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WALLMAN COMPACTIFICATION IN FTS

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1. Introduction. In this paper we construct a good extension of Wallman's compactification for all fuzzy topological spaces, and we prove that besides being a good extension of Wallman's original construction in TOP, our construction also extends the earlier construction in FNS which was given in [**1**]. FNS stands for the bireflective and coreflective subcategory of FTS consisting of all fuzzy neighborhood spaces.

Remarkably, in order to perform our construction we need to apply (in FTS) Shanin's generalization of Wallman's construction as given in [**10**]. The main advantages of our compactification over previous ones in, e.g., [**2, 3, 8**], are that, in the first place, it does not reduce to only a small class of fuzzy spaces (sometimes only to topologically generated spaces) and, in the second place, that it is given by a very explicit construction describing the closed fuzzy sets and a fortiori the convergence of prefilters in the compactification.

2. Preliminaries. We recall some concepts and notations which are required in this paper. I stands for the unit interval, $I_0 := I\backslash\{0\}$ and $I_1 := I\backslash\{1\}$. We restrict our attention to classical fuzzy sets (i.e., functions with domain some set X and codomain I) and classical fuzzy topological spaces in the sense of, e.g., [**4**]. Given a set X, a fuzzy topology on X is a collection $\Delta \subset I^X$ which is closed under the taking of finite infima, arbitrary suprema and which contains all constants. A map $f : (X, \Delta) \to (X', \Delta')$ between fuzzy topological spaces is said to be continuous if, for all $\mu' \in \Delta'$, we have $\mu' \circ f \in \Delta$. Fuzzy topological spaces and continuous maps form a topological category denoted FTS. This means, among other things, that initial and final structures exist in FTS. TOP is embedded in FTS as a full isomorphism closed subcategory which is at the same time bireflective and coreflective, i.e., the embedding has both a left and right adjoint. Given a fuzzy topological space (X, Δ) its coreflection in TOP is determined by the topology $\iota(\Delta) := \langle\{\mu^{-1}(]\alpha, 1]) \mid \mu \in \Delta, \alpha \in I\}\rangle$. If Γ is a collection of

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fuzzy sets, we denote Γ^c the collection of all pseudo-complements, i.e., $\mu \in \Gamma^c$ if and only if $1-\mu \in \Gamma$. If $\mathfrak{F}$ is a prefilter, then we recall that its character is the real number $c(\mathfrak{F}) := \inf_{\mu\in\mathfrak{F}} \sup_{x\in X} \mu(x)$. If $\mu \in I^X$, then we denote by $[\mu]$ the prefilter generated by $\{\mu\}$. If $\mathcal{F}$ is a filter on X, then $\omega(\mathcal{F})$ stands for the prefilter generated by $\{1_F | F \in \mathcal{F}\}$, i.e., $\mu \in \omega(\mathcal{F})$ if and only if there exists an $F \in \mathcal{F} : 1_F \leq \mu$.

Given a prefilter $\mathfrak{F}$ in a fuzzy topological space (X, Δ) the adherence of $\mathfrak{F}$ is defined as

$$\begin{aligned} \operatorname{adh} \mathfrak{F} : & X \to I \\ & x \to \inf_{\mu\in\mathfrak{F}} \bar{\mu}(x) \end{aligned}$$

where $^-$ stands for closure in (X, Δ). The numerical value $\operatorname{adh} \mathfrak{F}(x)$ is to be interpreted as the degree with which the point x is an adherence point of $\mathfrak{F}$. In the same way, one can define a limit of $\mathfrak{F}$, but we shall not have recourse to this in the present paper (see, e.g., [**5**]).

A subset A of a fuzzy space (X, Δ) is called dense if it is dense in the TOP-coreflection, i.e., if it fulfills the property that for all $\mu \in \Delta$, $\sup_{x\in A} \mu(x) = \sup_{x\in X} \mu(x)$. By a closed saturated 1-level prefilter, we mean a prefilter $\mathfrak{F}$ in Δ^c such that $c(\mathfrak{F}) = 1$ and which fulfills the property that if for some $\mu \in \Delta^c$ for all $\varepsilon \in I_0$, we have $(\mu+\varepsilon)\wedge 1 \in \mathfrak{F}$, then necessarily $\mu \in \mathfrak{F}$. For more details on convergence, we refer to [**4, 5**], and for the Wallman compactification in FNS, we refer to [**1**].

An extension (Y, Γ) of (X, Δ) (i.e., a space in which (X, Δ) is embedded) is called a compactification of X if X is densely embedded in Y and Y is compact.

For basic results on the Wallman-compactification in TOP and especially Shanin's generalization thereof, we refer to [**9, 10**].

3. The Wallman compactification in FTS. Let (X, Δ) be an arbitrary fuzzy space. We let

$$\mathcal{C}(\Delta) := \left\{ \cup_{i=1}^n \mu^{-1}[\alpha_i, 1] \mid n \in \mathbf{N}, \mu_i \in \Delta^c, \alpha_i \in I_1 \right\}.$$

It is clear that $\mathcal{C}(\Delta)$ is a basis for the closed fuzzy sets of $\iota(\Delta)$ and by definition it contains $\varnothing$ and X and is closed under the operation of taking finite unions. A subset $\mathcal{H}$ of $\mathcal{C}(\Delta)$ is called a $\mathcal{C}(\Delta)$-family if it has the F.I.P., and it is called a maximal $\mathcal{C}(\Delta)$-family if it is not properly contained in any other $\mathcal{C}(\Delta)$-family. In [**9**], it was shown that a maximal $\mathcal{C}(\Delta)$-family $\mathcal{M}$ fulfills the prime-property, i.e., if $A, B \in \mathcal{C}(\Delta)$ and $A \cup B \in \mathcal{M}$, then either $A \in \mathcal{M}$ or $B \in \mathcal{M}$. We shall use this fact freely in the sequel. A $\mathcal{C}(\Delta)$-family $\mathcal{H}$ is called vanishing if it has an empty intersection.

Given (X, Δ) and $\mathcal{C}(\Delta)$ as above, we shall put

$$V(X) := \{\text{vanishing maximal } \mathcal{C}(\Delta)\text{-families}\},$$

and

$$\hat{X} := X \cup V(X).$$

Notice that we do not suppose any separation-properties to be fulfilled in (X, Δ). We now extend the structure of X to $\hat{X}$ in the following way. Given $\mu \in \Delta^c$, we define

$$\hat{\mu} : \hat{X} \to I \begin{cases} x \to \mu(x) & x \in X \\ \mathcal{M} \to c(\mu, \mathcal{M}) & \mathcal{M} \in V(X) \end{cases}$$

where

$$c(\mu, \mathcal{M}) := \begin{cases} c([\mu] \vee \omega(\mathcal{M})) & \text{if } [\mu] \vee \omega(M) \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *For $\mu, \nu \in \Delta^c$ and α constant, the following properties hold*:

(1) $\hat{\alpha} = \alpha$

(2) $\mu \hat{\vee} \nu = \hat{\mu} \vee \hat{\nu}$

(3) $\mu \hat{\wedge} \nu = \hat{\mu} \wedge \hat{\nu}$.

Proof. The verification of (1) is of course trivial. For (2) and (3) first remark that if $\xi, \theta \in \Delta^c$ are such that $\theta \leq \xi$, then $\hat{\theta} \leq \hat{\xi}$ so we already have $\hat{\mu} \vee \hat{\nu} \leq \mu \hat{\vee} \nu$ and $\mu \hat{\wedge} \nu \leq \hat{\mu} \wedge \hat{\nu}$. Now let $\mathcal{M} \in V(X)$. If $\alpha \in I_1$ is such that $(\mu \vee \nu)^{-1}([\alpha, 1]) \in \mathcal{M}$, then either $\mu^{-1}([\alpha, 1]) \in \mathcal{M}$ or $\nu^{-1}([\alpha, 1]) \in \mathcal{M}$ and thus, under the assumption that $[\mu \vee \nu] \vee \omega(\mathcal{M})$

exists, we have

$$\begin{aligned}\mu\hat{\vee}\nu(\mathcal{M}) &= \sup\{\alpha \in I_1 \mid (\mu \vee \nu)^{-1}([\alpha, 1]) \in \mathcal{M}\} \\ &\le \sup\{\alpha \in I_1 \mid \mu^{-1}([\alpha, 1]) \in \mathcal{M}\} \vee \sup\{\alpha \in I_1 \mid \nu^{-1}([\alpha, 1]) \in \mathcal{M}\} \\ &= \hat{\mu}(\mathcal{M}) \vee \hat{\nu}(\mathcal{M}).\end{aligned}$$

In case $[\mu \vee \nu] \vee \omega(\mathcal{M})$ does not exist, we have

$$\mu\hat{\vee}\nu(\mathcal{M}) = \hat{\mu}(\mathcal{M}) = \hat{\nu}(\mathcal{M}) = 0.$$

Analogously, if $\alpha \in I_1$ and $\beta \in I_1$ are such that $\mu^{-1}([\alpha, 1]) \in \mathcal{M}$ and $\nu^{-1}([\beta, 1]) \in \mathcal{M}$, then by the maximality of $\mathcal{M}$ and the fact that

$$\mu^{-1}([\alpha, 1]) \cap \nu^{-1}([\beta, 1]) \subset (\mu \wedge \nu)^{-1}([\alpha \wedge \beta, 1])$$

we have $(\mu \wedge \nu)^{-1}([\alpha \wedge \beta, 1]) \in \mathcal{M}$ and thus, under the assumption that $[\mu] \vee \omega(\mathcal{M})$ and $[\nu] \vee \omega(\mathcal{M})$ exist, we have

$$\begin{aligned}\hat{\mu}(\mathcal{M}) \wedge \hat{\nu}(\mathcal{M}) &= \sup\{\alpha \wedge \beta \mid \mu^{-1}([\alpha, 1]) \in \mathcal{M} \text{ and } \nu^{-1}([\beta, 1]) \in \mathcal{M}\} \\ &\le \sup\{\gamma \in I_1 \mid (\mu \wedge \nu)^{-1}([\gamma, 1]) \in \mathcal{M}\} \\ &= \mu\hat{\wedge}\nu(\mathcal{M}).\end{aligned}$$

In case one of $[\mu] \vee \omega(\mathcal{M})$ and $[\nu] \vee \omega(\mathcal{M})$ does not exist, we have

$$\hat{\mu}(\mathcal{M}) \wedge \hat{\nu}(\mathcal{M}) = \mu\hat{\wedge}\nu(\mathcal{M}) = 0. \qquad \square$$

As a consequence of this proposition, the family $\{\hat{\mu} \mid \mu \in \Delta^c\}$ is a basis for the closed fuzzy sets of some fuzzy topology on $\hat{X}$, which we shall denote by $\hat{\Delta}$.

Theorem 3.2. *$(\hat{X}, \hat{\Delta})$ is a compactification of (X, Δ), in particular $(\hat{X}, \hat{\Delta})$ is ultracompact.*

Proof. That X is dense in $\hat{X}$ is of course an immediate consequence of Proposition 3.1 (1).

To see that $\hat{X}$ is ultracompact, we observe that

$$\mathcal{D} := \{\mu^{-1}([\alpha, 1]) \mid \mu \in \Delta^c, \alpha \in I_0\}$$

is a subbasis for the closed sets in $\iota(\hat{X})$.

Let $(F_j := \hat{\mu}_j^{-1}([\alpha_j, 1])_{j\in J}$ be a subfamily of $\mathcal{D}$ with $\cap_{j\in J} F_j = \varnothing$. The following assertion will be used at several places in the proof.

Assertion. If $\mathcal{M}$ is a maximal $\mathcal{C}(\Delta)$-family, $\mu \in \Delta^c$ and $\alpha \in I$, then $c([\mu] \vee \omega(\mathcal{M})) \geq \alpha$ if and only if for all $\beta < \alpha : \mu^{-1}([\beta, 1]) \in \mathcal{M}$.

Indeed, if $\alpha = 0$ or if $[\mu] \vee \omega(\mathcal{M})$ does not exist, there is nothing to prove. Otherwise, we have

$$\begin{aligned} c([\mu] \vee \omega(\mathcal{M})) \geq \alpha &\iff \forall \beta < \alpha, \forall M \in \mathcal{M} : 1_M \wedge \mu \not\leq \beta \\ &\iff \beta < \alpha, \forall M \in \mathcal{M} : \mu^{-1}([\beta, 1] \cap M) \neq \varnothing \end{aligned}$$

which by maximality of $\mathcal{M}$ proves our claim.

Consequently, it follows that for each $j \in J$ we have

$$F_j = P_j \cup Q_j$$

where

$$\begin{aligned} P_j &:= \mu_j^{-1}([\alpha_j, 1]) \\ Q_j &:= \{\mathcal{M} \mid \forall \beta < \alpha_j : \mu_j^{-1}([\beta_j, 1]) \in \mathcal{M}\} \end{aligned}$$

and where then $P_j \subset X$, $Q_j \subset V(X)$.

Therefore, for all $K \subset J$, we have

$$\bigcap_{j\in K} F_j = \Big(\bigcap_{j\in K} P_j\Big) \bigcup \Big(\bigcap_{j\in K} Q_j\Big)$$

and, in particular,

$$(*) \qquad \bigcap_{j\in J} P_j = \bigcap_{j\in J} Q_j = \varnothing.$$

Now we consider three cases:

$$(1) \qquad \exists K \in 2^{(J)} : \bigcap_{j\in K} P_j = \varnothing \quad \text{and} \quad \exists L \in 2^{(J)} : \bigcap_{j\in L} Q_j = \varnothing.$$

Then $K \cup L \in 2^{(J)}$ and $\cap_{j\in K\cup L} F_j = \varnothing$.

$$(2) \qquad \forall K \in 2^{(J)} : \bigcap_{j\in K} P_j \neq \varnothing.$$

This means that $(P_j)_{j\in J}$ is a $\mathcal{C}(\Delta)$-family which thus is contained in a maximal $\mathcal{C}(\Delta)$-family $\mathcal{M}$ and which by $(*)$ is vanishing. For each $j \in J$, we now have

$$\mu_j^{-1}([\alpha_j, 1]) = P_j \in \mathcal{M}$$

and, thus, again by the assertion $\hat{\mu}_j(\mathcal{M}) \geq \alpha_j$. This implies that $\mathcal{M} \in \cap_{j\in J} Q_j$ which by $(*)$ is a contradiction.

$$\forall L \in 2^{(J)} : \bigcap_{j\in L} Q_j \neq \varnothing. \tag{3}$$

If then $L \in 2^{(J)}$ and $\mathcal{M} \in \cap_{j\in L} Q_j$, and if, for each $j \in J$ we take $\beta_j < \alpha_j$, it follows again by the assertion that $\mu_j^{-1}([\beta_j, 1]) \in \mathcal{M}$ for all $j \in L$. Since this holds for all $L \in 2^{(J)}$ and all choices of $\beta_j < \alpha_j$, it follows that the whole family

$$\{\mu_j^{-1}([\beta, 1]) \mid j \in J, \beta < \alpha_j\}$$

is a $\mathcal{C}(\Delta)$-family and, therefore, is contained in some maximal $\mathcal{C}(\Delta)$-family $\mathcal{M}$. Now if $\mathcal{M}$ is nonvanishing, there exists $x \in X$ such that

$$\forall j \in J : \forall \beta < \alpha : \mu_j(x) \geq \beta,$$

and, thus, $\cap_{j\in J} P_j \neq \varnothing$, which is a contradiction. If $\mathcal{M}$ is vanishing, then it follows from its very definition that $\mathcal{M} \in Q_j$ for all $j \in J$ which is also a contradiction. This shows that of the three cases which we have considered, only the first one can occur, which applying Alexander's subbase Lemma means $\iota(\hat{X})$ is compact. □

Remarks 3.3. 1) Since Theorem 3.2 proves ultracompactness and not compactness of $\hat{X}$, the proof does not involve prefilters. If we had given a proof of compactness rather than of the stronger ultracompactness, then it would have been easily seen that $V(X)$ consists of two kinds of "new points." Those which are required in order to provide prefilters $\mathfrak{F}$ (on X) for which $\sup_{x\in X} \operatorname{adh} \mathfrak{F}(x) < c(\mathfrak{F})$, with a point $\mathcal{M}$ wherein the extension of $\mathfrak{F}$ to $\hat{X}$, say $\hat{\mathfrak{F}}$, fulfills $\operatorname{adh} \hat{\mathfrak{F}}(\mathcal{M}) = c(\mathfrak{F})$ and those which are required in order to provide prefilters $\mathfrak{F}$ for which $\sup_{x\in X} \operatorname{adh} \mathfrak{F}(x) = c(\mathfrak{F})$ but $\operatorname{adh} \mathfrak{F}(x) < c(\mathfrak{F})$ for all $x \in X$, with a point $\mathcal{M}$ wherein adh $\hat{\mathfrak{F}}(\mathcal{M}) = c(\mathfrak{F})$. Removing from $V(X)$ those $\mathcal{M}$ which are only required for the second type of prefilters leaves us with a smaller extension of X which is still compact but no longer necessarily ultracompact. This smaller extension is then a compactification which fulfills the property of coinciding with X if and only if X is compact, while the larger compactification which we have constructed coincides with X if and only if X is ultracompact.

2) Rather than proving that the construction of Theorem 3.2 coincides with the usual Wallman-compactification in case (X, Δ) is topologically generated, we shall prove that it even coincides with the construction in FNS given in [**1**] in case $(X, \Delta) \in |FNS|$, from which the aforementioned result will follow as a corollary.

4. Relation to the Wallman compactification in FNS. In [**1**] a Wallman compactification was constructed for symmetric [**11**], weakly-T_1 [**12**], fuzzy neighborhood spaces. The symmetry and weak-T_1 properties were imposed there in order to be able to identify the points of X with a special type of maximal closed saturated 1-level prefilter. However, we can generalize the construction of [**1**] to arbitrary spaces in FNS. This will make the link to the construction of the present paper clearer.

If $(X, \Delta) \in |FNS|$, then (as in [**1**]), we denote by $R(X)$ the collection of all maximal closed saturated 1-level prefilters $\mathfrak{F}$ on X such that adh $\mathfrak{F}(x) < 1$ for all $x \in X$. Further, we put $\tilde{X} := X \cup R(X)$. For any $\mathfrak{F} \in R(X)$ and $\mu \in \Delta^c$, we also put

$$I(\mu, \mathfrak{F}) := \{\varepsilon \in I \mid (\mu + \varepsilon) \wedge 1 \in \mathfrak{F}\},$$

and then we define

$$\tilde{\mu} : \tilde{X} \to I \begin{cases} x \to \mu(x) & x \in X \\ \mathfrak{F} \to 1 - \inf I(\mu, \mathfrak{F}) & \mathfrak{F} \in R(X). \end{cases}$$

The following results are now proved in exactly the same way as the corresponding ones in [**1**] and we therefore omit the proofs.

Proposition 4.1. *For $\mu, \nu \in \Delta^c$ and α constant the following properties hold*

1) $\tilde{\alpha} = \alpha$

2) $\mu \tilde{\vee} \nu = \tilde{\mu} \vee \tilde{\nu}$

(3) $\mu \tilde{\wedge} \nu = \tilde{\mu} \wedge \tilde{\nu}$.

Theorem 4.2. *If* $(X, \Delta) \in |FNS|$, *then* $(\tilde{X}, \tilde{\Delta})$ *is a compactification of* (X, Δ) *in* FNS, *in particular,* $(\tilde{X}, \tilde{\Delta})$ *is ultracompact.*

We shall now treat the relationship between $\hat{X}$ and $\tilde{X}$ if X is a fuzzy neighborhood space. Hereto we require some preliminary propositions.

Proposition 4.3. *If* $\mathfrak{F}$ *is a maximal closed saturated* 1*-level prefilter in* (X, Δ), *then*

$$\delta(\mathfrak{F}) := \{C \in \mathcal{C}(\Delta) \mid \exists\, \mu \in \mathfrak{F}, \exists\, \alpha \in I_1 : \mu^{-1}([\alpha, 1]) \subset C\}$$

is a maximal $\mathcal{C}(\Delta)$*-family and the following are equivalent*:

(1) *for all* $x \in X : \operatorname{adh} \mathfrak{F}(x) < 1$, *i.e.,* $\mathfrak{F} \in R(X)$

(2) $\delta(\mathfrak{F})$ *is vanishing, i.e.,* $\delta(\mathfrak{F}) \in V(X)$.

Proof. That $\delta(\mathfrak{F})$ is a $\mathcal{C}(\Delta)$-family is clear. Suppose it is not maximal, and let $\mathcal{M}$ be a maximal $\mathcal{C}(\Delta)$-family which strictly contains $\delta(\mathfrak{F})$. Then there exist $\xi \in \Delta^c$ and $\beta \in I_1$ such that

$$\xi^{-1}([\beta, 1]) \in \mathcal{M} \backslash \delta(\mathfrak{F}).$$

By Theorem 4.4 [**13**], it follows that

$$\theta := \xi \vee 1_{\xi^{-1}([\beta,1])} \in \Delta^c.$$

Now for any $\alpha \in I_1$ it is clear that we have

$$\theta^{-1}([\alpha, 1]) = \begin{cases} \xi^{-1}([\beta, 1]) & \text{if } \beta \leq \alpha \\ \xi^{-1}([\alpha, 1]) & \text{if } \alpha < \beta. \end{cases}$$

If $\mu \in \mathfrak{F}$, then for any $\alpha \in [\beta, 1[$ we have

$$\mu^{-1}([\alpha, 1]) \cap \theta^{-1}([\alpha, 1]) = \mu^{-1}([\alpha, 1]) \cap \xi^{-1}([\beta, 1]) \neq \varnothing$$

and, thus,

$$\sup_{x\in X} \mu \wedge \theta(x) = \sup\{\alpha \mid \mu^{-1}([\alpha, 1]) \cap \theta^{-1}([\alpha, 1]) \neq \varnothing\} = 1.$$

By maximality of $\mathfrak{F}$, this implies that $\theta \in \mathfrak{F}$ and, thus, $\xi^{-1}([\beta, 1]) = \theta^{-1}([\beta, 1]) \in \delta(\mathfrak{F})$, which is a contradiction. Consequently, $\delta(\mathfrak{F})$ is a maximal $\mathcal{C}(\Delta)$-family.

Finally, for any $x \in X$ we have that $\operatorname{adh}\mathfrak{F}(x) < 1$ if and only if $x \notin \cap_{F\in\delta(\mathfrak{F})} F$ which proves the equivalence of (1) and (2). $\square$

For ease in notation, in what follows, given $\mu \in \Delta^c$ and $\alpha \in I_1$, we shall put

$$\lambda(\mu, \alpha) := \mu \vee 1_{\mu^{-1}([\alpha,1])}.$$

Again, by Theorem 4.4 [**13**], we know that $\mu \in \Delta^c$ implies $\lambda(\mu, \alpha) \in \Delta^c$.

Proposition 4.4. *If $\mathcal{M}$ is a maximal $C(\Delta)$-family, then*

$$\mathcal{L}(\mathcal{M}) := \{\lambda(\mu, \alpha) \mid \mu^{-1}([\alpha, 1]) \in \mathcal{M}, \mu \in \Delta^c, \alpha \in I_1\}^{\sim}$$

is a maximal closed saturated 1-level prefilter and the following are equivalent:

(1) *For all $x \in X$: $\operatorname{adh}\mathcal{L}(\mathcal{M})(x) < 1$, i.e., $\mathcal{L}(\mathcal{M}) \in R(X)$*

(2) *$\mathcal{M}$ is vanishing, i.e., $\mathcal{M} \in V(X)$.*

Proof. That $\mathcal{L}(\mathcal{M})$ is a closed saturated 1-level prefilter is easily verified, and we leave this to the reader. Now put

$$T(\mathcal{M}) := \{(\mu, \alpha) \in \Delta^c \times I_1 \mid \mu^{-1}([\alpha, 1]) \in \mathcal{M}\}.$$

Let $\xi \in \Delta^c$ be such that for all $(\mu, \alpha) \in T(\mathcal{M})$, we have

$$\sup_{x\in X} \xi \wedge \lambda(\mu, \alpha)(x) = 1.$$

Now let $\varepsilon \in I_0$, and let $(\mu, \alpha) \in T(\mathcal{M})$. Then it follows that there exists an $x \in X$ such that $\xi \wedge \lambda(\mu, \alpha)(x) > \alpha \vee (1 - \varepsilon)$ and, consequently, $\xi^{-1}([1 - \varepsilon, 1]) \cap \mu^{-1}([\alpha, 1]) \neq \varnothing$. By maximality of

$\mathcal{M}$ this proves that $\xi^{-1}([1-\varepsilon, 1]) \in \mathcal{M}$. Thus we have that, for all $\varepsilon \in I_0 : (\xi, 1-\varepsilon) \in T(\mathcal{M})$. Consequently, since for all $\varepsilon \in I_0$, also

$$\lambda(\xi, 1-\varepsilon) \leq (\xi + \varepsilon) \wedge 1$$

and, since $\mathcal{L}(\mathcal{M})$ is saturated, this implies that $\xi \in \mathcal{L}(\mathcal{M})$, which in turn shows that $\mathcal{L}(\mathcal{M})$ is maximal. That (1) and (2) are equivalent again is clear. □

Proposition 4.5. *The maps* $\delta : R(X) \to V(X)$ *and* $\mathcal{L} : V(X) \to R(X)$ *are bijective inverses of each other.*

Proof. If $\mathcal{M} \in V(X)$, then for any $(\mu, \alpha) \in T(\mathcal{M})$, we have

$$\mu^{-1}([\alpha, 1]) = \lambda(\mu, \alpha)^{-1}([\alpha, 1]) \in \delta(\mathcal{L}(\mathcal{M})),$$

i.e., $\mathcal{M} \subset \delta(\mathcal{L}(\mathcal{M}))$ which by maximality implies $\mathcal{M} = \delta(\mathcal{L}(\mathcal{M}))$. If $\mathfrak{F} \in R(X)$, then for any $\mu \in \mathfrak{F}$ and $\varepsilon \in I_0$, we have $\lambda(\mu, 1-\varepsilon) \in \mathcal{L}(\delta, (\mathfrak{F}))$ and, thus, by saturedness of $\mathfrak{F}$, we have $\mathfrak{F} \subset \mathcal{L}(\delta(\mathfrak{F}))$ and, again, by maximality, $\mathfrak{F} = \mathcal{L}(\delta(\mathcal{F}))$. □

Theorem 4.6. *If* $(X, \Delta) \in |\mathrm{FNS}|$, *then the map*

$$(\tilde{X}, \tilde{\Delta}) \xrightarrow{\delta^*} (\hat{X}, \hat{\Delta})$$

defined by $\delta^*(x) := x$ *for all* $x \in X$ *and* $\delta^*(\mathfrak{F}) := \delta(\mathfrak{F})$ *for all* $\mathfrak{F} \in R(X)$ *is a homeomorphism, in particular,* $\tilde{\mu} = \hat{\mu} \circ \delta^*$ *holds for any* $\mu \in \Delta^c$.

Proof. From Proposition 4.5, we already know that δ^* is a bijection. Let $\mu \in \Delta^c$ and $\mathfrak{F} \in R(X)$. If $\varepsilon \in I_0$ is such that $(\mu + \varepsilon) \wedge 1 \in \mathfrak{F}$, then we have

$$\nu^{-1}([\alpha, 1]) \cap \mu^{-1}([\beta - \varepsilon, 1]) \neq \varnothing$$

for all $\alpha \in I_1$, $\beta \in I_1$ and $\nu \in \mathfrak{F}$. Consequently, $\mu^{-1}([\beta - \varepsilon, 1]) \in \delta(\mathfrak{F})$ for all $\beta \in I_1$ and this implies that

$$1 - \varepsilon \leq \hat{\mu}(\delta(\mathfrak{F})).$$

Thus, $\tilde{\mu} \leq \hat{\mu} \circ \delta^*$.

Conversely, if $\alpha \in I_1$ is such that $\mu^{-1}([\alpha, 1]) \in \delta(\mathfrak{F})$, then

$$\mu^{-1}([\alpha, 1]) \cap \xi^{-1}([\beta, 1]) \neq \varnothing$$

for all $\beta \in I_0$ and $\xi \in \mathfrak{F}$, i.e.,

$$\sup_{x \in X} (\mu + 1 - \alpha) \wedge \xi(x) = 1$$

for all $\xi \in \mathfrak{F}$. Consequently, $(\mu + 1 - \alpha) \wedge 1 \in \mathfrak{F}$, i.e.,

$$\alpha \leq \tilde{\mu}(\mathfrak{F}).$$

Thus $\hat{\mu} \circ \delta^* \leq \tilde{\mu}$, and we are done. □

Again, the next result is proved in exactly the same way as in [**1**], and so we omit the proof. For a topological space $(X, \mathcal{T})$, we put $(W(X), W(\mathcal{T}))$ for the Wallman compactification of $(X, \mathcal{T})$, i.e., $W(X)$ is the set of all maximal vanishing closed families together with the points of X and $W(\mathcal{T})$ is the topology with subbasis for the closed sets given by the family $\{F^* \mid F \text{ closed in } X\}$, where $F^* := F \cup \{\mathcal{F} \mid \text{maximal vanishing closed family such that } F \in \mathcal{F}\}$, (see, e.g., [**9**]).

Theorem 4.7. *If* $(X, \mathcal{T}) \in |\mathrm{TOP}|$*, then the compactification* $(\tilde{X}, \widetilde{\omega(\mathcal{T})})$ *and (the embedding in* FTS *of) the Wallman compactification* $(W(X), \omega(W(\mathcal{T})))$ *coincide.*

From this result and Theorem 4.6, we immediately deduce our next result.

Corollary 4.8. *If* $(X, \mathcal{T}) \in |\mathrm{TOP}|$*, then the compactification* $(\hat{X}, \widehat{\omega(\mathcal{T})})$ *and (the embedding in* FTS *of) the Wallman compactification* $(W(X), \omega(W(\mathcal{T})))$ *coincide.*

Remark 4.9. The smaller extension to which we alluded in Remark 3.3.1, also coincides with (the embedding of) the Wallman compactification $(W(X)), \omega(W(\mathcal{T})))$ in case X is topologically generated, since in this case compactness and ultracompactness are equivalent.

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