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SUBORDINATION FAMILIES

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ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a function F analytic in the unit disk Δ. Let $Hs(F)$ denote the closed convex hull of $s(F)$ and Λ denote the set of probability measures on $\partial\Delta$. Let $R = \{F : F \text{ is analytic in } \Delta \text{ and } Hs(F) = \{\int_{|x|=1} F(xz)\, d\mu(x) : \mu \in \Lambda\}\}$. Let R_Σ denote those functions analytic in Δ such that the set of support points of $s(F)$ is $\{F(xz) : |x| = 1\}$. In this paper we investigate R and R_Σ. We prove that if F is a univalent function in R and $(F(z) - F(0))/F'(0)$ has positive Hayman index, then F is in R_Σ.

Introduction. Let $\Delta = \{z : |z| < 1\}$ and let A denote the set of functions analytic in Δ. If $f \in A$, $f(z) = \sum_{n=0}^{\infty} a_n z^n (z \in \Delta)$, then let $\bar{m}(r, f) = \sum_{n=0}^{\infty} |a_n| r^n$. A is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. Let $B_0 = \{\phi : \phi \in A \text{ and } |\phi(z)| \le |z|\}$. A function f is called a support point of a compact subset $\mathcal{F}$ of A if $f \in \mathcal{F}$ and if there is a continuous, linear functional J on A so that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in \mathcal{F}\}$ and $\operatorname{Re} J$ is not constant on $\mathcal{F}$. We denote the set of support points of $\mathcal{F}$ by $\Sigma\mathcal{F}$, the set of extreme points of $\mathcal{F}$ by $E\mathcal{F}$ and the closed convex hull of $\mathcal{F}$ by $H\mathcal{F}$.

Let $D = \{f : f \in A \text{ and } \int_\Delta \int |f'(z)|^2 \, dy\, dx < +\infty \text{ where } z = x + iy\}$ and $S = \{f \in A : f(0) = 0, f'(0) = 1 \text{ and } f \text{ is univalent}\}$. If we set $M(r, f) = \max_{|z| \le r} |f(z)|$ then $\alpha = \lim_{r\to 1^-} (1 - r)^2 M(r, f) \le 1$ exists for each $f \in S$ and is called the Hayman index of f [**9**, p. 163, **17**, p. 141]. It is known that f has a unique direction of maximal growth $e^{i\theta_0}$ if $\alpha > 0$, in the sense that $\lim_{r\to 1^-} (1 - r)^2 |f(re^{i\theta_0})| = \alpha$ and $\lim_{r\to 1^-} (1 - r)^2 |f(re^{i\theta})| = 0$ for $\theta \ne \theta_0$ [**9**]. Let $S_\alpha = \{f : f \in S \text{ and } \alpha > 0\}$.

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If f and F are in A we say that f is subordinate to F (and write $f \prec F$) if there exists $\phi \in B_0$ such that $f = F \circ \phi$. We let $s(F) = \{F \circ \phi : \phi \in B_0\}$ and note that $s(F)$ is a compact subset of A. When F is univalent the relation $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

Let Λ denote the set of probability measures on $\partial\Delta$. We let $R = \{F \in A : Hs(F) = \{\int_{|x|=1} F(xz)\, d\mu : \mu \in \Lambda\}\}$ and $R_\Sigma = \{F \in A : \Sigma s(F) = \{F(xz) : |x| = 1\}\}$. Finally, let $U = \{F \in A : F \text{ is univalent}\}$. The systematic and general study of R and R_Σ was begun recently in [**15**]. It is known [**2**] that if $F \in H^1$, then $EHs(F) \supseteq \{F \circ \phi : \phi \in B_0, \phi \text{ inner}\}$. If $F \in R$, then $EHs(F) \subset \{F(xz) : |x| = 1\}$. So if $F \in H^1$, then $F \notin R$. In [**3**] Y. Abu-Muhanna proved that if $F \in U$, $\not{C}\backslash F(\Delta)$ is convex and $\partial F(\Delta)$ satisfies a certain smoothness condition at ∞, then $F \in R$. In [**15**] the authors were able to eliminate the smoothness condition at ∞ and thus complete the proof that, for any $F \in U$, if $\not{C}\backslash F(\Delta)$ is convex, then $F \in R$. This result was conjectured by Abu-Muhanna in [**3**]. Finally, Abu-Muhanna conjectured in [**3**] that if F is the universal covering map of Δ onto a domain D with $\not{C}\backslash D$ bounded and convex then $F \in R$. An example of this is provided by the function $F(z) = exp(1+z)/(1-z)$. J. Feng proved in [**10**] that this $F \in R$. The fact that $F \in R_\Sigma$ was proved in [**12**].

In [**15**] the authors point out that $R_\Sigma \subset R$ and prove that R is closed in A. The main thrust of [**15**] was to try to determine when functions are in R and for such functions to try to further ascertain when they are in R_Σ. In this paper we provide new conditions that guarantee membership in R_Σ given membership in R. In [**15**] the authors conjectured that $R\backslash R_\Sigma$ consists only of the univalent half plane mappings. We provide some supporting evidence for this conjecture.

Support points. In Theorem 1 we give new conditions that guarantee $\Sigma s(F) = \{F(xz) : |x| = 1\}$. This result generalizes [**14**, Theorem 1]. The proof is somewhat similar to that in [**14**] but we include it for completeness since it differs in some significant details.

Theorem 1. *Let $F \in R$ and suppose*

$$F(z) = \frac{G(z)}{(z - x_0)^\alpha} \qquad (|x_0| = 1)$$

where $G \in A$, $\lim_{\substack{z\to x_0\\ \text{nontangentially}}} |G(z)| = L \neq 0$, $|G(z)| = 0(1/(1-r)^{\beta})$ *where* $0 \leq \beta < \alpha$ *and* $\alpha > 1$. *Then* $F \in R_{\Sigma}$.

Proof. The inclusion $\{F(xz) : |x| = 1\} \subset \Sigma s(F)$ is known [**13**] and we need only prove that $\Sigma s(F) \subset \{F(xz) : |x| = 1\}$. Let $f \in \Sigma s(F)$. We have $f = F \circ \phi$ for $\phi \in \Sigma B_0$ [**1**]. By standard arguments [**5, 12**] we have

$$f(z) = F(\phi(z)) = \sum_{k=1}^{n} \lambda_k F(x_k z) \tag{2}$$

where $\lambda_k \geq 0$, $|x_k| = 1$ and $\sum_{k=1}^{n} \lambda_k = 1$. We see from (1) that $f(z) = G(\phi(z))/(\phi(z) - x_0)^{\alpha}$. Let $w_k = \bar{x}_k x_0 (k = 1, 2, \dots, n)$; then from (1), (2) and comparing singularities we see that $\phi(w_k) = x_0$ for $k = 1, 2, \dots, n$. We may write

$$\frac{1}{1 - \bar{x}_0 \phi(z)} = \sum_{k=1}^{m} t_k \frac{1}{1 - y_k z} \tag{3}$$

where $t_k \geq 0$, $|y_k| = 1$, and $\sum_{k=1}^{m} t_k = 1$ [**6**], [**13**, p. 100]. It follows that

$$\sum_{k=1}^{n} \lambda_k F(x_k z) = aG(\phi(z)) \bigg(\sum_{k=1}^{m} t_k \frac{1}{1 - y_k z} \bigg)^{\alpha} \tag{4}$$

where $a = (-x_0)^{-\alpha}$. By using (1) and comparing singularities on both sides of (4), we conclude $n = m$ and so we have

$$\sum_{k=1}^{n} \lambda_k F(x_k z) = aG(\phi(z)) \bigg(\sum_{k=1}^{m} t_k \frac{1}{1 - y_k z} \bigg)^{\alpha}. \tag{5}$$

Using (1) we can rewrite (5) as

$$a \sum_{k=1}^{n} \lambda_k \frac{G(x_k z)}{(1 - \bar{w}_k z)^{\alpha}} = G(\phi(z)) \bigg(\sum_{k=1}^{n} t_k \frac{1}{1 - y_k z} \bigg)^{\alpha}. \tag{6}$$

Thus, $w_k = \bar{y}_k$ for $k = 1, 2, \dots, n$. If we let $z = r\bar{y}_k$, multiply by $(1-r)^{\alpha}$, take absolute values and let $r \to 1^-$ in (6), we obtain

$$L\lambda_k = Lt_k^{\alpha}. \tag{7}$$

Since $\phi \in \Sigma B_0$, ϕ is a finite Blaschke product and so is analytic on $\partial\Delta$. Hence, $\{\phi(r\bar{y}_k) : 0 < r \leq 1\}$ is orthogonal to $\partial\Delta$ at $\phi(\bar{y}_k) = \phi(w_k) = x_0 = x_k w_k = x_k \bar{y}_k$. So it follows that $\phi(r\bar{y}_k) \to x_0$ nontangentially as $r \to 1^-$. Hence, $\lim_{r\to 1^-} |G(x_k\bar{y}_k r)| = L$ and $\lim_{r\to 1^-} |G(\phi(r\bar{y}_k))| = L$. Since $L \neq 0$ we conclude from (7) that $\lambda_k = t_k^\alpha$. Since $\alpha > 1$, $t_k^\alpha < t_k$ and if there are more than two nonzero λ_k's in (2), a contradiction to the fact that $\sum_{k=1}^n t_k = \sum_{k=1}^n \lambda_k = 1$ is easily obtained. This completes the proof. □

We next prove a type of Tauberian theorem which we need for technical reasons. The proof was provided by Y.J. Leung. Let $\hat{S}$ denote a symmetric Stolz angle with vertex $e^{i\theta}$ and having opening 2ϕ where $0 < \phi < \pi/2$. If $z \in \hat{S}$ we associate with z the unique number ζ that is a perpendicular projection of z onto the line segment having end points 0 and $e^{i\theta}$.

Theorem 2. *If $h \in A$ and $\bar{m}(r, h') = o(1/(1-r))$, then for any θ_0,*

$$\lim_{\substack{z\to e^{i\theta_0} \\ z\in\hat{S}}} |h(z) - h(\zeta)| = 0. \tag{8}$$

Proof. There is a positive number d so that if $z \in S$, then $|e^{i\theta_0} - z| \leq d(1 - |z|)$. As $|z - \zeta| \leq |e^{i\theta_0} - z|$ we have $|z - \zeta| \leq d(1 - |z|)$ wherever $z \in S$. Let $|z| = r$ and note that $|h(z) - h(\zeta)| \leq \sum_{n=1}^\infty |a_n||z^n - \zeta^n|$ where $h(z) = \sum_{n=0}^\infty a_n z^n$. It is easily seen that $|z^n - \zeta^n| \leq dnr^{n-1}(1-r)$ since $|\zeta| \leq |z|$. Hence, we have

$$|h(z) - h(\zeta)| \leq d(1-r)\sum_{n=1}^{\infty} n|a_n|r^{n-1} = d(1-r)\bar{m}(r, h'). \tag{9}$$

It follows directly from (9) and the assumption on $\bar{m}(r, h')$ that (8) holds. □

Remark . We do not assume the existence of either of the possible individual limits in (8).

Corollary 3. *If* $h \in A$, $\bar{m}(r, h') = o(1/(1-r))$ *and* $\lim_{\substack{\zeta \to e^{i\theta_0} \\ \text{radially}}} \operatorname{Re} h(\zeta) = L$, *then*

$$\lim_{\substack{z \to e^{i\theta_0} \\ z \in \hat{S}}} \operatorname{Re} h(z) = L. \tag{10}$$

Proof. Since $|\operatorname{Re} h(z) - \operatorname{Re} h(\zeta)| \leq |h(z) - h(\zeta)|$ it is easy to see that (10) follows from (8). □

Remark . If you assume $\lim_{\substack{\zeta \to e^{i\theta_0} \\ \text{radially}}} h(\zeta) = L$, then you conclude $\lim_{\substack{z \to e^{i\theta_0} \\ z \in \hat{S}}} h(z) = L$.

Theorem 4. *Let* $F \in R$ *and suppose* $F(z) = (z \exp g(z)/(z - x_0)^\alpha)$ *where* $x_0 = e^{i\theta_0}$, $\alpha > 1$, $g \in D$ *and* $\lim_{r \to 1^-} \operatorname{Re} g(re^{i\theta_0}) = L \neq -\infty$. *Then* $F \in R_\Sigma$.

Proof. It is a well-known classical fact that $\bar{m}(r, g') = o(1/(1-r))$. It follows from Corollary 3 that $\lim_{\substack{z \to e^{i\theta_0} \\ z \in \hat{S}}} \operatorname{Re} g(z) = L$. It was proved [**7**] that $\bar{m}(r, g) = o(\sqrt{\log(1/(1-r)})$. We claim $|\exp g(z)| = 0(1/(1-r)^\beta)$ for any $\beta > 0$. To see this, note that $|\exp g(x)| = \exp \operatorname{Re} g(z) \leq \exp \bar{m}(r, g)$ and that $\bar{m}(r, g) = o(\sqrt{\log(1/(1-r)}) = 0(\beta \log(1/(1-r))$ for any $\beta > 0$. Given α above we choose β so that $0 \leq \beta < \alpha$. It follows from Theorem 1 that $F \in R_\Sigma$. □

We next prove one of our main results. This Theorem supports the idea that any function in R with "large regular" growth is in R_Σ.

Theorem 5. *If* $F \in R \cap U$ *and* $(F(z) - F(0))/F'(0) \in S_\alpha$ *for some* $\alpha > 0$, *then* $F \in R_\Sigma$.

Proof. Let $f(z) = (F(z) - F(0))/F'(0)$ and note that it is sufficient to prove that $f \in R_\Sigma$. Since $f \in S_\alpha$ for some $\alpha > 0$, there exists a unique direction $e^{i\theta_0}$ such that $\lim_{r \to 1^-} ((1-r)^2/r)|f(re^{i\theta_0})| = \alpha$ [**9**]. If we write $g(z) = \log((1-e^{-i\theta_0}z)^2/z)f(z))$, then it follows from Bazilevich's Theorem [**9**, p. 160] that $g \in D$. So we have

$$f(z) = \frac{z}{(1-e^{-i\theta_0}z)^2} \exp g(z) \tag{11}$$

where $\lim_{r\to 1^-} \exp \operatorname{Re} g(re^{i\theta_0}) = \alpha > 0$. Hence, we deduce that $\lim_{r\to 1^-} \operatorname{Re} g(re^{i\theta_0}) = \log \alpha > -\infty$. It follows from Theorem 4 that $f \in R_\Sigma$ and this completes the proof. □

Remark . If $F \in R \cap U$ and $\mathbb{C}\backslash F(\Delta)$ is a Jordan arc going to ∞ that is contained in a half-strip, then it is known that $F \in S_\alpha$ for some $\alpha > 0$ [**17**, p. 311]. It follows from Theorem 5 that such an F is in R_Σ.

If $F \in S_\alpha$ for $\alpha > 0$ the representation (11) and the fact that $\bar{m}(r,g) = o(\sqrt{\log(1/1-r)})$ [**7**] permits an easy proof that

$$\lim_{r\to 1^-} \frac{\log|F(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}}} = 0 \tag{12}$$

uniformly for θ satisfying $|\theta - \theta_0| \geq \delta > 0$. W. Hayman and P.B. Kennedy first proved (12) by other methods in [**16**].

In addition to Theorem 1, Theorem 4 and Theorem 5, the next Theorem adds to the evidence suggesting that most functions in R are also in R_Σ. We let ED denote the set of extreme points of any set $D \subset \mathbb{C}$ and ∂D denote the boundary of D.

Theorem 6. *Suppose $F \in R$ and $\mathbb{C}\backslash \bar{F}(\Delta)$ is a bounded convex domain with the property that $E\partial F(\Delta) = \partial F(\Delta)$. Suppose $F(e^{i\theta}) \in \partial F(\Delta)$ whenever $F(e^{i\theta})$ exists. Then $F \in R_\Sigma$.*

Proof. There exists $w \in \mathbb{C}$ and $d > 0$ such that $|F(z) - w| \geq d$ for all $z \in \Delta$. It follows that $f(z) = 1/(F(z) - w) \in H^\infty$ and so $F(e^{i\theta})$ exists for almost all θ. Let $F \in \Sigma s(F)$. Then we have $f = F \circ \phi$ for $\phi \in \Sigma B_0$ [**1**]. As in the proof of Theorem 1, we need only prove that $\phi(z) = xz$ for some $|x| = 1$. Again, as in the proof of Theorem 1, we may write

$$f(z) = F(\phi(z)) = \sum_{k=1}^{n} \lambda_k F(x_k z) \tag{13}$$

where $|x_k| = 1$, $0 < \lambda_k \leq 1$, $\sum_{k=1}^{n} \lambda_k = 1$ [**5,12**]. We deduce from (13) that for almost all θ,

$$f(e^{i\theta}) = \sum_{k=1}^{n} \lambda_k F(x_k e^{i\theta}). \tag{14}$$

Now suppose $i \neq j$ and $x_i \neq x_j$. We first show that $F(x_i e^{i\theta}) \neq 0$ for each i and almost all θ. To see this, note that if $F(x_i e^{i\theta}) = 0$ for some i and for θ in a set of positive Lebesgue measure, then $f(x_i e^{i\theta}) = -(1/w)$ for all θ in this set and since $f \in H^\infty$ this would imply $g(x_i z) \equiv -(1/w)$. Hence, we would have $F(x_i z) \equiv 0$. This is impossible since $\mathbb{C} \backslash \overline{F(\Delta)}$ is a bounded convex domain.

Since $F(\phi(e^{i\theta})) \in \partial F(\Delta)$ whenever the limit exists and $E\partial F(\Delta) = \partial F(\Delta)$, it follows from (14) that $F(x_i e^{i\theta}) = F(x_j e^{i\theta})$ for almost all θ. So $g(x_i e^{i\theta}) = g(x_j e^{i\theta})$ for almost all θ and so we conclude that $g(x_i z) = g(x_j z)$ for all $z \in \Delta$. Hence, we have

$$F(x_i z) = F(x_j z) \tag{15}$$

for all $z \in \Delta$. It follows from (15) and Lemma 9 in [**15**] that $x_i = x_j$. We conclude that $f(z) = F(xz)$ for some $|x| = 1$ and so $\sum s(F) \subset \{F(xz) : |x| = 1\}$. Recalling [**13**] that $\Sigma s(F) \supset \{F(xz) : |x| = 1\}$, we see that $F \in R_\Sigma$ and this completes the proof. □

Remark . Theorem 6 provides a new proof that $F(z) = \exp(1+z)/(1-z) \in R_\Sigma$. This was first proved in [**12**] by different methods.

It is easily proved that if F is the universal covering map from Δ onto a domain D with $0 \in D$, $\mathbb{C} \backslash D$ bounded and convex, then $F(z) = \exp G(z)$ where G is in U. Since $G \in H^p$ for all $p < 1/2$ [**8**, p. 50] and $\exp(z)$ is locally conformal it will be the case that $F(e^{i\theta}) \in \partial F(\Delta)$ for almost all θ. If, as conjectured by Y. Abu-Muhanna [**3**], such an F is in R, then by Theorem 6, $F \in R_\Sigma$ whenever $E\partial F(\Delta) = \partial F(\Delta)$. This will be the case, for example, if F maps onto the complement of an appropriate ellipse.

It was conjectured in [**15**] that if F is a nonconstant function in R, $F(\Delta)$ is not a half plane, $\phi \in \Sigma B_0$ and (16) holds, then $\phi(z) = xz (|x| = 1)$. We next show that if the assumption $F \in R$ made in this conjecture is replaced by the assumption $F \in U \cap H^1$, then the conjecture is true.

Theorem 7. *Suppose* $F \in U \cap H^1$, $|x_k| = 1$, $0 < \lambda_k \leq 1$, $\sum_{k=1}^n \lambda_k = 1$, $\phi \in B_0$ *and*

$$F(\phi(z)) = \sum_{k=1}^{n} \lambda_k F(x_k z). \tag{16}$$

Then $\phi(z) = xz$ *for some* $|x| = 1$.

Proof. If we let $f(z) = (F(z) - F(0))/F'(0)$ we have $F \in S \cap H^1$ and (16) implies that

$$f(\phi(z)) = \sum_{k=1}^{n} \lambda_k f(x_k z). \tag{17}$$

Note that $f \circ \phi \in H^1$. Also, since ϕ is an inner function, Ryff's Theorem [**18**] implies that $||f(x_k z)||_1 = ||f(\phi(z))||_1 = ||f||_1$ for $k = 1, 2, \dots, n$ where $|| \quad ||$ denotes the H^1 norm. It follows from this fact, (17) and the condition for equality in Minkowski's inequality that for each pair j, k we have

$$\frac{f(x_k e^{i\theta})}{f(x_j e^{i\theta})} > 0 \tag{18}$$

for almost all θ. We recall that $f(z) = zg(z)$ where $|g(z)| \geq (1/(1 + |z|)^2)$ for $z \in \Delta$ [**17**, p. 21]. Since $1/g \in A$ the previous inequality implies $1/g(xz) \in H^\infty$ for any $|x| = 1$. Let $h(z) = f(x_k z)/f(x_j z)$ and note that

$$h(z) = \frac{x_k g(x_k z)}{x_j g(x_j z)} \qquad (z \in \Delta). \tag{19}$$

We conclude from (18), (19) and the fact that $g \in H^1$ that $h \in H^1$. Since we have $h(e^{i\theta}) > 0$ for almost all θ, we deduce that $h(z) \equiv c$ for all $z \in \Delta$ and some $c \in \mathbb{C}$ [**11**, p. 95]. We see that $c > 0$, and since $g(0) = 1$, we have $h(0) = c = (x_k/x_j)$. It follows that $c = 1$, $x_k = x_j$ and $\phi(z) = xz$ for some $|x| = 1$. This completes the proof. □

We finish by showing that if the assumption $F \in R$ made in Conjecture B in [**15**] is replaced by the assumption $F \in U$, then the conjecture is false. Let $F(z) = z/(1-z^2)$, $\phi \in \Sigma B_0$, $\phi(z) \neq xz$ and $\phi(z) = -\phi(-z)$ for all $z \in \Delta$. Note that $F \in U$ and F is not a univalent half plane mapping. We know that

$$\frac{1}{1-\phi(z)} = \sum_{k=1}^{n} t_k \frac{1}{1-x_k z} \tag{20}$$

where $|x_k| = 1$, $t_k \geq 0$, $n \geq 3$, $\sum_{k=1}^{n} t_k = 1$ [**13**, p. 100]. Since $\phi(z) = -\phi(-z)$, we obtain from (20)

$$\frac{1}{1+\phi(z)} = \frac{1}{1-\phi(-z)} = \sum_{k=1}^{n} t_k \frac{1}{1+x_k z}. \tag{21}$$

It follows from (20) and (21) that

$$F(\phi(z)) = \frac{1}{2}\,\frac{1}{1-\phi(z)} - \frac{1}{2}\,\frac{1}{1+\phi(z)} = \sum_{k=1}^{n} t_k \frac{x_k z}{1-(x_k z)^2}. \tag{22}$$

It is clear from (22) that the modified conjecture mentioned above is false. It follows from Lemma 7 in [**15**] that $F(z) = z/(1-z^2) \notin R$.

It is known that whenever $F \in A$ then $\{F(xz) : |x| = 1\} \subset EHs(F) \subset Es(F)$ [**13**]. Recently, Y. Abu-Muhanna and D.J. Hallenbeck proved that if $F \in U$, then $Es(F) \subset \{F \circ \phi : \phi \in EB_0\}$. Hence, whenever $F \in U$, we have the inclusions

$$\{F(xz) : |x| = 1\} \subset EHs(F) \subset \{F \circ \phi : \phi \in EB_0\}. \tag{23}$$

If $F(z) = z/(1-z^2)$ and ϕ is an odd function in $\Sigma B_0 \subset EB_0$, then (22) implies that $F \circ \phi \notin Es(F)$. So we have an explicit example of a function $F \in U$ such that both inclusions in (23) are strict. It was proved by K. Tkaczyńska in [**19**] that the inclusions in (23) are always strict whenever $F(\Delta)$ is convex, not a half plane, wedge or strip and $\partial F(\Delta)$ contains a line segment.

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