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LINEAR THIRD-ORDER DIFFERENCE EQUATIONS: OSCILLATORY AND ASYMPTOTIC BEHAVIOR

B. SMITH

Introduction. In several recent papers, the oscillatory and asymptotic behavior of solutions of second order difference equations have been discussed. For example, note the following papers [**1, 3, 5, 7**]. When compared to differential equations, the study of the oscillation properties of difference equations has received little attention for orders greater than two.

In this paper we will be concerned with the solutions of the linear third order difference equation

$$(E) \qquad \Delta^3 U_n + P_{n+1}\Delta U_{n+2} + Q_n U_{n+2} = 0,$$

where Δ denotes the differencing operation, $\Delta X_n = X_{n+1} - X_n$. The coefficient sequences are real sequences satisfying $P_n \geq 0$, $Q_n < 0$, $\Delta P_n - 2Q_n > 0$, $n \geq 1$ and $\sum^\infty (\Delta P_n - 2Q_n) = \infty$.

In [**6**] the equation

$$(1) \qquad \Delta^3 U_n - P_n U_{n+2} = 0,$$

is studied subject to the condition $P_n > 0$ for each $n \geq 1$. Therein: it is proved that (1) always has a nonoscillatory solution; a characterization of the existence of oscillatory solutions of (1), in terms of the behavior of nonoscillatory solutions is established; an example is given demonstrating (1) as having only nonoscillatory solutions.

In this work we prove (E) always has an oscillatory solution (Theorem 2.1). The theorem is a generalization of [**6**, Theorem 3.9], and extends to difference equations the result of Jones [**4**, Theorem 2] concerning linear differential equations. Moreover, a sufficient condition is given in terms of the sequences P_n, Q_n so that (E) has a solution satisfying

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n,$$

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n sufficiently large. Such a solution, where $U_n > 0$, is termed *strongly increasing* [**6**].

Primarily we use the terminology of Fort's book [**2**] in our discussion. A real sequence $U = \{U_n\}$ satisfying (E) for each $n \geq 1$ we term a *solution* of (E). By the *graph* of a solution U we mean the polygonal path connecting the points (n, U_n), $n \geq 1$. A point of contact of the graph of U with the real axis is a *node*. A solution of (E) is said to be *oscillatory* if it has arbitrarily large nodes; otherwise it is said to be *nonoscillatory*.

Section 1. For each solution $U = \{U_n\}$ of (E) define F, where

$$F[U_n] = F_n = (\Delta U_n)^2 - 2U_{n+1}\Delta^2 U_n - P_n U_{n+2}^2. \tag{2}$$

Computing the difference of F_n and making the substitution from (E), we find

$$\Delta F_n = (2Q_n - \Delta P_n)U_{n+2}^2 - (\Delta^2 U_n)^2 - P_{n+1}(\Delta U_{n+2})^2.$$

We have the following lemma.

Lemma 1.1. *If U is a solution of* (E), *the function defined by* (2) *is nonincreasing.*

Because F_n is nonincreasing, it follows that if U is a nontrivial solution of (E), F_n is eventually sign definite. Note the solution U of (E) with initial values

$$U_1 = a, \qquad U_2 = 2a, \qquad U_3 = 4a, \tag{3}$$

$a > 0$ and constant, is such that $F_n < 0$, $n \geq 1$. It is also true a solution of (E) exists where F_n remains positive for each $n \geq 1$. We have the theorem.

Theorem 1.2. *There exists a solution of* (E) *satisfying $F_n > 0$ for all $n \geq 1$.*

A proof is identical to that of [**6**, Theorem 2.4] and is omitted.

We conclude these preliminaries with a remark concerning the graph of a sequence U.

Remark 1.3. From the literature of differential equations, a function f defined by $f(x)$ is oscillatory provided $f(x)$ has arbitrarily large zeros; otherwise f is nonoscillatory. See [**8**] for example. The graph of a sequence U is the graph of a continuous function G defined by

$$G(x, U_n) = (\Delta U_n)(x - n) + U_n, \qquad n \le x \le n + 1, n \ge 1.$$

It is clear, under a horizontal translation of axes, the oscillatory character of G is invariant.

Section 2. We now state and prove our main results.

Theorem 2.1. *A solution U of* (E)*, where $F[U_n] > 0$, $n \ge 1$, is oscillatory.*

Proof. Let U be a solution of (E), where $F[U_n] > 0$ for each $n \ge 1$. Suppose U is nonoscillatory. Because (E) is linear, no generality is lost if we assume $U_n > 0$ for all $n \ge N$. For such a solution, $\{\Delta U_n\}$ is nonoscillatory, for if ΔU_n has a node at $x_0 = i$, $i > N$ an integer (cf. Remark 1.3), we see from (2),

$$F[U_i] = -2U_{i+1}\Delta^2 U_i - P_i U_{i+2}^2 > 0.$$

Hence, $\Delta^2 U_i < 0$. Consequently, at an arbitrary node for ΔU_n, the slope of $G(x, \Delta U_n)$ is negative, therefore $G(x, \Delta U_n)$ cannot change signs for $n > i$. Thus, ΔU_n is eventually of one sign. Suppose $\Delta U_n < 0$ for all large n. Then $\Delta^3 U_n > 0$ for all n sufficiently large. It follows that $\Delta^2 U_n$ is eventually sign definite. We cannot have $\Delta^2 U_n > 0$ for all large n, because $\Delta^j U_n \Delta^{j+1} U_n > 0$ for all n sufficiently large implies $\operatorname{sgn} \Delta^{j-1} U_n = \operatorname{sgn} \Delta^j U_n$ eventually. Similarly, $\Delta^2 U_n < 0$ together with $\Delta U_n < 0$ for all large n is impossible since $U_n > 0$, $n \ge N$. So we must have that an integer $M > N$ exists, where $\Delta U_n > 0$, $n \ge M$. Summing both sides of

$$\Delta F_n \le (2Q_n - \Delta P_n) U_{n+2}^2,$$

from M to $m-1$, we find

$$0 < F_m \leq F_M + U_{M+2}^2 \sum_{M}^{m-1} (2Q_n - \Delta P_n) \to -\infty$$

as $m \to \infty$, which is a contradiction. We conclude every solution U of (E), satisfying $F[U_n] > 0$, $n \geq 1$ is oscillatory. □

Example 2.2. The sequence defined by $U_n = 2^n$ is a solution of

$$\Delta^3 U_n + \frac{1}{2^{n+1}} \Delta U_{n+2} - \left[\frac{1}{2}\left(\frac{1}{2^n}\right) + \frac{1}{4}\right] U_{n+2} = 0.$$

This example illustrates our basic equation can have both oscillatory and nonoscillatory solutions. Example 2.2 further serves to illustrate the following general principle.

Theorem 2.3. *Suppose* $P_{n+1} + Q_n \leq 0$, $n \geq 1$. *Then* (E) *has a solution* U *satisfying*

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n,$$

$n \geq 1$.

Proof. Consider the solution U of (E) satisfying the relations (3). Clearly,

$$U_1 > 0, \qquad \Delta U_1 > 0, \qquad \Delta^2 U_1 > 0.$$

Suppose $U_k > 0$, $\Delta U_k > 0$, $\Delta^2 U_k > 0$ for some positive integer $k > 1$. From the identities

$$U_{k+1} = \Delta U_k + U_k,$$
$$\Delta U_{k+1} = \Delta^2 U_k + \Delta U_k,$$

we see $U_{k+1} > 0$ and $\Delta U_{k+1} > 0$. Now,

$$\Delta^2 U_{k+1} = \Delta^3 U_k + \Delta^2 U_k. \tag{4}$$

Making the substitution of (E) in (4), we find

$$(1 + P_{k+1}) \Delta^2 U_{k+1} = -(P_{k+1} + Q_k) \Delta U_{k+1} - Q_k U_{k+1} + \Delta^2 U_k.$$

It follows that $\Delta^2 U_{k+1} > 0$, and the theorem follows by mathematical induction. □

A consequence of Theorem 2.1 and Theorem 2.3 is the following result.

Theorem 2.4. *If $P_{n+1} + Q_n \le 0$, $n \ge 1$, equation* (E) *has both oscillatory and nonoscillatory solutions.*

We turn to our final results.

Theorem 2.5. *Let U be a solution of* (E) *satisfying $F_n > 0$, $n \ge 1$. Then the following are true*:

(i) $\sum^{\infty}(\Delta P_n - 2Q_n)U_{n+2}^2 < \infty$,

(ii) $\sum^{\infty}(\Delta^2 U_n)^2 < \infty$,

(iii) $\sum^{\infty} P_n(\Delta U_{n+1})^2 < \infty$.

Proof. Because U satisfies $F_n > 0$, $n \ge 1$, as a result of differencing F_n and summing from 1 to $m-1$, we have

$$0 < F_m = F_1 + \sum_{n=1}^{m-1}(2Q_n - \Delta P_n)U_{n+2}^2 - \sum_{n=1}^{m-1}(\Delta^2 U_n)^2 - \sum_{n=1}^{m-1} P_{n+1}(\Delta U_{n+2})^2.$$

Thus,

$$\sum_{n=1}^{m-1}(\Delta P_n - 2Q_n)U_{n+2}^2 + \sum_{n=1}^{m-1}(\Delta^2 U_n)^2 + \sum_{n=1}^{m-1} P_{n+1}(\Delta U_{n+2})^2 < F_1.$$

Letting m tend to infinity establishes each of (i), (ii) and (iii), since F_1 is independent of m. □

Theorem 2.6. *Let U be a nontrivial solution of* (E). *Suppose P_n is bounded, and $\liminf_{n\to\infty}(\Delta P_n - 2Q_n) > 0$. The following are equivalent*:

(iv) $F_n > 0$, $n \geq 1$,

(v) $\sum^\infty U_n^2 < \infty$,

(vi) $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \Delta U_n = \lim_{n\to\infty} \Delta^2 U_n = 0$,

(vii) $\lim_{n\to\infty} F_n = 0$.

Proof. That (iv) implies (v) follows from (i) Theorem 2.5.

The relations (vi) follow trivially from (v).

That (vi) implies (vii) follows from the boundedness of P_n.

Let U be a nontrivial solution of (E) such that $F[U_n] \to 0$ as $n \to \infty$. Since $F_n \to 0$, and F_n is decreasing, we must have $F_n > 0$ for each n. Hence, (vii) implies (iv) and the proof is complete. □

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Department of Mathematics, Texas Southern University, 3100 Cleburne Ave., Houston, TX 77004