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ON AN INEQUALITY FOR NONDECREASING SEQUENCES

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If (a_i) is a sequence of nonnegative real numbers, and if r and s are real numbers with $0 < r < s$, then we have

$$\left(\sum_{i=1}^{n} a_i^r\right)^{1/r} \geq \left(\sum_{i=1}^{n} a_i^s\right)^{1/s}, \tag{1}$$

and, under the additional assumption that (p_i) is a sequence of nonnegative weights satisfying $\sum_{i=1}^{n} p_i \leq 1$, we obtain

$$\left(\sum_{i=1}^{n} p_i a_i^r\right)^{1/r} \leq \left(\sum_{i=1}^{n} p_i a_i^s\right)^{1/s}. \tag{2}$$

Proofs for these well-known inequalities can be found, for instance, in [**1**, pp. 26–28].

In 1976 M.S. Klamkin and D.J. Newman [**2**] established an interesting variant of (1) for nondecreasing sequences: If $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ with $a_i - a_{i-1} \leq 1$ $(i = 1, \dots, n)$ and $s = 2r + 1$ with $r \geq 1$, then

$$\left((r+1)\sum_{i=1}^{n} a_i^r\right)^{1/(r+1)} \geq \left((s+1)\sum_{i=1}^{n} a_i^s\right)^{1/(s+1)}. \tag{3}$$

In 1981 A. Meir [**6**] published a weighted version of (3) which can be considered as an inverse inequality of (2) for nondecreasing sequences: Let $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $0 \leq p_0 \leq p_1 \leq \cdots \leq p_n$ satisfying

$$a_i - a_{i-1} \leq (p_i + p_{i-1})/2, \qquad i = 1, \dots, n. \tag{4}$$

If $s \geq 2r + 1$ and $r \geq 1$, then

$$\left((r+1)\sum_{i=1}^{n} p_i a_i^r\right)^{1/(r+1)} \geq \left((s+1)\sum_{i=1}^{n} p_i a_i^s\right)^{1/(s+1)}. \tag{5}$$

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Meir remarked that a slightly modified version of (5) can be obtained if the assumption (4) will be replaced by

$$a_i - a_{i-1} \le p_i, \qquad i = 1, \dots , n.$$

Then

$$(6) \quad \left((r+1)\sum_{i=1}^{n-1}((p_i + p_{i-1})/2)a_i^r\right)^{1/(r+1)} \ge \left((s+1)\sum_{i=1}^{n-1}((p_i + p_{i-1})/2)a_i^s\right)^{1/(s+1)}.$$

A refinement as well as an extension of (6) was given recently by G.V. Milovanović and I.Ž. Milovanović [**7**].

The aim of this note is to prove the following counterpart of Meir's result:

Theorem. *Let* $0 = a_0 \le a_1 \le \cdots \le a_n$ *and* $0 \le p_n \le p_{n-1} \le \cdots \le p_0$ *satisfying* $a_i - a_{i-1} \ge (p_i + p_{i-1})/2$, $i = 1, \dots , n$.

If $0 \le r \le 1$ *and* $r \le s \le 2r + 1$, *then*

$$(7) \qquad \left((r+1)\sum_{i=1}^{n} p_i a_i^r\right)^{1/(r+1)} \le \left((s+1)\sum_{i=1}^{n} p_i a_i^s\right)^{1/(s+1)}.$$

Proof. Let x and y be positive real numbers and let u and v be real numbers; we denote by $E(u, v; x, y)$ the mean value family

$$E(u, v; x, y) = \left[\frac{v}{u}\frac{x^u - y^u}{x^v - y^v}\right]^{1/(u-v)}, \qquad x \ne y, u \ne v, uv \ne 0.$$

(Many remarkable properties of the function E were published by E.B. Leach and M.C. Sholander [**3**–**5**].)

Since $E(u, v; x, y)$ is nondecreasing in u and v (see [**3**]), we conclude

$$E(r + 1, 1; a_i, a_{i-1}) \ge E(2r, r; a_i, a_{i-1})$$

which leads to

$$\begin{aligned}a_i^{r+1} - a_{i-1}^{r+1} &\geq \frac{r+1}{2}(a_i - a_{i-1})(a_i^r + a_{i-1}^r)\\ &\geq \frac{r+1}{2}\frac{p_i + p_{i-1}}{2}(a_i^r + a_{i-1}^r)\\ &\geq \frac{r+1}{2}(p_i a_i^r + p_{i-1}a_{i-1}^r), \qquad i = 1, \dots, n.\end{aligned}$$

Let $1 \leq j \leq n$; after summing for $i = 1, \dots, j$, we get

$$a_j^{r+1} \geq \frac{r+1}{2}(A_j + A_{j-1}) \tag{8}$$

with $A_j = \sum_{i=1}^{j} p_i a_i^r$.

Let $1 \leq t \leq 2$; then we have

$$E(2, 1; A_j, A_{j-1}) \geq E(t, 1; A_j, A_{j-1})$$

which leads to

$$((A_j + A_{j-1})/2)^{t-1} t(A_j - A_{j-1}) \geq A_j^t - A_{j-1}^t$$

and

$$\begin{aligned}(r+1)^{t-1}((A_j + A_{j-1})/2)^{t-1} t(A_j - A_{j-1})\\ \geq (r+1)^{t-1}(A_j^t - A_{j-1}^t).\end{aligned} \tag{9}$$

From (8) and (9) we conclude

$$\begin{aligned}tp_j a_j^{(r+1)(t-1)+r} &= a_j^{(r+1)(t-1)} t(A_j - A_{j-1})\\ &\geq [((r+1)/2)(A_j + A_{j-1})]^{t-1} t(A_j - A_{j-1})\\ &\geq (r+1)^{t-1}(A_j^t - A_{j-1}^t).\end{aligned}$$

Setting $t = (s+1)/(r+1)$ and summing for $j = 1, \dots, n$, we obtain

$$\frac{s+1}{r+1}\sum_{j=1}^{n} p_j a_j^s \geq (r+1)^{(s-r)/(r+1)}\left(\sum_{i=1}^{n} p_i a_i^r\right)^{(s+1)/(r+1)}$$

which is equivalent to inequality (7).

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