ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

NONSELF-ADJOINT DIFFERENTIAL OPERATORS IN DIRECT SUM SPACES

SOBHY E. IBRAHIM

1. Introduction. In [**8**] Everitt and Zettl considered the problem of characterizing all the self-adjoint operators which can be generated by formally symmetric Sturm-Liouville differential expressions M_p ($p = 1, 2$) defined on two intervals I_p ($p = 1, 2$) with boundary conditions at the endpoints. Their work was motivated by Sturm-Liouville problems which occur in the literature in which the coefficients have a singularity in the interior of the underlying interval. An interesting feature of their work is the possibility of generating self-adjoint operators in this way which are not expressible as the direct sums of self-adjoint operators defined in the separate intervals.

Our objective in this paper is to extend the results of Everitt and Zettl in [**8**] to the case where the differential expressions M_p are arbitrary and there is any finite number of intervals I_p, $p = 1, \dots, N$.

The operators involved are no longer symmetric but direct sums

$$T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p), T_0(M^+) = \bigoplus_{p=1}^{N} T_0(M_p^+),$$

where $T_0(M_p)$ is the minimal operator generated by M_p in I_p and M_p^+ denotes the formal adjoint of M_p, which form an adjoint pair of closed operators in $\oplus_{p=1}^{N} L^2_{w_p}(I_p)$. This fact allows us to use the abstract theory developed in [**1**] for the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$. Such an operator S satisfies $T_0(M) \subset S \subset [T_0(M^+)]^*$ and for some $\lambda \in \mathbf{C}$, $(S - \lambda I)$ is Fredholm with zero index. This class of operators is the counterpart of the class of maximal symmetric and self-adjoint operators in the case when $T_0(M)$ is symmetric. Using ideas and results from [**2**], we are also able to characterize all the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ in terms of the $L^2_{w_p}(I_p)$ solutions of

Received by the editors on October 15, 1989 and in revised form on April 4, 1990.

$M_p[u] = \lambda w_p u$. In order to prove this for all the cases that can occur in the intervals I_p, we need the analogue of the results in [**2**] for the case when the endpoints of the underlying interval are both singular. This is a result of special interest and extends one proved in [**15**] by Zai-Jiu Shang for formally symmetric and J-symmetric differential expressions.

2. Preliminaries. We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [**1**, Chapter III and **3**].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$, $R(T)$, respectively, and $N(T)$ will denote its null space. The *nullity* of T, written $\text{nul}\,(T)$, is the dimension of $N(T)$ and the *deficiency* of T, $\text{def}\,(T)$, is the codimension of $R(T)$ in H; thus, if T is densely defined and $R(T)$ is closed, then $\text{def}\,(T) = \text{nul}\,(T^*)$. The *Fredholm domain* of T is (in the notation of [**1**]) the open subset $\Delta_3(T)$ of $\mathbf{C}$ consisting of those values $\lambda \in \mathbf{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus, $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The *index* of $(T - \lambda I)$ is the number $\text{ind}\,(T - \lambda I) = \text{nul}\,(T - \lambda I) - \text{def}\,(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A, B in H are said to form an *adjoint pair* if $A \subset B^*$ and consequently $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$ where $(\cdot,\cdot)$ denotes the inner-product on H.

The *joint field of regularity* $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbf{C}$ which are such that $\lambda \in \Pi(A)$, the field of regularity of A, $\bar{\lambda} \in \Pi(B)$ and $\text{def}\,(A - \lambda I)$ and $\text{def}\,(B - \bar{\lambda} I)$ are finite. An adjoint pair A, B is said to be *compatible* if $\Pi(A, B) \neq \varnothing$. Recall that $\lambda \in \Pi(A)$ if and only if there exists a positive constant $K(\lambda)$ such that

$$||(A - \lambda I)x|| \geq K(\lambda)||x|| \quad \text{for all } x \in D(A),$$

or equivalently, on using the closed-graph theorem, $\text{nul}\,(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

Definition 2.1. A closed operator S in H is said to be *regularly solvable* with respect to the compatible adjoint pair A, B if $A \subset S \subset B^*$

and $\Pi(A, B) \cap \Delta_4(S) \neq \varnothing$, where

$$\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), \operatorname{ind}(S - \lambda I) = 0\}.$$

The terminology *regularly solvable* comes from Visik's paper [**16**].

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrix A_p on an interval I_p, where I_p denotes an interval with left endpoint a_p and right endpoint b_p $(-\infty \leq a_p < b_p \leq \infty)$, $p = 1, 2, \ldots, N$. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $(n \times n)$-matrices $A_p = \{a^p_{rs}\}$ whose entries are complex-valued functions on I_p which satisfy the following conditions:

(2.1)
$$\begin{cases} a^p_{rs} \in L^1_{\mathrm{loc}}(I_p) & & 1 \leq r,\, s \leq n,\, n \geq 2, \\ a^p_{r,r+1} \neq 0 & \text{a.e. on } I_p & 1 \leq r \leq n-1 \\ a^p_{rs} = 0 & \text{a.e. on } I_p & 2 \leq r+1 < s \leq n, \;\; p = 1, \ldots, N. \end{cases}$$

For $A_p \in Z_n(I_p)$ the *quasi-derivatives* associated with A_p are defined by

(2.2)
$$\begin{cases} y^{[0]} := y, & \\ y^{[r]} := (a^p_{r,r+1})^{-1}\{(y^{[r-1]})' - \sum_{s=1}^{r} a^p_{rs} y^{[s-1]}\} & 1 \leq r \leq n-1, \\ y^{[n]} := (y^{[n-1]})' - \sum_{s=1}^{n} a^p_{ns} y^{[s-1]}, & p = 1, \ldots, N, \end{cases}$$

where the prime $'$ denotes differentiation.

The *quasi-differential expression* M_p associated with A_p is given by

$$M_p[y] := i^n y^{[n]}, \qquad p = 1, \ldots, N, \tag{2.3}$$

this being defined on the set

$$V(M_p) := \{y : y^{[r-1]} \in AC_{\mathrm{loc}}(I_p),\; r = 1, \ldots, n;\; p = 1, \ldots, N\} \tag{2.4}$$

where $AC_{\mathrm{loc}}(I_p)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The *formal adjoint* M_p^+ of M_p is defined by the matrix $A_p^+ \in Z_n(I_p)$ given by

$$A_p^+ := -J^{-1}_{n\times n} A_p^* J_{n\times n}, \tag{2.5}$$

where A_p^* is the conjugate transpose of A_p and $J_{n\times n}$ is the nonsingular $n \times n$ matrix

$$J_{n\times n} = ((-1)^r \delta_{r,n+1-s})_{\substack{1\le r\le n,\\ 1\le s\le n}}, \tag{2.6}$$

δ being the Kronecker delta. If $A_p^+ = \{a_{rs}^p\}^+$, then it follows that

$$\{a_{rs}^p\}^+ = (-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}^p. \tag{2.7}$$

The quasi-derivatives associated with A_p^+ are therefore

(2.8)

$$\begin{cases} y_+^{[0]} := y \\ y_+^{[r]} := (\bar{a}_{n-r,n-r+1}^p)^{-1}\{(y_+^{[r-1]})' - \sum_{s=1}^r (-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}^p y_+^{[s-1]}, \\ \qquad\qquad 1\le r\le n-1\}, \\ y_+^{[n]} := (y_+^{[n-1]})' - \sum_{s=1}^n (-1)^{n+s+1}\bar{a}_{n-s,s+1,1}^p y_+^{[s-1]}, \end{cases}$$

and

$$M_p^+[y] := i^n y_+^{[n]}, \qquad p = 1,\dots,N, \tag{2.9}$$

for all y in

(2.10)

$$V(M_p^+) := \{y : y_+^{[r-1]} \in AC_{\mathrm{loc}}(I_p),\ r = 1,\dots,n;\ p = 1,\dots,N\}.$$

Note that $(A_p^+)^+ = A_p$ and so $(M_p^+)^+ = M_p$. We refer to [**6, 7** and **17**] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V(M_p)$, $v \in V(M_p^+)$ and $\alpha,\beta \in I_p$, $p = 1,\dots,N$, we have Green's formula,

$$\int_\alpha^\beta \{\bar{v}M_p[u] - u\overline{M_p^+[v]}\}\,dx = [u,v]_p(\beta) - [u,v]_p(\alpha), \tag{2.11}$$

where

$$\begin{aligned} [u,v]_p(x) &= \left(i^n \sum_{r=0}^{n-1}(-1)^{n+r+1}u^{[r]}(x)\bar{v}_+^{[n-r-1]}(x)\right) \\ &= (-i)^n[u(x),\dots,u^{[n-1]}(x)]J_{n\times n}\begin{bmatrix}\bar{v}(x)\\ \vdots \\ \bar{v}_+^{[n-1]}(x)\end{bmatrix}; \end{aligned} \tag{2.12}$$

see [**17**, Corollary 1].

Given functions $f_{jp} \in V(M_p)$, $g_{kp} \in V(M_p^+)$, $j = 1, \dots, r$, $k = 1, \dots, s$; $p = 1, \dots, N$ we obtain from (2.12) that

$$\begin{aligned}(2.13)\quad ([f_{jp}, g_{kp}]_p(x))_{\substack{1\le j\le r\\ 1\le k\le s}} \\ = (-i)^n \left(f_{jp}^{[\ell-1]}(x)\right)_{\substack{1\le j\le r\\ 1\le \ell\le n}} J_{n\times n} \left((\bar{g}_{kp})_+^{[\ell-1]}(x)\right)_{\substack{1\le \ell\le n\\ 1\le k\le s}}.\end{aligned}$$

Let w_p be a function which satisfies

$$(2.14)\quad w_p > 0 \qquad \text{a.e. on } I_p, \qquad w_p \in L^1_{\text{loc}}(I_p), \qquad p = 1, \dots, N.$$

The equation

$$(2.15)\qquad M_p[y] = \lambda w_p y, \qquad \lambda \in \mathbf{C}$$

on I_p is said to be *regular* at the left endpoint a_p if it is finite and $X \in (a_p, b_p)$,

$$(2.16)\quad a_p \in \mathbf{R}, \qquad w_p, a_{rs}^p \in L^1[a_p, X], \qquad r, s = 1, \dots, n;\ p = 1, \dots, N.$$

Otherwise, (2.15) is said to be *singular* at a_p. Similarly, we define the terms regular and singular at b_p. If (2.15) is regular at both endpoints then it is said to be regular; in this case we have

$$(2.17)\quad a_p, b_p \in \mathbf{R}, \qquad w_p, a_{rs}^p \in L^1(a_p, b_p) \qquad r, s = 1, \dots, n;\ p = 1, \dots, N.$$

Note that, in view of (2.7), an endpoint of I_p is regular for (2.15) if and only if it is regular for the equation

$$(2.18)\qquad M_p^+[y] = \bar{\lambda} w_p y \qquad \lambda \in \mathbf{C}.$$

Let $H_p = L^2_{w_p}(a_p, b_p)$ denote, for $p = 1, \dots, N$ the usual weighted L^2-space with inner-product

$$(2.19)\qquad (f, g)_p := \int_{I_p} f(x)\overline{g(x)} w_p(x)\, dx, \qquad p = 1, \dots, N,$$

and norm $||f|| := (f,f)^{1/2}_{w_p}$; this is a Hilbert space on identifying functions which differ only on null sets. Set

(2.20)
$$\begin{cases} D(M_p) := \{u : u \in V(M_p), u \text{ and } \frac{1}{w_p}M_p[u] \in L^2_{w_p}(a_p,b_p) \\ \qquad\qquad\qquad\qquad\qquad\qquad p = 1,\dots,N\}, \\ D(M_p^+) := \{v : v \in V(M_p^+), v \text{ and } \frac{1}{w_p}M_p^+[v] \in L^2_{w_p}(a_p,b_p), \\ \qquad\qquad\qquad\qquad\qquad\qquad p = 1,\dots,N\}. \end{cases}$$

Note that, at a regular endpoint, say a_p, $u^{[r-1]}(a_p)(v_+^{[r-1]}(a_p))$ is defined for all $u \in V(M_p)(v \in V(M_p^+))$, $r = 1,2,\dots,n$. The manifolds $D(M_p)$, $D(M_p^+)$ of $L^2_{w_p}(a_p,b_p)$ are the domains of the so-called *maximal operators* $T(M_p)$, $T(M_p^+)$, respectively, defined by

$$T(M_p)u := \frac{1}{w_p}M_p[u], \qquad u \in D(M_p)$$

and

$$T(M_p^+)v := \frac{1}{w_p}M_p^+[v], \qquad v \in D(M_p^+).$$

For the regular problem the *minimal operators* $T_0(M_p)$, $T_0(M_p^+)$ are the restrictions of $\frac{1}{w_p}M_p[\cdot]$ and $\frac{1}{w_p}M_p^+[\cdot]$ to the subspaces,

$$(2.21)\quad \begin{cases} D_0(M_p) := \{u : u \in D(M_p),\ u^{[r-1]}(a_p) = u^{[r-1]}(b_p) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad r = 1,\dots,n\} \\ D_0(M_p^+) := \{v : v \in D(M_p^+),\ v_+^{[r-1]}(a_p) = v_+^{[r-1]}(b_p) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad r = 1,\dots,n\}, \end{cases}$$

respectively. The subspaces $D_0(M_p)$, $D_0(M_p^+)$ are dense in $L^2_{w_p}(a_p,b_p)$ and $T_0(M_p)$, $T_0(M_p^+)$ are closed operators (see [**17**, Section 3]). In the singular problem we first introduce operators $T_0'(M_p)$, $T_0'(M_p^+)$, $T_0'(M_p)$ being the restriction of $1/w_pM_p[\cdot]$ to

$$(2.22)\quad D_0'(M_p) := \{u : u \in D(M_p), \operatorname{supp} u \subset (a_p,b_p),\ p = 1,\dots,N\},$$

and with $T_0'(M_p^+)$ defined similarly. These operators are densely-defined and closable in $L^2_{w_p}(a_p,b_p)$ and we define the minimal operators $T_0(M_p)$, $T_0(M_p^+)$ to be their respective closures (cf. [**17**, Section 5]). We denote the domains of $T_0(M_p)$ and $T_0(M_p^+)$ by $D_0(M_p)$ and $D_0(M_p^+)$, respectively. It can be shown that, if (2.15) is regular at a_p,

(2.23)
$$\begin{cases} u \in D_0(M_p) \Longrightarrow u^{[r-1]}(a_p) = 0, & r = 1, \dots, n;\ p = 1, \dots, N \\ v \in D_0(M_p^+) \Longrightarrow v_+^{[r-1]}(a_p) = 0, & r = 1, \dots, n;\ p = 1, \dots, N. \end{cases}$$

Moreover, in both the regular and singular problems, we have

(2.24)
$$[T_0(M_p)]^* = T(M_p^+), \qquad [T(M_p)]^* = T_0(M_p^+), \qquad p = 1, \dots, N;$$

see [**17**, Section 5] in the case when $M_p = M_p^+$ and compare with the treatment in [**1**, Section III.10.3] in the general case.

In the case of two singular endpoints, the problem on (a_p, b_p) is effectively reduced to the problems with one singular endpoint on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, where $c_p \in (a_p, b_p)$. We denote by $T(M_p; a_p)$, $T(M_p; b_p)$ the maximal operators with domains $D(M_p; a_p)$ and $D(M_p; b_p)$ respectively and denote by $T_0(M_p; a_p)$ and $T_0(M_p; b_p)$ the closures of the operators $T_0'(M_p; a_p)$ and $T_0'(M_p; b_p)$ defined in (2.22) on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, respectively.

Let $\tilde{T}_0'(M_p)$ be the orthogonal sum $\tilde{T}_0'(M_p) = T_0'(M_p; a_p) \oplus T_0'(M_p; b_p)$ in

$$L^2_{w_p}(a_p, b_p) = L^2_{w_p}(a_p, c_p) \oplus L^2_{w_p}(c_p, b_p), \qquad p = 1, \dots, N;$$

$\tilde{T}_0'(M_p)$ is densely defined and closable in $L^2_{w_p}(a_p, b_p)$ and its closure is given by

$$\tilde{T}_0(M_p) = T_0(M_p; a_p) \oplus T_0(M_p; b_p), \qquad p = 1, \dots, N.$$

Also,

$$\operatorname{nul}[\tilde{T}_0(M_p) - \lambda I] = \operatorname{nul}[T_0(M_p; a_p) - \lambda I] + \operatorname{nul}[T_0(M_p; b_p) - \lambda I],$$
$$\operatorname{def}[\tilde{T}_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p; a_p) - \lambda I] + \operatorname{def}[T_0(M_p; b_p) - \lambda I],$$

and $R[\tilde{T}_0(M_p) - \lambda I]$ is closed if and only if $R[T_0(M_p; a_p) - \lambda I]$ and $R[T_0(M_p; b_p) - \lambda I]$ are both closed. These results imply in particular that,

$$\Pi[\tilde{T}_0(M_p)] = \Pi[T_0(M_p; a_p)] \cap \Pi[T_0(M_p; b_p)], \qquad p = 1, \dots, N.$$

We refer to [**1**, Section III.10.4, **3** and **12**] for more details.

Next, we state the following result; the proof is similar to that in [**1**, Section III.10.4].

Theorem 2.2. $\tilde{T}_0(M_p) \subset T_0(M_p)$, $T(M_p) \subset T(M_p; a_p) \oplus T(M_p; b_p)$ *and*

$$\dim \{D[T_0(M_p)]/D[\tilde{T}_0(M_p)]\} = n.$$

If $\lambda \in \Pi[\tilde{T}_0(M_p)] \cap \Delta_3[T_0(M_p) - \lambda I]$, *then*

$$\operatorname{ind}[T_0(M_p) - \lambda I] = n - \operatorname{def}[T_0(M_p; a_p) - \lambda I] - \operatorname{def}[T_0(M_p; b_p) - \lambda I],$$

and in particular, if $\lambda \in \Pi[T_0(M_p)]$,

$$\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p; a_p) - \lambda I] + \operatorname{def}[T_0(M_p; b_p) - \lambda I] - n, \tag{2.25}$$

$p = 1, \dots, N$.

Remark 2.3. It can be shown that

$$\begin{aligned} D[\tilde{T}_0(M_p)] &= \{u : u \in D[T_0(M_p)] \text{ and } u^{[k-1]}(c_p) = 0, k = 1, \dots, n\}, \\ D[\tilde{T}_0(M_p^+)] &= \{v : v \in D[T_0(M_p^+)] \text{ and } v_+^{[k-1]}(c_p) = 0, k = 1, \dots, n\}; \end{aligned} \tag{2.26}$$

see [**1**, Section III.10.4].

Let H be the direct sum,

$$H = \bigoplus_{p=1}^{N} H_p = \bigoplus_{p=1}^{N} L^2_{w_p}(I_p). \tag{2.27}$$

Elements of H will be denoted by $\underset{\sim}{f} = \{f_1, \dots, f_N\}$ with $f_1 \in H_1, \dots, f_N \in H_N$.

When $I_i \cap I_j = \varnothing$, $i \neq j$, $i, j = 1, \dots, N$, the direct sum space $\oplus_{p=1}^N L^2_{w_p}(I_p)$ can be naturally identified with the space $L^2_w(\cup_{p=1}^N I_p)$, where $w = w_p$ on I_p, $p = 1, \dots, N$. This remark is of particular significance when $\cup_{p=1}^N I_p$ may be taken as a single interval; see [**8** and **12**].

We now establish by [**8, 13** and **14**] some further notation.

$$\text{(2.28)} \qquad \begin{cases} D_0(M) = \oplus_{p=1}^N D_0(M_p), & D(M) = \oplus_{p=1}^N D(M_p), \\ D_0(M^+) = \oplus_{p=1}^N D_0(M_p^+), & D(M^+) = \oplus_{p=1}^N D(M_p^+), \end{cases}$$

$$\text{(2.29)} \qquad \begin{cases} T_0(M)\underset{\sim}{f} = \{T_0(M_1)f_1, \dots, T_0(M_N)f_N\}, \\ \qquad f_1 \in D_0(M_1), \dots, f_N \in D_0(M_N), \\ T_0(M^+)\underset{\sim}{g} = \{T_0(M_1^+)g_1, \dots, T_0(M_N^+)g_N\}, \\ \qquad g_1 \in D_0(M_1^+), \dots, g_N \in D_0(M_N^+). \end{cases}$$

Also,

$$\text{(2.30)} \qquad \begin{cases} T(M)\underset{\sim}{f} = \{T(M_1)f_1, \dots, T(M_N)f_N\}, \\ \qquad f_1 \in D(M_1), \dots, f_N \in D(M_N), \\ T(M^+)\underset{\sim}{g} = \{T(M_1^+)g_1, \dots, T(M_N^+)g_N\}, \\ \qquad g_1 \in D(M_1^+), \dots, g_N \in D(M_N^+), \end{cases}$$

(2.31)

$$[\underset{\sim}{f}, \underset{\sim}{g}] = \sum_{p=1}^N \{[f_p, g_p]_p(b_p) - [f_p, g_p]_p(a_p)\}, \qquad \underset{\sim}{f} \in D(M), \underset{\sim}{g} \in D(M^+);$$

$$\text{(2.32)} \qquad (\underset{\sim}{f}, \underset{\sim}{g}) = \sum_{p=1}^N (f_p, g_p)_p,$$

where $\underset{\sim}{f} = \{f_1, \dots, f_N\}$, $\underset{\sim}{g} = \{g_1, \dots, g_N\}$ and $(\cdot, \cdot)_p$ the inner-product defined in (2.19).

Note that $T_0(M)$ is a closely densely defined operator in H.

We summarize a few additional properties of $T_0(M)$ in the form of a lemma.

Lemma 2.4. *We have*

(a)
$$[T_0(M)]^* = \bigoplus_{p=1}^{N} [T_0(M_p)]^* = \bigoplus_{p=1}^{N} T(M_p^+)$$
$$[T_0(M^+)]^* = \bigoplus_{p=1}^{N} [T_0(M_p^+)]^* = \bigoplus_{p=1}^{N} T(M_p).$$

In particular,

$$D[T_0(M)]^* = D[T(M^+)] = \bigoplus_{p=1}^{N} D[T(M_p^+)],$$
$$D[T_0(M^+)]^* = D[T(M)] = \bigoplus_{p=1}^{N} D[T(M_p)].$$

(b)
$$\operatorname{nul}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \operatorname{nul}[T_0(M_p) - \lambda I],$$
$$\operatorname{nul}[T_0(M^+) - \bar{\lambda} I] = \sum_{p=1}^{N} \operatorname{nul}[T_0(M_p^+) - \bar{\lambda} I].$$

(c) The deficiency indices of $T_0(M)$ are given by

$$\operatorname{def}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I], \quad \text{for all } \lambda \in \Pi[T_0(M)],$$
$$\operatorname{def}[T_0(M^+) - \bar{\lambda} I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I], \quad \text{for all } \bar{\lambda} \in \Pi[T_0(M^+)].$$

Proof. Part (a) follows immediately from the definition of $T_0(M)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions. □

Lemma 2.5. *Let* $T_0(M) = \oplus_{p=1}^N T_0(M_p)$ *be a closely densely defined operator on* H. *Then*

$$\Pi[T_0(M)] = \bigcap_{p=1}^N \Pi[T_0(M_p)]. \tag{2.33}$$

Proof. The proof follows from Lemma 2.4 and since $R[T_0(M) - \lambda I]$ is closed, if and only if $R[T_0(M_p) - \lambda I]$, $p = 1, \dots, N$ are closed. □

Lemma 2.6. *If* S_p, $p = 1, \dots, N$ *are regularly solvable with respect to* $T_0(M_p)$ *and* $T_0(M_p^+)$, *then*

$$S = \oplus_{p=1}^N S_p$$

is regularly solvable with respect to $T_0(M)$ *and* $T_0(M^+)$.

Proof. The proof follows from Lemmas 2.4 and 2.5. □

3. The case of one interval with two singular endpoints. In this section we shall consider our interval to be $I = (a, b)$. We denote by $T(M)$ and $T_0(M)$ the maximal and minimal operators. We see from (2.24) that $T_0(M) \subset T(M) = [T_0(M^+)]^*$ and, hence, $T_0(M)$, $T_0(M^+)$ form an adjoint pair of closed densely defined operators in $L_w^2(a, b)$.

Lemma 3.1. *For* $\lambda \in \Pi[T_0(M), T_0(M^+)]$, $\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I]$ *is constant and*

$$0 \le \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] \le 2n. \tag{3.1}$$

In the problem with one singular endpoint,

$$n \le \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \lambda I] \le 2n \quad \textit{for } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

In the regular problem,

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = 2n, \quad \textit{for } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

Proof. For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we obtain from (2.24) and (2.25) that

$$\begin{aligned}
&\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\bar{\lambda} I]\\
&\quad=\{\operatorname{def}[T_0(M;a)-\lambda I]+\operatorname{def}[T_0(M;b)-\lambda I]-n\}\\
&\qquad+\{\operatorname{def}[T_0(M^+;a)-\bar{\lambda} I]+\operatorname{def}[T_0(M^+;b)-\bar{\lambda} I]-n\}\\
&\quad=\{\operatorname{nul}[T(M^+;a)-\bar{\lambda} I]+\operatorname{nul}[T(M^+;b)-\bar{\lambda} I]-n\}\\
&\qquad+\{\operatorname{nul}[T(M;a)-\lambda I]+\operatorname{nul}[T(M;b)-\lambda I]-n\}\\
&\quad\le 2(2n-n)=2n,
\end{aligned}$$

with equality in the regular problem. In the problem with one singular endpoint it is proved in [**2**] that $\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\bar{\lambda} I] \ge n$. For the problem with two singular endpoints, we have

$$\begin{aligned}
&\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\bar{\lambda} I]\\
&\quad=\{\operatorname{def}[T_0(M;a)-\lambda I]+\operatorname{def}[T_0(M^+;a)-\bar{\lambda} I]\}\\
&\qquad+\{\operatorname{def}[T_0(M;b)-\lambda I]+\operatorname{def}[T_0(M^+;b)-\bar{\lambda} I]\}-2n\\
&\quad\ge 2n-2n=0.
\end{aligned}$$

The Lemma is therefore proved. □

For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we define r, s and m as follows:

$$\begin{aligned}
r=r(\lambda) :&= \operatorname{def}[T_0(M)-\lambda I]\\
&=\operatorname{def}[T_0(M;a)-\lambda I]+\operatorname{def}[T_0(M;b)-\lambda I]-n\\
&=r_1+r_2-n,\\
s=s(\lambda) :&= \operatorname{def}[T_0(M^+)-\bar{\lambda} I]\\
&=\operatorname{def}[T_0(M^+;a)-\bar{\lambda} I]+\operatorname{def}[T_0(M^+;b)-\bar{\lambda} I]-n\\
&=s_1+s_2-n,
\end{aligned} \tag{3.2}$$

and

$$m := r+s.$$

Since

$$r=r_1+r_2-n, \qquad s=s_1+s_2-n,$$

then

$$\begin{aligned} m &= r + s \\ &= (r_1 + r_2 - n) + (s_1 + s_2 - n) \\ &= (r_1 + s_1) + (r_2 + s_2) - 2n \\ &= m_1 + m_2 - 2n. \end{aligned} \tag{3.3}$$

Also, since

$$n \le m_i \le 2n, \qquad i = 1, 2,$$

then by Lemma 3.1, we have that

$$0 \le m \le 2n. \tag{3.4}$$

For $\Pi[T_0(M), T_0(M^+) \neq \varnothing]$ the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ are characterized by the following theorem.

Theorem 3.2. *For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, let r and m be defined by* (3.2)*, and let ψ_j $(j = 1, \dots, r)$, ϕ_k $(k = r+1, \dots, m)$ be arbitrary functions satisfying*:

(i) *$\{\psi_j : j = 1, \dots, r\} \subset D(M)$ is linearly independent modulo $D_0(M)$ and $\{\phi_k : k = r+1, \dots, m\} \subset D(M^+)$ is linearly independent modulo $D_0(M^+)$;*

(ii) *$[\psi_j, \phi_k](a) - [\psi_j, \phi_k](b) = 0$, $j = 1, \dots, r$; $k = r+1, \dots, m$.*

Then the set

$$\{u : u \in D(M), [u, \phi_k](a) - [u, \phi_k](b) = 0,\ k = r+1, \dots, m\} \tag{3.5}$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and

$$\{v : v \in D(M^+), [\psi_j, v](a) - [\psi_j, v](b) = 0,\ j = 1, \dots, r\} \tag{3.6}$$

is the domain of S^*; moreover, $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, then with r and m defined by (3.2) *there exist functions ψ_j, $j = 1, \dots, r$, ϕ_k, $k = r+1, \dots, m$,* *which satisfy* (i) *and* (ii) *and are such that* (3.5) *and* (3.6) *are the domains of* S *and* S^**, respectively.*

S *is self-adjoint if and only if* $M^+ = M$, $r = s$ *and* $\phi_k = \psi_{k-r}$, $k = r+1, \dots, m$; S *is* J *self-adjoint if* $M = JM^+J$ (J *is a complex conjugate*), $r = s$ *and* $\phi_k = \bar{\psi}_{k-r}$, $k = r+1, \dots, m$.

Proof. This is a consequence of [**1**, Theorem III.3.6] and we give only a brief sketch of the proof.

Let D_1, D_2 denote the sets in (3.5) and (3.6), respectively, and define

$$D_1' := \left\{ u : u = u_0 + \sum_{j=1}^{r} c_j \psi_j \quad \text{for some } u_0 \in D_0(M) \text{ and } c_j \in \mathbf{C} \right\},$$

$$D_2' := \left\{ v : v = v_0 + \sum_{k=r+1}^{m} c_k \phi_k \quad \text{for some } v_0 \in D_0(M^+) \text{ and } c_k \in \mathbf{C} \right\}.$$

It is easily checked that $D_1' \subset D_1$ and $D_2' \subset D_2$. Since D_1 is determined by $m - r = s$, linear conditions and $D(M)/D_0(M)$ has dimension m, it follows that $D_1/D_0(M)$ has dimension $m - s = r$ and, consequently, $D_1' = D_1$. Similarly, $D_2' = D_2$. Let S, S^+ denote the restrictions of $T(M)$, $T(M^+)$ to D_1, D_2, respectively. On using $D_i = D_i'$, $i = 1, 2$, it is straightforward to prove that $S^+ = S^*$ and $S = (S^+)^*$; in particular, this implies that S is a closed r-dimensional extension of $T_0(M)$ and as $\lambda \in \Delta_3(T_0(M))$ it follows that $\lambda \in \Delta_3(S)$ (see [**1**, Theorem IX.4.1]). Furthermore, from [**1**, Theorem III.3.1],

$$\begin{aligned} r = \dim \{D(S)/D_0(M)\} &= \operatorname{ind}(S - \lambda I) + \operatorname{def}[T_0(M) - \lambda I] \\ &= \operatorname{ind}(S - \lambda I) + r, \end{aligned}$$

whence $\operatorname{ind}(S - \lambda I) = 0$ and $\lambda \in \Delta_4(S)$.

To prove the converse we first observe that the sesquilinear form $\beta[\cdot, \cdot]$ in [**1**, Theorem III.3.6] is now given by

$$\begin{aligned} \beta[u, v] &= (T(M)u, v) - (u, T(M^+)v) \\ &= [u, v](b) - [u, v](a), \qquad u \in D(M),\ v \in D(M^+); \end{aligned}$$

note that, since a, b are two singular endpoints, then the limits

$$[u, v](a) := \lim_{x \to a^+} [u, v](x), \qquad [u, v](b) := \lim_{x \to b^-} [u, v](x),$$

exist for all $u \in D(M)$, $v \in D(M^+)$ by (2.11). For $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, we obtain from [**1**, Theorem III.3.1] that $D(S)/D_0(M)$ and $N([T(M^+)-\bar{\lambda}I][S-\lambda I]$ have dimension def $[T_0(M)-\lambda I] =: r$ while $D(S^*)/D_0(M^+)$ and $N([T(M)-\lambda I][S^*-\bar{\lambda}I])$ have dimension def $[T_0(M^+)-\bar{\lambda}I] =: s$. The second part of the theorem follows from [**1**, Theorem III.3.6] on choosing $\{\psi_j : j = 1,\dots,r\}$, $\{\phi_k : k = r+1,\dots,m\}$ to be bases of $N([T(M^+)-\bar{\lambda}I][S-\lambda I])$ and $N([T(M)-\lambda I][S^*-\bar{\lambda}I])$, respectively. The last part of the theorem is immediate. □

For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, define r, s and m as in (3.2) and (3.3). Let $\{\Psi_j^a : j = 1,\dots,s_1\}$, $\{\Phi_k^a : k = s_1+1,\dots,m_1\}$ be bases for $N[T(M;a)-\lambda I]$, $N[T(M^+;a)-\bar{\lambda}I]$, respectively; thus, $\Psi_j^a, \Phi_k^a \in L_w^2(a,c)$, $j = 1,\dots,s_1, k = s_1+1,\dots,m_1$, and

$$M[\Psi_j^a] = \lambda w \Psi_j^a, \qquad M^+[\Phi_k^a] = \bar{\lambda} w \Phi_k^a \quad \text{on } (a,c). \tag{3.7}$$

Similarly, let $\{\Psi_j^b : j = 1,\dots,s_2\}$, $\{\Phi_k^b : k = s_2+1,\dots,m_2\}$ be bases for $N[T(M;b)-\lambda I]$ and $N[T(M^+;b)-\bar{\lambda}I]$, respectively; thus, $\Psi_j^b, \Phi_k^b \in L_w^2(c,b)$ and

$$M[\Psi_j^b] = \lambda w \Psi_j^b, \qquad M^+[\Phi_k^b] = \bar{\lambda} w \Phi_k^b \quad \text{on } (c,b). \tag{3.8}$$

Since $[T_0(M^+;a)-\bar{\lambda}I]$ and $[T_0(M^+;b)-\bar{\lambda}I]$ have closed ranges, so do their adjoints $[T(M;a)-\lambda I]$ and $[T(M;b)-\lambda I]$ and, moreover, $R[T(M;a)-\lambda I]^\perp = N[T_0(M^+;a)-\bar{\lambda}I] = \{0\}$ and $R[T(M;b)-\lambda I]^\perp = N[T_0(M^+;b)-\bar{\lambda}I] = \{0\}$. Hence, $R[T(M;a)-\lambda I] = L_w^2(a,c)$ and $R[T(M;b)-\lambda I] = L_w^2(c,b)$. Similarly, $R[T(M^+;a)-\bar{\lambda}I] = L_w^2(a,c)$ and $R[T(M^+;b)-\bar{\lambda}I] = L_w^2(c,b)$. We can therefore define the following:

$$\begin{aligned} x_j^a &:= \Psi_j^a, && j = 1,\dots,s_1, \\ [T(M;a)-\lambda I]x_j^a &:= \Phi_j^a, && j = s_1+1,\dots,m_1, \\ [T(M^+;a)-\bar{\lambda}I]y_j^a &:= \Psi_j^a, && j = 1,\dots,s_1, \\ y_j^a &:= \Phi_j^a, && j = s_1+1,\dots,m_1; \end{aligned} \tag{3.9}$$

$$\begin{aligned} x_j^b &:= \Psi_j^b, && j = 1,\dots,s_2, \\ [T(M;b)-\lambda I]x_j^b &:= \Phi_j^b, && j = s_2+1,\dots,m_2, \\ [T(M^+;b)-\bar{\lambda}I]y_j^b &:= \Psi_j^b, && j = 1,\dots,s_2, \\ y_j^b &:= \Phi_j^b, && j = s_2+1,\dots,m_2. \end{aligned} \tag{3.10}$$

Next we state the following results; the proofs are similar to those in [**2**, Section 4 and **11**].

Lemma 3.3. *The sets* $\{x_j^a : j = 1, \dots, m_1\}$, $\{x_k^b : k = 1, \dots, m_2\}$ *are bases of* $N([T(M^+; a) - \bar{\lambda}I][T(M; a) - \lambda I])$ *and* $N([T(M^+; b) - \lambda I][T(M; b) - \lambda I])$, *respectively;* $\{y_j^a : j = 1, \dots, m_1\}$ *and* $\{y_k^b : k = 1, \dots, m_2\}$ *are bases of* $N([T(M; a) - \lambda I][T(M^+; a) - \bar{\lambda}I]$ *and* $N([T(M; b) - \lambda I][T(M^+; b) - \bar{\lambda}I])$, *respectively.*

On applying [**1**, Theorem III.3.1], we obtain

Corollary 3.4. *Any* $z^a \in D(M; a)$ *and* $(z^a)^+ \in D(M^+; a)$ *have the unique representations*

(3.11)
$$z^a = z_0^a + \sum_{j=1}^{m_1} a_j x_j^a \qquad z_0^a \in D_0(M; a),\ a_j \in \mathbf{C},$$

(3.12)
$$(z^a)^+ = (z_0^a)^+ + \sum_{j=1}^{m_1} b_j y_j^a \qquad (z_0^a)^+ \in D_0(M^+; a),\ b_j \in \mathbf{C}.$$

Also, any $z^b \in D(M; b)$ *and* $(z^b)^+ \in D(M^+; b)$ *have the unique representations*

$$z^b = z_0^b + \sum_{k=1}^{m_2} c_k x_k^b \qquad z_0^b \in D_0(M; b),\ c_k \in \mathbf{C}. \tag{3.13}$$

$$(z^b)^+ = (z_0^b)^+ + \sum_{k=1}^{m_2} d_k y_k^b \qquad (z_0^b)^+ \in D_0(M^+; b),\ d_k \in \mathbf{C}. \tag{3.14}$$

A central role in the argument is played by

Lemma 3.5. *Let*

$$E_{m_1 \times m_1} := ([x_j^a, y_k^a](a))_{\substack{1 \le j \le m_1, \\ 1 \le k \le m_1}} \tag{3.15}$$

$$E_{m_2 \times m_2} := ([x_j^b, y_k^b](b))_{\substack{1 \le j \le m_2, \\ 1 \le k \le m_2}} \tag{3.16}$$

and

$$E^{1,2}_{s_1\times r_1} := ([x^a_j, y^a_k](a))_{\substack{1\le j\le s_1\\ s_1+1\le k\le m_1}}, \tag{3.17}$$

$$E^{1,2}_{s_2\times r_2} := ([x^b_j, y^b_k](b))_{\substack{1\le j\le s_2\\ s_2+1\le k\le m_2}}. \tag{3.18}$$

Then

$$\operatorname{rank} E^{1,2}_{s_i\times r_i} = \operatorname{rank} E_{m_i\times m_i} = m_i - n, \qquad i = 1, 2. \tag{3.19}$$

In view of Lemma 3.5 and since $r_i, s_i \ge m_i - n$, $i = 1, 2$, we may suppose, without loss of generality, that the matrices

$$E^{1,2}_{(m_1-n)\times(m_1-n)} = ([x^a_j, y^a_k](a))_{\substack{1\le j\le m_1-n\\ n+1\le k\le m_1}} \tag{3.20}$$

and

$$E^{1,2}_{(m_2-n)\times(m_2-n)} = ([x^b_j, y^b_k](b))_{\substack{1\le j\le m_2-n\\ n+1\le k\le m_2}} \tag{3.21}$$

satisfy

$$\operatorname{rank} E^{1,2}_{(m_i-n)\times(m_i-n)} = m_i - n, \qquad i = 1, 2. \tag{3.22}$$

If we partition $E_{m_i\times m_i}$, $i = 1, 2$, as

$$E_{m_i\times m_i} = \left[\begin{array}{c:c} E^{1,1}_{(m_i-n)\times n} & E^{1,2}_{(m_i-n)\times(m_i-n)} \\ \hdashline E^{2,1}_{n\times n} & E^{2,2}_{n\times(m_i-n)} \end{array}\right] \tag{3.23}$$

and set

$$\begin{cases} E^1_{(m_i-n)\times m_i} = E^{1,1}_{(m_i-n)\times n} \oplus E^{1,2}_{(m_i-n)\times(m_i-n)} \\ E^2_{n\times m_i} = E^{2,1}_{n\times n} \oplus E^{2,2}_{n\times(m_i-n)}, \end{cases} \tag{3.24}$$

$$\begin{cases} F^1_{m_i\times n} = E^{1,1}_{(m_i-n)\times n} \oplus^T E^{2,1}_{n\times n} \\ F^2_{m_i\times(m_i-n)} = E^{1,2}_{(m_i-n)\times(m_i-n)} \oplus^T E^{2,2}_{n\times(m_i-n)}, \end{cases} \tag{3.25}$$

then (3.22) yields the results,

$$\text{rank}\, E^1_{(m_i-n)\times m_i} = \text{rank}\, F^2_{m_i\times(m_i-n)} = m_i - n, \qquad i = 1, 2. \tag{3.26}$$

Lemma 3.6. *Let $D_1(M;a)$ be the linear span of $\{z_i^a : i = 1, \dots, n\}$ where $z_i^a \in D(M;a)$ satisfy the following conditions for $k = 1, \dots, n$ and some $c \in (a,b)$,*

$$\begin{cases} (z_i^a)^{[k-1]}(a) = \delta_{ik}, & (z_i^a)^{[k-1]}(c) = 0, \\ z_i^a(t) = 0 & \text{for } t \geq c. \end{cases} \tag{3.27}$$

and let $D_2(M;a)$ be the linear span of $\{x_i^a : i = 1, \dots, m_1 - n\}$ with (3.22) *satisfied. Then*

$$D(M;a) = D_0(M;a) \dot{+} D_1(M;a) \dot{+} D_2(M;a).$$

Similarly,

$$D(M;b) = D_0(M;b) \dot{+} D_1(M;b) \dot{+} D_2(M;b).$$

If $D_1(M^+;a)$ and $D_2(M^+;a)$ are the linear spans of $\{(z_i^a)^+ : i = 1, \dots, n\}$ and $\{y_k^a : k = n+1, \dots, m_1\}$, respectively, then

$$D(M^+;a) = D_0(M^+;a) \dot{+} D_1(M^+;a) \dot{+} D_2(M^+;a). \tag{3.28}$$

Similarly,

$$D(M^+;b) = D_0(M^+;b) \dot{+} D_1(M^+;b) \dot{+} D_2(M^+;b). \tag{3.29}$$

We shall now characterize all the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ in terms of boundary conditions featuring $L^2_w(a,b)$-solutions of the equations $M[u] = \lambda w u$ and $M^+[v] = \bar{\lambda} w v$, $\lambda \in \mathbf{C}$.

Theorem 3.7. *Let $\lambda \in \Pi[T_0(M), T_0(M^+)]$, let r, s and m be defined by* (3.2) *and let x_i^a, $i = 1, \dots, m_1$, y_i^a, $i = 1, \dots, m_1$, x_j^b, $j = 1, \dots, m_2$, and y_j^b, $j = 1, \dots, m_2$, be defined by* (3.9) *and* (3.10) *respectively and arranged to satisfy* (3.22). *Let* $M_{s\times(m_1-n)}, N_{s\times(m_2-n)}$, $K_{r\times(m_1-n)}$ *and* $L_{r\times(m_2-n)}$ *be numerical matrices which satisfy the following conditions*:

(i) $\operatorname{Rank}\{M_{s\times(m_1-n)} \oplus N_{s\times(m_2-n)}\} = s$, $\operatorname{Rank}\{K_{r\times(m_1-n)} \oplus L_{r\times(m_2-n)}\} = r$,

(ii) $\{K_{r\times(m_1-n)}E^{1,2}_{(m_1-n)\times(m_1-n)}M^T_{s\times(m_1-n)}\}_{r\times s} = \{L_{r\times(m_2-n)}E^{1,2}_{(m_2-n)\times(m_2-n)}N^T_{s\times(m_2-n)}\}_{r\times s}$.

Then the set of all $u \in D[T(M)]$ *such that*

$$\text{(3.30)}\quad M_{s\times(m_1-n)}\begin{bmatrix}[u,y^a_{n+1}](a)\\ \vdots\\ [u,y^a_{m_1}](a)\end{bmatrix} - N_{s\times(m_2-n)}\begin{bmatrix}[u,y^b_{n+1}](b)\\ \vdots\\ [u,y^b_{m_2}](b)\end{bmatrix} = 0_{s\times 1}$$

is the domain of an operator S *which is regularly solvable with respect to* $T_0(M)$ *and* $T_0(M^+)$ *and* $D(S^*)$ *is the set of all* $v \in D[T(M^+)]$ *which are such that*

(3.31)

$$K_{r\times(m_1-n)}\begin{bmatrix}[x^a_1,v](a)\\ \vdots\\ [x^a_{m_1-n},v](a)\end{bmatrix} - L_{r\times(m_2-n)}\begin{bmatrix}[x^b_1,v](b)\\ \vdots\\ [x^b_{m_2-n},v](b)\end{bmatrix} = 0_{r\times 1}.$$

Proof. Let

$$\text{(3.32)}\qquad M_{s\times(m_1-n)} = (\alpha_{ik})_{\substack{r+1\le i\le m\\ n+1\le k\le m_1}},\ N_{s\times(m_2-n)} = (\beta_{i\ell})_{\substack{r+1\le i\le m\\ n+1\le \ell\le m_2}}$$

and set

$$\text{(3.33)}\quad g^a_i = \sum_{k=n+1}^{m_i}\bar{\alpha}_{ik}y^a_k,\qquad g^b_i = \sum_{\ell=n+1}^{m_2}\bar{\beta}_{i\ell}y^b_\ell,\qquad i = r+1,\dots,m.$$

Then $g_i \in D[T(M^+)]$, where

$$g_i = \begin{cases} g^a_i & \text{in } (a,c],\\ g^b_i & \text{in } [c,b),\quad i = r+1,\dots,m.\end{cases}$$

By [**17**, Theorem 8], we may choose ϕ_i, $(i=r+1,\dots,m)\in D[T(M^+)]$,

$$\phi_i = \begin{cases} \phi_i^a & \text{in } (a,c], \\ \phi_i^b & \text{in } [c,b), \end{cases}$$

such that for $a'\in(a,c)$ and $k=1,\dots,n$,

$$(\phi_i^a)_+^{[k-1]}(c)=0, \qquad (\phi_i^a)_+^{[k-1]}(a')=(g_i^a)_+^{[k-1]}(a'), \\ \phi_i^a = g_i^a \quad \text{on } (a,a'], \quad i=r+1,\dots,m, \tag{3.34}$$

and for $b'\in(c,b)$,

$$(\phi_i^b)_+^{[k-1]}(c)=0, \qquad (\phi_i^b)_+^{[k-1]}(b')=(g_i^b)_+^{[k-1]}(b'), \\ \phi_i^b = g_i^b \quad \text{on } [b',b), \quad i=r+1,\dots,m. \tag{3.35}$$

This gives

$$M_{s\times(m_1-n)}\begin{bmatrix} [u,y_{n+1}^a](a) \\ \vdots \\ [u,y_{m_1}^a](a)\end{bmatrix} = \begin{bmatrix} [u,\sum_{k=n+1}^{m_1}\bar{\alpha}_{r+1,k}y_k^a](a) \\ \vdots \\ [u,\sum_{k=n+1}^{m_1}\bar{\alpha}_{m,k}y_k^a](a)\end{bmatrix} = \begin{bmatrix} u,\phi_{r+1}^a](a) \\ \vdots \\ [u,\phi_m^a](a)\end{bmatrix}.$$

Similarly,

$$N_{s\times(m_2-n)}\begin{bmatrix} [u,y_{n+1}^b](b) \\ \vdots \\ [u,y_{m_2}^b](b)\end{bmatrix} = \begin{bmatrix} [u,\phi_{r+1}^b](b) \\ \vdots \\ [u,\phi_m^b](b)\end{bmatrix}.$$

The boundary condition (3.30) therefore coincides with that in (3.5). Similarly, (3.31) coincides with (3.6) on making the following choices:

$$K_{r\times(m_1-n)} = (\gamma_{jk})_{\substack{1\le j\le r \\ 1\le k\le m_1-n}}, \qquad L_{r\times(m_2-n)} = (\varepsilon_{j\ell})_{\substack{1\le j\le r \\ 1\le \ell\le m_2-n}}, \tag{3.36}$$

$$h_j^a = \sum_{k=1}^{m_1-n}\gamma_{jk}x_k^a, \qquad h_j^b = \sum_{\ell=1}^{m_2-n}\varepsilon_{j\ell}x_\ell^b. \tag{3.37}$$

Then $h_j \in D[T(M)]$, where

$$h_j = \begin{cases} h_j^a & \text{in } (a,c], \\ h_j^b & \text{in } [c,b), \quad j = 1, \dots, r, \end{cases}$$

and ψ_j $(j = 1, \dots, r) \in D[T(M)]$,

$$\psi_j = \begin{cases} \psi_j^a & \text{in } (a,c], \\ \psi_j^b & \text{in } [c,b) \end{cases}$$

such that for $a' \in (a,c)$ and $k = 1, \dots, n$,

$$(3.38) \qquad \begin{gathered} (\psi_j^a)^{[k-1]}(c) = 0, \qquad (\psi_j^a)^{[k-1]}(a') = (h_j^a)^{[k-1]}(a'), \\ \psi_j^a = h_j^a \quad \text{on } (a,a'], \quad j = 1, \dots, r, \end{gathered}$$

and for $b' \in (c,b)$,

$$(3.39) \qquad \begin{gathered} (\psi_j^b)^{[k-1]}(c) = 0, \qquad (\psi_j^b)^{[k-1]}(b') = (h_j^b)^{[k-1]}(b'), \\ \psi_j^b = h_j^b \quad \text{on } [b',b), \quad j = 1, \dots, r. \end{gathered}$$

It remains to show that the above functions ϕ_k, $k = r+1, \dots, m$, and ψ_j, $j = 1, \dots, r$, satisfy conditions (i) and (ii) in Theorem 3.2. First, suppose that $\{\psi_j : j = 1, \dots, r\}$ is not linearly independent modulo $D_0(M)$, that is, there exist constants $c_1, \dots, c_r$, not all zero, such that $u = \sum_{j=1}^r c_j \psi_j \in D[T_0(M)]$. Since $u^{[k-1]}(c) = 0$ for $k = 1, \dots, n$ and $u \in D[T_0(M)]$, then $u \in D[\tilde{T}_0(M)]$ and $u|(a,c] \in D[T_0(M;a)]$. Hence, by Green's formula (2.11), we have for all $v \in D[T(M^+;a)]$ that $[u,v](a) - [u,v](c) = 0$. This implies that $[u,v](a) = [u,v](c) = 0$. Similarly, $u|[c,b) \in D[T_0(M;b)]$ and we have

$$[u,v](b) = [u,v](c) = 0 \quad \text{for all } v \in D[T(M^+;b)].$$

Hence,

$$\begin{aligned} 0_{1\times m_1} &= ([u, y_1^a](a), \dots, [u, y_{m_1}^a](a)) \\ &= \left(\left[\sum_{j=1}^r c_j \sum_{k=1}^{m_1-n} \gamma_{jk} x_k^a, y_1^a \right](a), \dots, \left[\sum_{j=1}^r c_j \sum_{k=1}^{m_1-n} \gamma_{jk} x_k^a, y_{m_1}^a \right](a) \right) \\ &= (c_1, \dots, c_r) K_{r\times(m_1-n)} E^1_{(m_1-n)\times m_1}, \end{aligned}$$

on using the notation in (3.24). In view of (3.26), we conclude that,

$$(c_1, \dots, c_r) K_{r\times(m_1-n)} = 0_{1\times(m_1-n)}. \tag{3.40}$$

Similarly,

$$\begin{aligned} 0_{1\times m_2} &= ([u, y_1^b](b), \dots, [u, y_{m_2}^b](b)) \\ &= \left(\left[\sum_{j=1}^{r} c_j \sum_{\ell=1}^{m_2-n} \varepsilon_{j\ell} x_\ell^b, y_1^b \right](b), \dots, \left[\sum_{j=1}^{r} c_j \sum_{\ell=1}^{m_2-n} \varepsilon_{j\ell} x_\ell^b, y_{m_2}^b \right](b) \right) \\ &= (c_1, \dots, c_r) L_{r\times(m_2-n)} E^1_{(m_2-n)\times m_2}, \end{aligned}$$

on using the notation in (3.24). In view of (3.26), we conclude that

$$(c_1, \dots, c_r) L_{r\times(m_2-n)} = 0_{1\times(m_2-n)}. \tag{3.41}$$

We obtain from (3.40) and (3.41) that

$$(c_1, \dots, c_r)\{K_{r\times(m_1-n)} \oplus L_{r\times(m_2-n)}\} = 0_{1\times m}, \qquad m = m_1 + m_2 - 2n,$$

which contradicts the assumption that $\{K_{r\times(m_1-n)} \oplus L_{r\times(m_2-n)}\}$ has rank r. It follows similarly that $\{\phi_k : k = r+1, \dots, m\} \subset D[T(M^+)]$ is linearly independent modulo $D_0(M^+)$.

Finally, we prove (ii) in Theorem 3.2.

$$\begin{aligned} ([\psi_j^a, \phi_k^a](a))_{\substack{1\le j\le r \\ r+1\le k\le m}} &= \left(\left[\sum_{\ell=1}^{m_1-n} \gamma_{j\ell} x_\ell^a, \sum_{p=n+1}^{m_1} \bar{\alpha}_{kp} y_p^a \right](a) \right)_{\substack{1\le j\le r \\ r+1\le k\le m}} \\ &= (\gamma_{j\ell})_{\substack{1\le j\le r \\ 1\le \ell\le m_1-n}} ([x_\ell^a, y_p^a](a))_{\substack{1\le \ell\le m_1-n \\ n+1\le p\le m_1}} \\ &\quad \cdot (\alpha_{kp})_{\substack{r+1\le k\le m \\ n+1\le p\le m_1}} \\ &= K_{r\times(m_1-n)} E^{1,2}_{(m_1-n)\times(m_1-n)} M^T_{s\times(m_1-n)}. \end{aligned} \tag{3.42}$$

Similarly,

$$([\psi_j^b, \phi_k^b](b))_{\substack{1\le j\le r \\ r+1\le k\le m}} = L_{r\times(m_2-n)} E^{1,2}_{(m_2-n)\times(m_2-n)} N^T_{s\times(m_2-n)}. \tag{3.43}$$

From (ii), (3.42) and (3.43) it follows that condition (ii) in Theorem 3.2 is satisfied. The proof is therefore complete. □

The converse of Theorem 3.7 is

Theorem 3.8. *Let S be regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$, let $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, let r, s and m be defined by* (3.2) *and* (3.3) *and suppose that* (3.22) *is satisfied. Then there exist numerical matrices $K_{r\times(m_1-n)}, L_{r\times(m_2-n)}, M_{s\times(m_1-n)}$ and $N_{s\times(m_2-n)}$ such that conditions* (i) *and* (ii) *in Theorem* 3.8 *are satisfied and $D(S)$ is the set of $u \in D(M)$ satisfying* (3.30) *while $D(S^*)$ is the set of $v \in D(M^+)$ satisfying* (3.31).

Proof. Let $\{\psi_i : i = 1, \dots, r\} \subset D(M)$, $\{\phi_j : j = r+1, \dots, m\} \subset D(M^+)$ and set

$$\psi_i = \begin{cases} \psi_i^a & \text{in } (a,c], \\ \psi_i^b & \text{in } [c,b) \end{cases} \quad \text{and} \quad \phi_j = \begin{cases} \phi_j^a & \text{in } (a,c], \\ \phi_j^b & \text{in } [c,b), \end{cases}$$

are satisfying the second part of Theorem 3.2. From (3.28) and (3.29) we have that

$$\phi_j^a = y_{j0}^a + \sum_{k=1}^{n} \eta_{jk}(z_k^a)^+ + \sum_{k=n+1}^{m_1} \alpha_{jk} y_k^a, \qquad j = r+1, \dots, m, \tag{3.44}$$

for some $y_{j0}^a \in D[T_0(M^+; a)]$ and complex constants η_{jk}, α_{jk}; and

$$\phi_j^b = y_{j0}^b + \sum_{k=1}^{n} \xi_{jk}(z_k^b)^+ + \sum_{k=n+1}^{m_2} \beta_{jk} y_k^b, \qquad j = r+1, \dots, m, \tag{3.45}$$

for some $y_{j0}^b \in D[T_0(M^+; b)]$ and complex constants ξ_{jk}, β_{jk}. Since $y_{j0}^a \in D[T_0(M^+; a)]$ and $y_{j0}^b \in D[T_0(M^+; b)]$, then $y_{j0} \in D[\tilde{T}_0(M^+)]$, where

$$y_{j0} = \begin{cases} y_{j0}^a & \text{in } (a,c], \\ y_{j0}^b & \text{in } [c,b). \end{cases}$$

Hence, for all $u \in D[T(M)]$,

$$[u, y_{j0}^a](a) = [u, y_{j0}^a](c) = 0 \quad \text{and} \quad [u, y_{j0}^b](b) = [u, y_{j0}^b](c) = 0.$$

Also,

$$[u, (z_k^a)^+](a) = [u, (z_k^b)^+](b) = 0, \qquad k = 1, \dots, n.$$

Let

(3.46)

$$M_{s\times(m_1-n)} = (\bar{\alpha}_{jk})_{\substack{r+1\le j\le m \\ n+1\le k\le m_1}}, \qquad N_{s\times(m_2-n)} = (\bar{\beta}_{jk})_{\substack{r+1\le j\le m \\ n+1\le k\le m_2}}.$$

Then, from (3.44),

$$\begin{bmatrix} [u,\phi^a_{r+1}](a) \\ \vdots \\ [u,\phi^a_m](a) \end{bmatrix} = \begin{bmatrix} \left[u, \sum_{k=n+1}^{m_1} \alpha_{r+1,k} y^a_k\right](a) \\ \vdots \\ \left[u, \sum_{k=n+1}^{m_1} \alpha_{mk} y^a_k\right](a) \end{bmatrix} = M_{s\times(m_1-n)} \begin{bmatrix} [u, y^a_{n+1}](a) \\ \vdots \\ [u, y^a_{m_1}](a) \end{bmatrix}.$$

Similarly, from (3.45),

$$\begin{bmatrix} [u,\phi^b_{r+1}](b) \\ \vdots \\ [u,\phi^b_m](b) \end{bmatrix} = N_{s\times(m_2-n)} \begin{bmatrix} [u, y^b_{n+1}(b) \\ \vdots \\ [u, y^b_{m_2}](b) \end{bmatrix}.$$

Therefore, we have shown that the boundary conditions (3.30) coincide with those in (3.5); similarly (3.31) and the conditions in (3.6) can be shown to coincide if we choose

(3.47)

$$K_{r\times(m_1-n)} = (\gamma_{ik})_{\substack{1\le i\le r \\ 1\le k\le m_1-n}} \quad \text{and} \quad L_{r\times(m_2-n)} = (\varepsilon_{i\ell})_{\substack{1\le i\le r \\ 1\le \ell\le m_2-n}}$$

where the γ_{ik} and $\varepsilon_{i\ell}$ are the constants uniquely determined by the decompositions,

(3.48)

$$\psi^a_i = x^a_{i0} + \sum_{k=1}^{n} \zeta_{ik} z^a_k + \sum_{k=1}^{m_1-n} \gamma_{ik} x^a_k, \qquad i = 1, \ldots, r,$$

(3.49)

$$\psi^b_i = x^b_{i0} + \sum_{k=1}^{n} \xi_{ik} z^b_k + \sum_{\ell=1}^{m_2-n} \varepsilon_{i\ell} x^b_\ell, \qquad i = 1, \ldots, r,$$

derived from Lemma 3.6.

We next prove that (i) and (ii) are consequences of conditions (i) and (ii) in Theorem 3.2. Suppose that rank $\{K_{r\times(m_1-n)}\oplus L_{r\times(m_2-n)}\}<r$. Then, there exist constants $c_1,\dots,c_r$, not all zero, such that

$$(c_1,\dots,c_r)\{K_{r\times(m_1-n)}\oplus L_{r\times(m_2-n)}\}=0_{1\times m}. \tag{3.50}$$

This implies that

$$\begin{aligned}0_{1\times(m_1-n)}&=(c_1,\dots,c_r)K_{r\times(m_1-n)}\\&=\left(\sum_{i=1}^r c_i\gamma_{i1},\dots,\sum_{i=1}^r c_i\gamma_{i,m_1-n}\right),\end{aligned}$$

and, also,

$$\begin{aligned}0_{1\times(m_2-n)}&=(c_1,\dots,c_r)L_{r\times(m_2-n)}\\&=\left(\sum_{i=1}^r c_i\varepsilon_{i1},\dots,\sum_{i=1}^r c_i\varepsilon_{i,m_2-n}\right).\end{aligned}$$

Consequently, on substituting in (3.48) and (3.49), respectively, we obtain

$$\psi_i^a=x_{i0}^a+\sum_{k=1}^n\zeta_{ik}z_k^a,\qquad i=1,\dots,r, \tag{3.51}$$

$$\psi_i^b=x_{i0}^b+\sum_{k=1}^n\xi_{ik}z_k^b,\qquad i=1,\dots,r. \tag{3.52}$$

Let $u=\sum_{i=1}^r c_i\psi_i$, then by (3.51) and (3.52) we have that

$$\begin{aligned}u^a&=\sum_{i=1}^r c_ix_{i0}^a+\sum_{i=1}^r\sum_{k=1}^n c_i\zeta_{ik}z_k^a,\\u^b&=\sum_{i=1}^r c_ix_{i0}^b+\sum_{i=1}^r\sum_{k=1}^n c_i\xi_{ik}z_k^b,\end{aligned}$$

where $u=\{u^a,u^b\}$, $u^a\in D[T(M;a)]$ and $u^b\in D[T(M;b)]$. For arbitrary $v=\{v^a,v^b\}\in D[T(M^+)]$, $v^a\in D[T(M^+;a)]$ and $v^b\in D[T(M^+;b)]$, we have that

$$\begin{aligned}[u,v](a)&=0,\\ [u,v](b)&=0.\end{aligned}$$

Hence, by Green's formula, $u \in D[T_0(M)]$, and we have that $\{\psi_i : i = 1, \dots, r\}$ is linearly dependent modulo $D_0(M)$, contrary to assumption. We have therefore proved that $\{K_{r\times(m_1-n)} \oplus L_{r\times(m_2-n)}\}$ has rank r. The proof of rank $\{M_{s\times(m_1-n)} \oplus N_{s\times(m_2-n)}\} = s$ is similar.

On using (3.44) and (3.48) and the facts that $z_k^a = (z_k^a)^+ = 0$ on $(a, a'](a' \in (a, c))$, $(k = 1, \dots, n)$ and $[u^a, v^a](a) = 0$ if either $u^a \in D(M; a)$ and $v^a \in D_0(M^+; a)$ or $u^a \in D_0(M, a)$ and $v^a \in D(M^+; a)$, we obtain

$$\left([\psi_i^a, \phi_j^a](a)\right)_{\substack{1\le i\le r\\ r+1\le j\le m}} = \left(\left[\sum_{k=1}^{m_1-n} \gamma_{ik} x_k^a, \sum_{\ell=n+1}^{m_1} \alpha_{i\ell} y_k^a\right](a)\right)_{\substack{1\le i\le r\\ r+1\le j\le m}}$$
$$= K_{r\times(m_1-n)} E^{1,2}_{(m_1-n)\times(m_1-n)} M^T_{s\times(m_1-n)}.$$

Similarly, we obtain

$$\left([\psi_i^b, \phi_j^b](b)\right)_{\substack{1\le i\le r\\ r+1\le j\le m}} = \left(\left[\sum_{\ell=1}^{m_2-n} \varepsilon_{i\ell} x_\ell^b, \sum_{k=n+1}^{m_2} \beta_{jk} y_k^b\right](b)\right)_{\substack{1\le i\le r\\ r+1\le j\le m}}$$
$$= L_{r\times(m_2-n)} E^{1,2}_{(m_2-n)\times(m_2-n)} N^T_{s\times(m_2-n)}.$$

The proof is therefore complete. □

Finally, assume that M is formally J-symmetric, that is, $M^+ = JMJ$, where J is complex conjugation. The operator $T_0(M)$ is then J-symmetric and $T_0(M)$ and $T_0(M^+) = JT_0(M)J$ from an adjoint pair with $\Pi[T_0(M), T_0(M^+)] = \Pi[T_0(M)]$. Since $M[u] = \lambda w u$ if and only if $M^+[\bar{u}] = \bar{\lambda} w \bar{u}$, it follows from (3.4) that for all $\lambda \in \Pi[T_0(M)]$, $\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \bar{\lambda} I]$ is constant ℓ, say, and so in (3.2), $r = s = \ell$ with $0 \le \ell \le n$; note that Frentzen proved in [**9**] that a formally J-symmetric expression M generated by a Shin-Zettl matrix must be of even order. If S is a J-self-adjoint extension of $T_0(M)$, then $S^* = JSJ$ and consequently $v \in d(S^*)$ if and only if $\bar{v} \in D(S)$. In the case when M is formally J-symmetric, for a complex conjugation J, Theorems 3.7 and 3.8 include Theorem 5.5 in [**15**].

4. The general theorems. The results in this section are generalizations of the main theorems in Section 3 for any N intervals $I_p = (a_p, b_p)$, $p = 1, \dots, N$.

For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we define r, s and m as follows:
(4.1)

$$\begin{aligned} r = r(\lambda) &:= \operatorname{def}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I] \\ &= \sum_{p=1}^{N} \operatorname{nul}[T(M_p^+) - \bar{\lambda} I] = \sum_{p=1}^{N} r_p \\ s = s(\lambda) &:= \operatorname{def}[T_0(M^+) - \bar{\lambda} I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] \\ &= \sum_{p=1}^{N} \operatorname{nul}[T(M_p) - \lambda I] = \sum_{p=1}^{N} s_p \end{aligned}$$

and

$$\begin{aligned} m := r + s &= \sum_{p=1}^{N} r_p + \sum_{p=1}^{N} s_p \\ &= \sum_{p=1}^{N} (r_p + s_p) = \sum_{p=1}^{N} m_p. \end{aligned}$$

By Lemma 3.1, m is constant on $\Pi[T_0(M), T_0(M^+)]$ and

$$nN \le m \le 2nN. \tag{4.2}$$

For $\Pi[T_0(M), T_0(M^+)] \neq \varnothing$ the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ are characterized by the following theorem.

Theorem 4.1. *For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, let r and m be defined by (4.1) and let $\underset{\sim}{\psi}_j$, $j = 1, \dots, r$, $\underset{\sim}{\phi}_k$, $k = r+1, \dots, m$, be arbitrary functions satisfying*:

(i) *$\{\underset{\sim}{\psi}_j : j = 1, \dots, r\} \subset D(M)$ is linearly independent modulo $D_0(M)$ and $\{\underset{\sim}{\phi}_k : k = r+1, \dots, m\} \subset D(M^+)$ is linearly independent modulo $D_0(M^+)$;*

(ii) *$[\underset{\sim}{\psi}_j, \underset{\sim}{\phi}_k] = \sum_{p=1}^{N} \{[\psi_{jp}, \psi_{kp}]_p(b_p) - [\psi_{jp}, \phi_{kp}]_p(a_p)\} = 0$, $j = 1, \dots, r$; $k = r+1, \dots, m$.*

Then the set

(4.3)
$$\Big\{\underset{\sim}{u} : \underset{\sim}{u} \in D(M), [\underset{\sim}{u}, \underset{\sim}{\phi}_k] = \sum_{p=1}^{N} \{[u_p, \phi_{kp}]_p(b_p) - [u_p, \phi_{kp}]_p(a_p)]\} = 0, \qquad k = r+1, \dots, m\Big\}$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and

(4.4)
$$\{\underset{\sim}{v} : \underset{\sim}{v} \in D(M^+), [\underset{\sim}{\psi}_j, \underset{\sim}{v}] = \sum_{p=1}^{N} \{[\psi_{jp}, v_p]_p(b_p) - [\psi_{jp}, v_p]_p(a_p)\} = 0, \qquad j = 1, \dots, r\}$$

is the domain of S^*, moreover $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, then with r and m defined by (4.1), *there exist functions $\underset{\sim}{\psi}_j$, $j = 1, \dots, r$, $\underset{\sim}{\phi}_k$, $k = r+1, \dots, m$, which satisfy* (i) *and* (ii) *and are such that* (4.3) *and* (4.4) *are the domains of S and S^*, respectively.*

S is self-adjoint if and only if $M = M^+$, $r = s$ and $\underset{\sim}{\phi}_k = \underset{\sim}{\psi}_{k-r}$, $k = r+1, \dots, m$; S is J-self-adjoint if and only if $M = JM^+J$, $r = s$ and $\underset{\sim}{\phi}_k = \underset{\sim}{\psi}_{k-r}$, $k = r+1, \dots, m$.

Proof. The proof is entirely similar to that of Theorem 3.2 and [**2**, Theorem 3.2] and is therefore omitted. □

The regularly solvable operators are determined by boundary conditions imposed at the endpoints of each of the intervals I_p. The type of these boundary conditions depends on the nature of the problem in the interval I_p. There are four possibilities for each p, $p = 1, \dots, N$.

Case (i). $I_p = [a_p, b_p]$, i.e., the case of two regular endpoints. In this case, we put $r_p = s_p = n$, $p = 1, \dots, N$, in (4.1). Then for each p,

$$\operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] = 2n \quad \text{for } \lambda \in \Pi[T_0(M_p), T_0(M_p^+)].$$

By (4.3) and (4.4), if we put

$$\alpha_{jk}^p = -(\bar{\phi}_{jp})_+^{[n-k]}(a_p), \qquad \beta_{jk}^p = (\bar{\phi}_{jp})_+^{[n-k]}(b_p);$$
$$\tau_{jk}^p = -\psi_{jp}^{[n-k]}(a_p), \qquad \delta_{jk}^p = \psi_{jp}^{[n-k]}(b_p),$$

$(j, k = 1, \dots, n; p = 1, \dots, N)$. Then the boundary conditions in this case on the functions $u_p \in D(M_p)$ are

$$B_p(u_p, I_p) = M_{n\times n}^p \underset{\sim}{u}_p(a_p) + N_{n\times n}^p \underset{\sim}{u}_p(b_p) = 0, \tag{4.5}$$

where

$$M_{n\times n}^p = ((-1)^k \alpha_{jk}^p)_{\substack{1\le j\le n,\\ 1\le k\le n}} \qquad N_{n\times n}^p = ((-1)^k \beta_{jk}^p)_{\substack{1\le j\le n;\\ 1\le k\le n}}$$
$$\underset{\sim}{u}_p(\cdot) = (u_p(\cdot), \dots, u_p^{[n-1]}(\cdot))^T, \quad T \text{ for a transposed matrix,}$$

and on the functions $v_p \in D(M_p^+)$ are

$$B_p^*(v_p, I_p) = K_{n\times n}^p \underset{\sim}{\bar{v}}_p(a_p) + L_{n\times n}^p \underset{\sim}{\bar{v}}_p(b_p) = 0, \tag{4.6}$$

where

$$K_{n\times n}^p = ((-1)^{n+1-k} \tau_{jk}^p)_{\substack{1\le j\le n,\\ 1\le k\le n}} \qquad L_{n\times n}^p = ((-1)^{n+1-k} \delta_{jk}^p)_{\substack{1\le j\le n;\\ 1\le k\le n}}$$
$$\underset{\sim}{\bar{v}}_p(\cdot) = (\bar{v}_p(\cdot), \dots, (\bar{v}_p)_+^{[n-1]}(\cdot))^T,$$

and $\alpha_{jk}^p, \beta_{jk}^p, \tau_{jk}^p, \delta_{jk}^p$, $j, k = 1, \dots, n$; $p = 1, \dots, N$, are complex numbers satisfying

$$M_{n\times n}^p J (K_{n\times n}^p)^T = N_{n\times n}^p J (L_{n\times n}^p)^T. \tag{4.7}$$

The above boundary conditions determine the domains of the operators which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$ for each p; see [**1**, Theorem III.10.6 and **10**, Theorem II.2.12] for more details.

In the other three cases, the operators S_p, $p = 1, \dots, N$, which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$ are determined in terms of boundary conditions featuring $L^2_{w_p}(a_p, b_p)$-solutions of the equations (2.15) and (2.18).

Case (ii). $I_p = [a_p, b_p)$, i.e., the case of the problem with the left-hand endpoint of I_p assumed to be regular but the right-hand endpoint may be either regular or singular. For $\lambda \in \Pi[T_0(M_p), T_0(M_p^+)]$, define r_p, s_p and m_p as in (4.1). Let $\{\psi_{jp} : j = 1, \dots, s_p, p = 1, \dots, N\}$, $\{\phi_{kp} : k = s_p + 1, \dots, m_p; p = 1, \dots, N\}$ be bases for $N[T(M_p - \lambda I]$ and $N[T(M_p^+) - \bar{\lambda} I]$, respectively. Thus, $\psi_{jp}, \phi_{kp} \in L^2_{w_p}(a_p, b_p)$, $j = 1, \dots, s_p, k = s_p + 1, \dots, m_p$, and

$$M_p[\psi_{jp}] = \lambda w_p \psi_{jp}, \qquad M_p^+[\phi_{kp}] = \bar{\lambda} w_p \phi_{kp}, \qquad p = 1, \dots, N.$$

We can therefore define the following functions x_{jp}, y_{jp}, $j = 1, \dots, m_p$, $p = 1, \dots, N$.

$$\begin{aligned} x_{jp} &:= \psi_{jp}, & j &= 1, \dots, s_p, \\ [T(M_p) - \lambda I] x_{jp} &:= \phi_{jp}, & j &= s_p + 1, \dots, m_p, \end{aligned} \tag{4.8}$$

$$\begin{aligned} [T(M_p^+) - \bar{\lambda} I] y_{jp} &:= \psi_{jp}, & j &= 1, \dots, s_p, \\ y_{jp} &:= \phi_{jp}, & j &= s_p + 1, \dots, m_p, \end{aligned} \tag{4.9}$$

and these functions are arranged to satisfy (3.22) for each p. Let

$$M^p_{s_p \times n} J^{-1}_{n \times n} = -i^n (\alpha^p_{jk})_{\substack{r_p+1 \le j \le m_p \\ 1 \le k \le n}}, N^p_{s_p \times (m_p - n)} = (\beta^p_{jk})_{\substack{r_p+1 \le j \le m_p \\ n+1 \le k \le m_p}}$$

and set

$$g_{jp} := \sum_{k=n+1}^{m_p} \bar{\beta}^p_{jk} y_{kp}, \qquad j = r_p + 1, \dots, m_p;\ p = 1, \dots, N.$$

Then $g_{jp} \in D(M_p^+)$ and, by [**17**, Theorem 8] we may choose ϕ_{jp}, $j = r_p + 1, \dots, m_p \in D(M_p^+)$ such that for $k = 1, \dots, n$ and some $c_p \in (a_p, b_p)$,

$$\begin{gathered} (\phi_{jp})_+^{[k-1]}(a_p) = \bar{\alpha}^p_{jk}, \qquad (\phi_{jp})_+^{[k-1]}(c_p) = (g_{jp})_+^{[k-1]}(c_p), \\ \phi_{jp} = g_{jp} \quad \text{on } [c_p, b_p), \quad p = 1, \dots, N. \end{gathered}$$

Similarly, let

$$K^p_{r_p\times n}J^{-1}_{n\times n}=(-i)^n(\gamma^p_{jk})_{\substack{1\le j\le r_p\\1\le k\le n}},L^p_{r_p\times(m_p-n)}=(\varepsilon^p_{jk})_{\substack{1\le j\le r_p\\1\le k\le m_p-n}},$$

$$h_{jp}:=\sum_{k=1}^{m_p-n}\varepsilon^p_{jk}x_{kp},\qquad j=1,\dots,r_p;\ p=1,\dots,N,$$

and ψ_{jp}, $j=1,\dots,r_p,p=1,\dots,N\in D(M_p)$ such that

$$\psi^{[k-1]}_{jp}(a_p)=\tau^p_{jk},\qquad \psi^{[k-1]}_{jp}(c_p)=h^{[k-1]}_{jp}(c_p),$$
$$\psi_{jp}=h_{jp}\quad\text{on } [c_p,b_p).$$

Then the boundary conditions in this case on the functions $u_p\in D(M_p)$ are

(4.10)

$$B_p(u_p,I_p)=M^p_{s_p\times n}\begin{bmatrix}u_p(a_p)\\ \vdots\\ u_p^{[n-1]}(a_p)\end{bmatrix}-N^p_{s_p\times(m_p-n)}\begin{bmatrix}[u_p,y_{n+1},p]_p(b_p)\\ \vdots\\ [u_p,y_{m_p,p}]_p(b_p)\end{bmatrix}=0_{s_p\times1},$$

and on the functions $v_p\in D(M_p^+)$ are

$$B^*_p(v_p,I_p)=K^p_{r_p\times n}\begin{bmatrix}\bar v_p(a_p)\\ \vdots\\ (\bar v_p)_+^{[n-1]}(a_p)\end{bmatrix}-L^p_{r_p\times(m_p-n)}\begin{bmatrix}[x_{1p},v_p]_p(b_p)\\ \vdots\\ [x_{m_p-n,p},v_p]_p(b_p)\end{bmatrix}=0_{r_p\times1},\tag{4.11}$$

which determine the domains of the operators which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$ for each p, where $M^p_{s_p\times n},N^p_{s_p\times(m_p-n)},K^p_{r_p\times n}$ and $L^p_{r_p\times(m_p-n)}$ are numerical matrices which satisfy the following conditions:

$$\begin{aligned}&\operatorname{Rank}\{K^p_{r_p\times n}\oplus L^p_{r_p\times(m_p-n)}\}=r_p,\\&\operatorname{Rank}\{M^p_{s_p\times n}\oplus N^p_{s_p\times(m_p-n)}\}=s_p,\end{aligned}\tag{4.12}$$

$$\{L^p_{r_p\times(m_p-n)}E^{1,2}_{(m_p-n)\times(m_p-n)}(N^p_{s_p\times(m_p-n)})^T + (-i)^n K^p_{r_p\times n}J_{n\times n}(M^p_{s_p\times n})^T\} = 0_{r_p\times s_p}, \qquad p=1,\dots,N; \tag{4.13}$$

see [**2**, Theorems 5.1 and 5.2] for more details.

Case (iii). $I_p = (a_p, b_p]$; it is similar to case (ii) with the right-hand endpoint of I_p assumed to be regular, but the left-hand endpoint may be either regular or singular. The boundary conditions in this case on the functions $u_p \in D(M_p)$ are

$$B_p(u_p, I_p) = M^p_{s_p\times n}\begin{bmatrix} u_p(b_p) \\ \vdots \\ u_p^{[n-1]}(b_p)\end{bmatrix} - N^p_{s_p\times(m_p-n)}\begin{bmatrix}[u_p, y_{n+1,p}]_p(a_p) \\ \vdots \\ [u_p, y_{m_p,p}]_p(a_p)\end{bmatrix} = 0_{s_p\times 1} \tag{4.14}$$

and on the functions $v_p \in D(M_p^+)$ are

$$B_p^*(v_p, I_p) = K^p_{r_p\times n}\begin{bmatrix}\bar{v}_p(b_p) \\ \vdots \\ (\bar{v}_p)_+^{[n-1]}(b_p)\end{bmatrix} - L^p_{r_p\times(m_p-n)}\begin{bmatrix}[x_{1p}, v_p]_p(a_p) \\ \vdots \\ [x_{m_p-n,p}, v_p]_p(a_p)\end{bmatrix} = 0_{r_p\times 1}, \tag{4.15}$$

which determine the domains of the operators which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$ for each p, where $M^p_{s_p\times n}, N^p_{s_p\times(m_p-n)}, K^p_{r_p\times(m_p-n)}$ and $L^p_{r_p\times(m_p-n)}$ are numerical matrices which satisfy (4.12) and (4.13), respectively.

Case (iv). $I_p = (a_p, b_p)$, i.e., the case of two singular endpoints of I_p.

By (4.1), (3.2), (3.3) and (3.4) we have that, for $\lambda \in \Pi[T_0(M), T_0(M^+)]$,

$$\begin{aligned} r = r(\lambda) &:= \operatorname{def}\,[T_0(M) - \lambda I] = \sum_{p=1}^{N} r_p \\ &= \sum_{p=1}^{N} (r_p^1 + r_p^2 - n), \\ s = s(\lambda) &:= \operatorname{def}\,[T_0(M^+) - \bar{\lambda} I] = \sum_{p=1}^{N} s_p \\ &= \sum_{p=1}^{N} (s_p^1 + s_p^2 - n) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} m := r + s &= \sum_{p=1}^{N} \{(r_p^1 + s_p^1) + (r_p^2 + s_p^2) - 2n\} \\ &= \sum_{p=1}^{N} (m_p^1 + m_p^2 - 2n). \end{aligned}$$

Since

$$n \le m_p^i \le 2n, \qquad i = 1, 2,$$

then $0 \le m \le 2nN$. Also, we can define the functions $x_{ip}^{a_p}, y_{ip}^{a_p}$, $i = 1, \dots, m_p^1$, $p = 1, \dots, N$, and $x_{jp}^{b_p}, y_{jp}^{b_p}$, $j = 1, \dots, m_p^2$, $p = 1, \dots, N$, in a similar way to those functions which are defined in (3.9) and (3.10), respectively. These functions are arranged to satisfy (3.22) for each p. Let $M_{s_p \times (m_p^1 - n)}$, $N^p_{s_p \times (m_p^2 - n)}$, $K^p_{r_p \times (m_p^1 - n)}$ and $L^p_{\tau_p \times (m_p^2 - n)}$ are numerical matrices defined similarly to those in (3.32) and (3.36), respectively. Then the boundary conditions in this case on the functions $u_p \in D(M_p)$ are

$$\begin{aligned} B_p(u_p, I_p) = M^p_{s_p \times (m_p^1 - n)} &\begin{bmatrix} [u_p, y^{a_p}_{n+1,p}]_p(a_p) \\ \vdots \\ [u_p, y^{a_p}_{m_p^1,p}]_p(a_p) \end{bmatrix} \\ - N^p_{s_p \times (m_p^2 - n)} &\begin{bmatrix} [u_p, y^{b_p}_{n+1,p}]_p(b_p) \\ \vdots \\ [u_p, y^{b_p}_{m_p^2,p}]_p(b_p) \end{bmatrix} = 0_{s_p \times 1} \end{aligned} \tag{4.17}$$

and on the functions $v_p \in D(M_p^+)$ are

$$\text{(4.18)}\quad \begin{aligned} B_p^*(v_p, I_p) = K^p_{r_p\times(m_p^1-n)} &\begin{bmatrix} [x^{a_p}_{1p}, v_p]_p(a_p) \\ \vdots \\ [x^{a_p}_{m_p^1-n,p}, v_p]_p(a_p) \end{bmatrix} \\ - L^p_{r_p\times(m_p^2-n)} &\begin{bmatrix} [x^{b_p}_{1p}, v_p]_p(b_p) \\ \vdots \\ [x^{b_p}_{m_p^2-n,p}, v_p]_p(b_p) \end{bmatrix} = 0_{r_p\times 1}, \end{aligned}$$

which determine the domains of the operators which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$ for each p, where $M^p_{s_p\times(m_p^1-n)}$, $N^p_{s_p\times(m_p^2-n)}$, $K^p_{r_p\times(m_p^1-n)}$ and $L^p_{r_p\times(m_p^2-n)}$ are numerical matrices which satisfy the following conditions:

$$\text{(4.19)}\quad \begin{aligned} \text{Rank}\,\{M^p_{s_p\times(m_p^1-n)} \oplus N^p_{s_p\times(m_p^2-n)}\} = s_p, \\ \text{Rank}\,\{K^p_{r_p\times(m_p^1-n)} \oplus L^p_{r_p\times(m_p^2-n)}\} = r_p, \end{aligned}$$

$$\text{(4.20)}\quad \begin{aligned} &(\{K^p_{r_p\times(m_p^1-n)} E^{1,2}_{(m_p^1-n)\times(m_p^1-n)} (M^p_{s_p\times(m_p^1-n)})^T\})_{r_p\times s_p} \\ &\quad = (\{L^p_{r_p\times(m_p^2-n)} E^{1,2}_{(m_p^2-n)\times(m_p^2-n)} (N^p_{s_p\times(m_p^2-n)})^T\})_{r_p\times s_p}; \end{aligned}$$

see Theorems 3.7 and 3.8 in the single interval for more details.

Next, the characterization of all operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ in terms of boundary conditions featuring $L^2_{w_p}(a_p, b_p)$-solutions of the equations (2.15) and (2.18) for any N intervals $I_p = (a_p, b_p)$, $p = 1, \dots, N$, is covered by the following theorems.

Theorem 4.1. *Let $\lambda \in \Pi[T_0(M), T_0(M^+)]$ and let r, s and m be as given in* (4.1). *Then the set of all $u = \{u_p\} \in D(M)$ such that*

$$\text{(4.21)}\quad \sum_{p=1}^{N} B_p(u, I_p) = 0,$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $D(S^*)$ is the set of all $v = \{v_p\} \in D(M^+)$ which are such that

$$\sum_{p=1}^{N} B_p^*(v, I_p) = 0. \tag{4.22}$$

In (4.21) and (4.22), $B_p(u, I_p)$ and $B_p^*(v, I_p)$ take one of the forms in (4.5), (4.10), (4.14), (4.17); (4.6), (4.11), (4.15), (4.18), respectively, depending on the nature of the problem in the interval I_p.

The converse of Theorem 4.1 is

Theorem 4.2. *Let S be regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$, let $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$ and let r, s and m be defined by* (4.1). *Then $D(S)$ is the set of $u \in D(M)$ satisfying* (4.21) *while $D(S^*)$ is the set of $v \in D(M^+)$ satisfying* (4.22).

Remark 4.3. Theorems 4.1 and 4.2 follow from the following results for the case of a single interval: [**1**, Theorem III.10.6 and **10**, Theorem II.2.12] for the regular problem, [**2**, Theorems 5.1 and 5.2] for the case of one singular endpoint and Theorems 3.7 and 3.8 in the case when both endpoints are singular.

5. Discussion. In [**8**] Everitt and Zettl discussed the possibility of generating self-adjoint operators which are not expressible as the direct sums of self-adjoint operators defined in the separate intervals. In this section we extend this case to the case of general ordinary differential operators, i.e., we discuss the possibiilty of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals $I_p = (a_p, b_p)$, $p = 1, 2$. We will refer to these operators as "new regularly solvable operators." If a_p is a regular endpoint and b_p is singular, then by [**1**, Theorem III.10.13] the sum

$$\operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] = n \quad \text{for } \lambda \in \Pi[T_0(M_p), T_0(M_p^+)],$$

$p = 1, 2$ if and only if the term in (4.3) at the endpoint b_p is zero. By Lemma 3.1, for $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we get in all cases,

$$0 \le \text{def}\,[T_0(M) - \lambda I] + \text{def}\,[T_0(M^+) - \bar{\lambda} I] \le 4n, \tag{5.1}$$

while

$$2n \le \text{def}\,[T_0(M) - \lambda I] + \text{def}\,[T_0(M^+) - \bar{\lambda} I] \le 4n, \tag{5.2}$$

when each interval has at most one singular endpoint, and

$$\text{def}\,[T_0(M) - \lambda I] + \text{def}\,[T_0(M^+) - \bar{\lambda} I] = 4n, \tag{5.3}$$

for the case when all endpoints are regular.

Let

$$\text{def}\,[T_0(M) - \lambda I] + \text{def}\,[T_0(M^+) - \bar{\lambda} I] = d,$$

and

$$\text{def}\,[T_0(M_p) - \lambda I] + \text{def}\,[T_0(M_p^+) - \bar{\lambda} I] = d_p, \qquad p = 1, 2.$$

Then, by part (c) in Lemma 2.4, we have that $d = d_1 + d_2$.

We now consider some of the possibilities.

Example 1. $d = 0$. This is the minimal case in (5.1) and can only occur when all four endpoints are singular. In this case $T_0(M)|$ is itself regularly solvable and has no proper regularly solvable extensions, see [**1**, Chapter III and **3**].

Example 2. $d = n$ with $d_1 = 0$ and $d_2 = n$ or $d_1 = n$ and $d_2 = 0$. In this case we must have three singular endpoints and one regular. There are no new regularly solvable extensions and we have $S = T_0(M_1) \oplus S_2$ or $S = S_1 \oplus T_0(M_2)$ where S_1 and S_2 are regularly solvable extensions of $T_0(M_1)$ and $T_0(M_2)$, respectively, i.e., all regularly solvable extensions of $T_0(M)$ can be obtained by forming sums of regularly solvable extensions of $T_0(M_1)$ and $T_0(M_2)$; see (i) of Example 3 below.

Example 3. Two singular endpoints and $d = 2n$. We consider two cases:

(i) Both singular endpoints are from the same interval, say I_1. Then

$$S = T_0(M_1) \oplus S_2,$$

where S_2 is a regularly solvable extension of $T_0(M_2)$ and generates all regularly solvable extensions of $T_0(M)$.

(ii) There is one regular and one singular endpoint in each interval and $d_1 = d_2 = n$. Then *mixing* can occur and we get new regularly solvable extensions of $T_0(M)$. For the sake of definiteness assume that the endpoints a_1 and b_2 are singular endpoints and b_1, a_2 are regular endpoints. The other cases are entirely similar.

For $\underset{\sim}{u} \in D(M)$, $\underset{\sim}{\phi_j} \in D(M^+)$ with $\underset{\sim}{u} = \{u_1, u_2\}$, $\underset{\sim}{\phi_j} = \{\phi_{j1}, \phi_{j2}\}$, condition (4.3) reads

$$\begin{aligned} 0 = [\underset{\sim}{u}, \underset{\sim}{\phi_j}] &= [u_1, \phi_{j1}]_1(b_1) - [u_1, \phi_{j1}]_1(a_1) \\ &\quad + [u_2, \phi_{j2}]_2(b_2) - [u_2, \phi_{j2}]_2(a_2), \qquad j = 1, \dots, n. \end{aligned} \tag{5.4}$$

Also, for $\underset{\sim}{v} \in D(M^+)$, $\underset{\sim}{\psi_j} \in D(M)$ with $\underset{\sim}{v} = \{v_1, v_2\}$, $\underset{\sim}{\psi_j} = \{\psi_{j1}, \psi_{j2}\}$, condition (4.4) reads

$$\begin{aligned} 0 = [\underset{\sim}{\psi_j}, \underset{\sim}{v}] &= [\psi_{j1}, v_1]_2(b_1) - [\psi_{j1}, v_1]_1(a_1) \\ &\quad + [\psi_{j2}, v_2]_2(b_2) - [\psi_{j2}, v_2]_2(a_2), \qquad j = 1, \dots, n \end{aligned} \tag{5.5}$$

and condition (ii) in Theorem 4.1 reads

$$\begin{aligned} 0 = [\underset{\sim}{\psi_j}, \underset{\sim}{\phi_k}] &= [\psi_{j1}, \phi_{k1}]_1(b_1) - [\psi_{j1}, \phi_{k1}]_1(a_1) \\ &\quad + [\psi_{j2}, \phi_{k2}]_2(b_2) - [\psi_{j2}, \phi_{k2}]_2(a_2), \qquad j, k = 1, \dots, n. \end{aligned} \tag{5.6}$$

By [**1**, Theorem III.10.13], the terms involving the singular endpoints a_1 and b_2 are zero so that (5.4), (5.5) and (5.6) reduce to

$$[u_1, \phi_{j1}]_1(b_1) - [u_2, \phi_{j2}]_2(a_2) = 0, \qquad j = 1, \dots, n, \tag{5.7}$$

$$[\psi_{j1}, v_1]_1(b_1) - [\psi_{j2}, v_2]_2(a_2) = 0, \qquad j = 1, \dots, n, \tag{5.8}$$

and

$$[\psi_{j1}, \phi_{k1}]_1(b_1) - [\psi_{j2}, \phi_{k2}]_2(a_2) = 0, \qquad j, k = 1, \dots, n. \tag{5.9}$$

Thus, the boundary conditions are not separated for the two intervals and hence the regularly solvable operator cannot be expressed as a direct sum of regularly solvable operators defined in the separate intervals I_1 and I_2.

We refer to Everitt and Zettl's paper [**8**] for more examples and more details.

Acknowledgment. I am grateful to Professor W.D. Evans of the School of Mathematics at University of Wales College of Cardiff for suggesting this problem and for reading the manuscript carefully and making several helpful comments.

REFERENCES

1. D.E. Edmunds and W.D. Evans, *Spectral theory and differential operators*, Oxford University Press, 1987.

2. W.D. Evans and Sobhy E. Ibrahim, *Boundary conditions for general ordinary differential operators and their adjoints*, Proc. Roy. Soc. Edinburgh, 1989, to appear.

3. W.D. Evans, *Regularly solvable extensions of non-self-adjoint ordinary differential operators*, Proc. Roy. Soc. Edinburgh, Sect. A., **97** (1984), 79–95.

4. W.N. Everitt, *Integrable square solutions of ordinary differential equations*, Quart. J. Math. Oxford Sect. 2, **14** (1963), 170–180.

5. ———, *Singular differential equations* II: *Some self-adjoint even order cases*, Quart. J. Math. Oxford Sect. (2), **18** (1967), 13–32.

6. W.N. Everitt and D. Race, *Some remarks on linear ordinary quasi-differential equations*, Proc. London Math. Soc. (3) **54** (1987), 300–320.

7. W.N. Everitt and A. Zettl, *Generalized symmetric ordinary differential expressions* I: *The general theory*, Nieuw Arch. Wisk. (3) **XXVII** (1979), 363–397.

8. ———, *Sturm-Liouville differential operators in direct sum spaces*, Rocky Mountain J. Math. **16** (1986), 497–516.

9. H. Frentzen, *On J-symmetric quasi-differential expressions with matrix-valued coefficients*, Quaestiones Math. **10** (1986), 153–164.

10. Sobhy E. Ibrahim, *Problems associated with differential operators*, Ph.D. thesis (1989), Faculty of Science, Benha University, Egypt.

11. Sun Jiong, *On the self-adjoint extensions of symmetric ordinary differential operators with middle deficiency indices*, Acta Math. Sinica (N.S) **1** (1986), 152–167.

12. A.M. Krall and A. Zettl, *Singular self-adjoint Sturm-Liouville problems*, Differential Integral Equations **1** (1988), 423–432.

13. ———, *Singular self-adjoint Sturm-Liouville problems* II: *Interior singular points*, SIAM J. Math. Anal. **19** (1988), 1135–1141.

14. S.J. Lee, *On boundary conditions for ordinary linear differential operators*, J. London Math. Soc. Ser. 2, **12** (1976), 447–454.

15. Zai-Jiu Shang, *On J-selfadjoint extensions of J-symmetric ordinary differential operators*, J. Differential Equations **73** (1988), 153–177.

16. M.I. Visik, *On general boundary problems for elliptic differential equations*, Amer. Math. Soc. Transl. Ser. 2 **24** (1963), 107–172.

17. A. Zettl, *Formally self-adjoint quasi-differential operators*, Rocky Mountain J. Math. **5** (1975), 453–474.

University of Benha, Department of Mathematics, Faculty of Science, Benha, Egypt