must show that points of $X - R$ have Lindelöf closed neighborhood of X. Take an open neighborhood U of x_α such that $\operatorname{cl} U \cap \{x_\alpha : \alpha < \omega_1\} = \{x_\alpha\}$. Since R is hereditarily Lindelöf, it is easy to show that $\operatorname{cl} U$ is Lindelöf. Thus, X is locally Lindelöf.

Since R has a countable basis, the character of X does not exceed 2^ω.

Since R is hereditarily separable, it is easy to show that the tightness of X does not exceed ω.

To show perfectness of X, let U be an open set of X. Since R is locally compact and hereditarily Lindelöf, $U \cap R$ is the countable union of compact sets. Furthermore, $U - R$ is closed in X. Thus, U is the countable union of closed sets of X.

To show that X has the property S, let $\{x_\alpha : \alpha < \omega_1\}$ be a right separated subset of X, identify this set with ω_1. Then $\omega_1 \cap R$ is σ-discrete (in fact countable), since R is hereditarily Lindelöf. Furthermore, $\omega_1 - R$ is closed discrete in X. Thus ω_1 is σ-discrete.

It is easy to show ω_1-Lindelöfness of X using the hereditarily Lindelöfness of R.

This completes the proof. □

Remark. The above proof implies that the assertion, that every perfectly normal, locally Lindelöf, ω_1-Lindelöf space of character $\leq 2^\omega$ which has the property S is Lindelöf, implies $2^\omega < 2^{\omega_1}$.

Corollary 5.4. $[2^\omega < 2^{\omega_1}]$. *In every perfectly normal, locally compact, connected space, it has the property S if and only if it is Lindelöf.*

Proof. Because perfect, locally compact spaces are first countable, character and tightness are countable. □

Corollary 5.5. $[2^\omega < 2^{\omega_1}]$. *Let X be a perfectly normal, locally Lindelöf, locally connected space of countable tightness and character $\leq 2^\omega$. Then X has the property S if and only if X is paracompact.*

Proof. Apply 5.3 in each component of X. □

Remark. Note that by 4.2, the resulting assertion of the above corollary, that local Lindelöfness is replaced by local compactness, is valid without $2^\omega < 2^{\omega_1}$.

The author would like to express his thanks to the referee of this paper for his valuable comments.

REFERENCES

1. Z. Balogh, *Locally nice space under Martin's Axiom*, Comment Math. Univ. Carol. **24** (1983), 63–85.

2. ———, *Paracompactness in locally Lindelöf spaces*, Canadian J. Math. **38** (1986), 719–727.

3. D.K. Burke, *Covering properties*, in *Hand book of set theoretic topology*, North-Holland, Amsterdam, 1984.

4. A. Charlesworth, R. Hodel and F.D. Tall, *On a theorem of Jones and Heath concerning discrete subspaces*, Colloq. Math. **34** (1975), 33–37.

5. K.J. Devlin and S. Shelah, *A weak version of* $\diamond$ *which follows* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. **29** (1978), 239–247.

6. R. Engelking, *General topology*, Polish Scientific Publishers, Warsawa, Poland, 1977.

7. G. Gruenhage, *Paracompactness and subparacompactness in perfectly normal locally compact spaces*, Russian Math. Surveys **35** (1980), 49–55.

8. G. Gruenhage and P. Daniels, *A perfectly normal, locally compact, noncollectionwise normal space from* $\diamond^*$, Proc. Amer. Math. Soc. **95** (1985), 115–118.

9. K. Kunen, *Set theory*, North-Holland, Amsterdam, 1980.

10. D. Lane, *Paracompactness in perfectly normal, locally connected, locally compact spaces*, Proc. Amer. Math. Soc. **80** (1980), 693–696.

11. P. Nyikos, *Covering properties on* σ*-scattered spaces*, Topology Proceedings **2** (1977), 509–542.

12. ———, *The theory of nonmetrizable manifolds*, in *Hand book of set theoretic topology*, North-Holland, Amsterdam (1984), 633–684.

13. M.E. Rudin, *The undecidability of the existence of a perfectly normal manifold*, Houston J. Math. **5** (1979), 249–252.

14. J. Roitman, *Basic S and L*, in *Hand book of set theoretic topology*, North-Holland, Amsterdam (1984), 295–326.

15. Z. Szentmiklóssy, *S-spaces and L-spaces under Martin's axiom*, Coll. Math. Soc. János Bolyai, Budapest, Hungary (1978), 1139–1145 (North-Holland, Amsterdam, 1980).

16. F.D. Tall, *Normality versus collectionwise normality*, in *Hand book of set theoretic topology*, North-Holland, Amsterdam (1984), 685–732.

17. A.D. Taylor, *Diamond principles, ideals and the normal Moore problem*, Canadian J. Math. **33** (1981), 282–296.

18. S. Watson, *Locally compact normal metaLindelöf spaces may not be paracompact. An application of uniformization and Suslin lines*, Proc. Amer. Math. Soc. **98** (1986), 676–680.

19. W. Weiss, *Versions of Martin's Axiom*, in *Hand book of set theoretic topology*, North-Holland, Amsterdam (1984), 827–886.

Department of Mathematics, College of Education, Ryukyu University, Nishihara-cho, Okinawa, Japan

Current address: Department of Mathematics, Faculty of Education, Oita University, Dannoharu Oita 870-11, Japan

e-mail: nkemoto@cc.oita-u.ac.jp