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A GRENANDER AND SZEGÖ LIMIT THEOREM FOR TOEPLITZ OPERATORS ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. This paper is a generalization of earlier results of Szegö and Grenander concerning asymptotic distributions of eigenvalues of Toeplitz operators on a Hilbert space of square integrable functions defined on real numbers to locally compact abelian groups.

Introduction. Let G be a nondiscrete locally compact abelian group, and let Γ be its noncompact dual group. Let m and μ be Haar measures on G and Γ, respectively, normalized so that the Fourier inversion theorem holds. Let $(\mathcal{D}, \leq)$ be a directed set such that $\{\pi(\varepsilon)\}_{\varepsilon\in\mathcal{D}}$ is a net whose values are those Borel sets of Γ with compact closures. Let $D(\varepsilon, x) = \int_\Gamma \chi\pi_{(\varepsilon)}(\gamma)(x,\gamma)\,d\mu(\gamma)$, a Fourier inversion transform of a characteristic function $\chi\pi_{(\varepsilon)}$ on Γ, where (x,γ) is a character on Γ for $f \in L^1(G)$. Let

$$K^{(\varepsilon)}(x,y) = \int_G D(\varepsilon, x-z) f(z) D(\varepsilon, z-y)\,dm(z).$$

We then obtain the following integral operator $U_f^{(\varepsilon)}$ on $L^2(G)$, known as a Toeplitz operator.

$$(1)\qquad U_f^{(\varepsilon)}(\varphi)(x) = \int_G K^{(\varepsilon)}(x,y)\varphi(y)\,dm(y) \qquad \text{for all } \varphi \in L^2(G).$$

We shall establish the following main result of the paper.

Theorem 1. *Let f be real-valued in $L^1(G)$ and let $U_f^{(\varepsilon)}$ be its corresponding self-adjoint Toeplitz operator as defined in (1). Let $Q(\varepsilon,\gamma) = |D(\varepsilon,\cdot)|^2 = \int_G |D(\varepsilon,x)|^2(-x,\gamma)\,dm(x)$, the Fourier transform of $|D(\varepsilon,\cdot)|^2$, and let $N[(a,b); U_f^{(\varepsilon)}]$ be the set of eigenvalues of $U_f^{(\varepsilon)}$ within a closed interval $[a,b]$. Then*

$$(1.1)\qquad \lim_{\varepsilon\to\infty} Q(\varepsilon,\gamma)/\mu(\pi(\varepsilon)) = 1$$

is a necessary and sufficient condition for

$$(1.2)\qquad \lim_{\varepsilon\to\infty} N[[a,b];U_f^{(\varepsilon)}]/\mu(\pi(\varepsilon)) = m\{x|a \le f(x) \le b\},$$

where the limit (1.1) *is in the sense of convergence in measure on each compact set of* Γ, *and where the assumption* $m\{x|f(x) = a \text{ or } b\} = 0$ *and* $0 \notin [a,b]$ *are provided.*

The above result is a generalization of earlier consideration of asymptotic eigenvalue distribution of such Toeplitz operators defined on the real line by Szegö and Grenander [**1**]. Later, H. Kreiger [**5**] in 1965 obtained its generalization to a class of locally compact abelian groups whose dual is compactly generated but noncompact and offered a sufficient condition for the theorem. The author in this paper further finds that the condition (1.1) is indeed a necessary and sufficient condition. The ideas and methods introduced to prove the theorem are closely related to previous works of several authors. See [**3,4,6**] in this connection.

We first establish a proof of a sufficient condition by considering the case when the generating function f of $U_f^{(\varepsilon)}$ is a characteristic function, then by a simple argument subsequently carried to a class of real simple functions and finally for an arbitrary real-valued function in $L^1(G)$.

For the proof of Theorem 1 we shall introduce some extended results from integral operators listed as corollaries to a well-known Mercer's theorem [**7**].

Corollary 1. *Let X be a locally compact Hausdorff space. Let m be a regular measure on X. Let K be the Hilbert Schmidt operator on $L^2(X)$ with a kernel $K(x,y)$ satisfying the following properties*:

(i) $K(x,y) = K(y,x)$,

(ii) $K(x,y)$ *is positive semi-definite and continuous,*

(iii) $\lim_{x\to x'}\{\int_X |K(x,y) - K(x',y)|^2\, dm(y)\}^{1/2} = 0$ for all $x' \in X$,

(iv) $\int_X K(x,x)\, dm(x) < \infty$.

Let $\{\lambda_k\}_{k=1}^{\infty}$ be the nonzero eigenvalues of K repeated according to their multiplicities where for reasons of future convenience we assume *that* $\lambda_1 \geq \lambda_2 \geq \cdots$ *. We assert that*

$$\sum_{i=1}^{\infty} \lambda_i \leq \int_X K(x,x)\,dm(x) \tag{1}$$

Proof. It is clear from the property (ii) that all the eigenvalues of K are nonnegative. Let $\varphi_i(x)$, $i = 1, 2, \ldots$ be a complete orthonormal set of eigenfunctions corresponding to eigenvalues λ_i, $i = 1, 2, \ldots$. We assert first that each $\varphi_i(x)$ is a continuous function. In fact, using the property (iii) and the Schwartz inequality, we obtain

$$\begin{aligned}|\varphi_1(x) - \varphi_1(x')| &= \left|\frac{1}{\lambda_1}\int_X [K(x,y) - K(x',y)]\varphi_i(y)\,dm(y)\right| \\ &\leq \frac{1}{\lambda_1}||K(x,\cdot) - K(x',\cdot)||_2||\varphi_i||_2.\end{aligned}$$

Our assertion follows immediately. Consequently, the *remainders*

$$K_n(x,y) = K(x,y) - \sum_{i=1}^{\infty} \lambda_i\varphi_i(x)\varphi_i(y), \qquad n = 1, 2, \ldots ,$$

are also continuous functions. Since we have

$$K_n(x,y) = \sum_{i=n+1}^{\infty} \lambda_i\varphi_i(x)\varphi_i(y)$$

in the sense of mean convergence, it follows that

$$\int_X\int_X K_n(x,y)f(y)f(x)\,dm(x)\,dm(y) = \sum_{i=n+1}^{\infty} \lambda_i|(\varphi_i, f)|^2 \geq 0 \tag{2}$$

for every element f of $L^2(X)$, where (,) denotes the inner product of $L^2(X)$. From this we deduce that $K_n(x,x) \geq 0$. In fact, if we had $K_n(x_0,x_0) < 0$, we would have by continuity $K_n(x,y) < 0$ in a neighborhood $V \times V$ of (x_0,x_0), where V is a neighborhood of x_0. Setting $f(x) = 1$ for $x \in V$ and $f(x) = 0$ elsewhere, integral (2) would become negative. Thus a contradiction would occur. Hence, we have

$$K_n(x,x) = K(x,x) - \sum_{i=1}^{n} \cdot\lambda_i\varphi_i(x)\overline{\varphi_i(x)} \geq 0$$

for $n = 1, 2, \ldots$. From this we conclude that the series of positive terms $\sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\overline{\varphi_i(x)}$ is convergent and that its sum is $\leq K(x,x)$. Integrating the series we obtain

$$\sum_{i=1}^{\infty} \lambda_i \leq \int_X K(x,x)\, dm(x). \qquad \square$$

Corollary 2. *For each* $n = 1, 2, \ldots$, *let* K_n *be a Hilbert Schmidt operator with kernel* $K_n(x,y)$ *satisfying all the properties* (i)–(iv) *in Corollary* 1. *Let* $K(x,y)$ *be measurable on* $X \times X$ *and such that*

(i) $\lim_{n\to\infty} \int_X \int_X |K(x,y) - K_n(x,y)|^2\, dm(x)\, dm(y) = 0$

(ii) $\lim_{n\to\infty} \int_X K_n(x,x)\, dm(x) = \int_X K(x,x)\, dm(x) < \infty$.

Then K, *the operator with kernel* $K(x,y)$ *is compact and positive semi-definite and if* $\{\lambda_i\}_{i=1}^{\infty}$ *represents its positive eigenvalues repeated according to their multiplicities and arranged so that* $\lambda_1 \geq \lambda_2 \geq \cdots$. *Then*

$$\sum_{i=1}^{\infty} \lambda_1 \leq \int_X K(x,x)\, dm(x). \tag{3}$$

Proof. By (i) of (2), K_n converges to K as $n \to \infty$ in the Hilbert Schmidt topology and a fortiori in the uniform operator topology. Thus, if $\{\lambda_n, i\}_{i=1}^{\infty}$ represents for each $n = 1, 2$, its positive eigenvalues repeated according to their multiplicities and arranged in decreasing order, we have

$$\lim_{n\to\infty} \lambda_{n,i} = \lambda_i, \qquad i = 1, 2, \ldots . \tag{4}$$

By Corollary 1,

$$\sum_{i=1}^{\infty} \lambda_{n,i} \leq \int_X K_n(x,x)\, dm(x), \qquad n = 1, 2, \ldots . \tag{5}$$

By (ii) of (2) and Fatou's lemma applied to (4), we obtain (3). $\square$

Corollary 3. *If $\omega(x)$ is a bounded measurable function on X and if $K(x,y)$ is the kernel of a Hilbert Schmidt operator satisfying all properties* (i)–(iv) *in Corollary* 1, *then* (1) *holds for the operator with the kernel $\omega(x)K(x,y)\omega(y)$.*

Proof. Choose a sequence $\{\omega_n(x)\}_{n=1}^{\infty}$ of $C(X)$ which is uniformly bounded and converging to $\omega(x)$ a.e. For each n fixed, $\omega_n(x)K(x,y)\omega_n(y)$ satisfies properties (i)–(iv) of Corollary 1. To check (i), (ii) and (iv) is trivial. It suffices to show (iii). For $x, x' \in X$, we have

$$\begin{aligned}
&||\omega_n(x')K(x',\cdot)\omega_n(\cdot) - \omega_n(x)K(x,\cdot)\omega_n(\cdot)||_2 \\
&\quad\le ||\omega_n(x')K(x',\cdot)\omega_n(\cdot) - \omega_n(x')K(x,\cdot)\omega_n(\cdot)||_2 \\
&\qquad + ||\omega_n(x')K(x,\cdot)\omega_n(\cdot) - \omega_n(x)K(x,\cdot)\omega_n(\cdot)||_2 \\
&\quad\le ||\omega_n||_\infty^2 ||K(x',\cdot) - K(x,\cdot)||_2 + |\omega_n(x') - \omega_n(x)|\,||K(x,\cdot)\omega_n(\cdot)||_2.
\end{aligned}$$

Since $\omega_n(x)$ is continuous and $K(x,y)$ satisfies all the properties (i)–(iv) in Corollary 1, it follows from the inequality above that $\omega_n(x)K(x,y)\omega_n(y)$ satisfies (iii) of Corollary 1 for each n. Furthermore, $\{\omega_n(x)K(x,y)\omega_n(y)\}_{n=1}^{\infty}$ satisfies properties (i) and (ii) of Corollary 2. In fact, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_X\int_X |\omega(x)K(x,y)\omega(y) - \omega_n(x)K(x,y)\omega_n(y)|^2\,dm(x)\,dm(y) \\
&\quad\le \lim_{n\to\infty}\left\{\int_X\int_X |(\omega(x)-\omega_n(x))K(x,y)\omega(y)|\,dm(x)\,dm(y)\right. \\
&\qquad \left.+ \int_X\int_X |\omega_n(x)K(x,y)(\omega(y)-\omega_n(y))|^2\,dm(x)\,dm(y)\right\} \\
&\quad\le \lim_{n\to\infty}\int_X |\omega(x)-\omega_n(x)|^2\left(\int_X |K(x,y)|^2\,dm(y)\right)||\omega||_\infty^2\,dm(x) \\
&\qquad + \lim_{n\to\infty}\int_X |\omega(y)-\omega_n(y)|^2\left(\int_X |K(x,y)|^2\,dm(x)\right)||\omega_n||_\infty^2\,dm(y) \\
&\quad= \int_X \lim_{n\to\infty}|\omega(x)-\omega_n(x)|^2\left(\int_X |K(x,y)|^2\,dm(y)\right)||\omega||_\infty^2\,dm(y) \\
&\qquad + \int_X \lim_{n\to\infty}|\omega(y)-\omega_n(y)|^2\left(\int_X |K(x,y)|^2\,dm(x)\right)||\omega_n||_\infty^2\,dm(y) \\
&\quad= 0.
\end{aligned}$$

This proves (i) of Corollary 2. To prove (ii) of Corollary 2 is trivial and the results follow immediately. □

In the following we start to prove Theorem 1 for its sufficient condition (1.1) when the generating function of the operators is of characteristic type.

Let χ_Ω be the characteristic function of a Borel subset Ω of G such that $0 < m(\Omega) < \infty$. Let $U_\Omega^{(\varepsilon)}$ be the integral operator on $L^1(G)$ corresponding to χ_Ω. That is,

$$U_\Omega^{(\varepsilon)}(\varphi(x)) = \int_G K^{(\varepsilon)}(x,y)\varphi(y)\,dm(y), \qquad \varphi \in L^2(G),$$

where $K^{(\varepsilon)}(x,y) = \int_G D(\varepsilon, x-z)\chi_\Omega(z)D(\varepsilon, z-y)\,dm(z)$.

Lemma 1. *For $U_\Omega^{(\varepsilon)}$ and $K^{(\varepsilon)}(x,y)$ defined as above, we have*

(i) *$K^{(\varepsilon)}(x,y)$ is a positive-definite continuous function on $G \times G$.*

(ii) $\lim_{x\to x'} ||K^{(\varepsilon)}(x,\cdot) - K^{(\varepsilon)}(x',\cdot)||_2 = 0,$

(iii) *$U_\Omega^{(\varepsilon)}$ is of Hilbert Schmidt type.*

Proof. Clearly, $K^{(\varepsilon)}(x,y)$ is continuous. Furthermore, if $\varphi \in L^2(G)$, then

$$\begin{aligned}
&\int_G\int_G K^{(\varepsilon)}(x,y)\varphi(x)\varphi(y)\,dm(x)\,dm(y)\\
&\quad= \int_G \varphi(x)\int_G D(\varepsilon, x-z)\chi_\Omega(z)D(\varepsilon, z-y)\,dm(z)\varphi(y)\,dm(y)\,dm(x)\\
&\quad= \int_G\int_G D(\varepsilon, x-z)\varphi(x)\,dm(x)\int_G D(\varepsilon, z-y)\varphi(y)\,dm(y)\chi_\Omega(z)\,dm(z)\\
&\quad= \int_G \left|\int_G D(\varepsilon, z-u)\varphi(u)\,dm(u)\right|^2 \chi_\Omega(z)\,dm(z) \geq 0.
\end{aligned}$$

These steps are easily justified by Fubini's theorem. This shows that $U_\Omega^{(\varepsilon)}$ is positive-definite. Noting that $D(\varepsilon, x)$ is the Fourier inverse transform of $\chi_{\pi(\varepsilon)}$, we have

$$\begin{aligned}|K^{(\varepsilon)}(x,y) - K^{(\varepsilon)}(x',y)| \\ &= \bigg| \int_G D(\varepsilon, x-z)\chi_\Omega(z) D(\varepsilon, z-y)\,dm(z) \\ &\quad - \int_G D(\varepsilon, x'-z)\chi_\Omega(z) D(\varepsilon, z-y)\,dm \bigg| \\ &\le \int_G |D(\varepsilon, x-z) - D(\varepsilon, x'-z)|\chi_\Omega(z)|D(\varepsilon, z-y)|\,dm(z) \\ &= ||D_x(\varepsilon,\cdot) - D_{x'}(\varepsilon,\cdot)||_\infty (\chi_\Omega * |D(\varepsilon,\cdot)|(y)),\end{aligned}$$

where $|| \quad ||_\infty$ denotes the uniform norm on $C_0(G)$, the space of continuous functions on G vanishing at infinity, and where for each x fixed $D_x(\varepsilon, z)$ denotes the function $D(\varepsilon, x-z)$. Since $D_x(\varepsilon, z) \in C_0(G)$, it follows that

$$\lim_{x \to x'} ||D_x(\varepsilon,\cdot) - D_{x'}(\varepsilon,\cdot)||_\infty = 0.$$

Thus, our assertion follows from the inequality

$$||K^{(\varepsilon)}(x,\cdot) - K^{(\varepsilon)}(x',\cdot)||_2 \le ||D_x(\varepsilon,\cdot) - D_{x'}(\varepsilon,\cdot)||_\infty ||\chi_\Omega * |D(\varepsilon,\cdot)|||_2.$$

Here we have used the fact that $\chi_\Omega \in L^1(G)$ and $D(\varepsilon,\cdot) \in L^2(G)$. Also, since it is the Fourier transform of $\chi_{\pi(\varepsilon)}(\nu)$ so that, by Young's inequality, $||\chi_\Omega * |D(\varepsilon,\cdot)|\,||_2 \le ||\chi_\Omega||_1 ||D(\varepsilon,\cdot)||_2$, see [**8**]. We finally have (1.1)

$$\begin{aligned}&\int_G \int_G |K^{(\varepsilon)}(x,y)|^2\,dm(x)\,dm(y) \\ &= \int_G\int_G\int_G D(\varepsilon, x-z)\chi_\Omega(z) D(\varepsilon, z-y)\,dm(z) \\ &\qquad \int_G D(\varepsilon, x-w)\chi_\Omega D(\varepsilon, w-y) dm(w)\,dm(x)\,dm(y) \\ &= \int_G \int_G \chi_\Omega(z)\chi_\Omega(w)\,dm(z)\,dm(w) \int_G D(\varepsilon, x-z) D(\varepsilon, x-w)\,dm(x) \\ &\quad \cdot \int_G D(\varepsilon, z-y) D(\varepsilon, w-y)\,dm(y) \\ &= \int_G \int_G \chi_\Omega(z)\chi_\Omega(w)|D(\varepsilon, z-w)|^2\,dm(z)\,dm(w).\end{aligned}$$

Note that we have used the following identity to obtain the last expression:

$$D(\varepsilon, u-v) = \int_G D(\varepsilon, u-t)D(\varepsilon, v-t)\, dm(t). \tag{1.2}$$

The integral $\int_G \int_G \chi_\Omega(z)\chi_\Omega(w)|D(\varepsilon, z-w)|^2\, dm(z)\, dm(w)$ in (1.1) is finite since $|D(\varepsilon, z-w)|^2$ is bounded. This shows that $U_\Omega^{(\varepsilon)}$ is a Hilbert Schmidt operator. □

Lemma 2. *Let $\lambda(\varepsilon, k)$, $k = 1, 2, \ldots$, be the eigenvalues of $U_\Omega^{(\varepsilon)}$ repeated according to their multiplicities. Then*

(i) $0 \leq \lambda(\varepsilon, k) \leq 1$, $k = 1, 2, \ldots,$

(ii) $\sum_k \lambda(\varepsilon, k) \leq m(\Omega)\mu(\pi(\varepsilon))$;

in addition, if condition (1.1) *of Theorem* 1 *holds, then the following condition is valid*:

(iii) $\sum_k \lambda^2(\varepsilon, k) = m(\Omega)\mu(\pi(\varepsilon))(1 - \Delta(\varepsilon))$,

where $0 \leq \Delta(\varepsilon) \leq 1$, *and where* $\lim_{\varepsilon\to\infty} \Delta(\varepsilon) = 0$.

Proof. It is apparent that $U_\Omega^{(\varepsilon)} = P^{(\varepsilon)} M_\Omega P^{(\varepsilon)}$, where $P^{(\varepsilon)}$ is defined by $P^{(\varepsilon)}\varphi = D(\varepsilon, \cdot) * \varphi$ and where M_Ω is defined by $M_\Omega \varphi = \chi_\Omega \varphi$ for $\varphi \in L^2(G)$. We have $D(\varepsilon, \cdot) * D(\varepsilon, \cdot) = D(\varepsilon, \cdot)$ which follows from (1.2). Using this property, we obtain

$$[P^{(\varepsilon)}]^2 \cdot \varphi = D(\varepsilon, \cdot) * D(\varepsilon, \cdot) * \varphi = D(\varepsilon, \cdot) * \varphi = P^{(\varepsilon)}\varphi.$$

A self-adjoint property can also be obtained easily. This shows that $P^{(\varepsilon)}$ is a projection. Clearly M_Ω is also a projection. Thus,

$$||U_\Omega^{(\varepsilon)}|| \leq ||P^{(\varepsilon)}||\, ||M_\Omega||\, ||P^{(\varepsilon)}|| \leq 1.$$

That (i) holds is an immediate consequence of this fact. Since Lemma 2 holds for $U_\Omega^{(\varepsilon)}$, we apply Mercer's theorem (weak form), i.e., Corollary

1, to obtain

$$
\begin{aligned}
\sum \lambda(\varepsilon,k) &\le \int_G K^{(\varepsilon)}(x,x)\,dm(x) \\
&= \int_G\int_G D(\varepsilon,x-z)\chi_\Omega(z)D(\varepsilon,z-x)\,dm(z)\,dm(x) \qquad (2.1)\\
&= \int_G \chi_\Omega(z)\int_G D(\varepsilon,x-z)D(\varepsilon,z-x)\,dm(x)\,dm(z).
\end{aligned}
$$

Observing (1.2), we have

$$\mu(\pi(\varepsilon)) = D(\varepsilon,0) = \int_G D(\varepsilon,x-z)D(\varepsilon,z-x)\,dm(x).$$

It follows that

$$\sum_k \lambda(\varepsilon,k) \le m(\Omega)\mu(\pi(\varepsilon)).$$

This proves (ii). Moreover, since $U_\Omega^{(\varepsilon)}$ is of Hilbert Schmidt type, using the result from the standard theory of integral equations [**7**], we obtain

$$\int_G\int_G |K^{(\varepsilon)}(x,y)|^2\,dm(x)\,dm(y) = \sum_{k=1}^{\infty}\lambda^2(\varepsilon,k). \qquad (2.2)$$

It follows from (1.1), (2.2) and the Plancherel theorem that

$$
\begin{aligned}
\sum_k \lambda^2(\varepsilon,k) &= \int_G\int_G \chi_\Omega(z)\chi_\Omega(w)|D(\varepsilon,z-w)|^2\,dm(z)\,dm(w)\\
&= \int_G \chi_\Omega(z)\cdot\chi_\Omega * |D(\varepsilon,\cdot)|^2(z)\,dm(z)\\
&= \int_\Gamma |\chi_\Omega^\wedge(\nu)|^2 Q(\varepsilon,\nu)\,d\mu(\nu)\\
&= \mu(\pi(\varepsilon))\int_\Gamma |\chi_\Omega^\wedge(\nu)|^2\frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\,d\mu(\nu) \qquad (2.3)\\
&= \mu(\pi(\varepsilon))\int_\Gamma |\chi_\Omega^\wedge(\nu)|^2\left\{1-\left[1-\frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right]\right\}d\mu(\nu)\\
&= \mu(\pi(\varepsilon))\int_\Gamma |\chi_\Omega^\wedge(\nu)|^2\,d\mu(\nu)\\
&\quad - \mu(\pi(\varepsilon))\int_\Gamma |\chi_\Omega^\wedge(\nu)|^2\left[1-\frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right]d\mu(\nu)\\
&= \mu(\pi(\varepsilon))m(\Omega)(1-\Delta(\varepsilon)),
\end{aligned}
$$

where

$$\Delta(\varepsilon) = m(\Omega)^{-1} \int_{\Gamma} |\chi_{\Omega}^{\wedge}(\nu)|^2 \left[1 - \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right] d\mu(\nu).$$

If the condition (1.1) of Theorem 1 holds, then we assert that

$$\lim_{\varepsilon\to\infty} \Delta(\varepsilon) = 0. \tag{2.4}$$

To prove (2.4), we use the following argument. Given a small positive number η, there exists a compact set C in Γ such that $\int_{\Gamma/C} |\chi_{\Omega}(\nu)|^2 \, d\mu(\nu) < \eta m(\Omega)$. Since $(Q(\varepsilon,\nu))/(\mu(\pi(\varepsilon)))$ converges to 1 in measure on C as $\varepsilon \to \infty$, there exists an $\varepsilon_0 \in \mathcal{D}$ such that $\mu(I_\eta^{(\varepsilon)}(C)) < \eta m(\Omega)^{-1}$ for $\varepsilon \geq \varepsilon_0$, where $I_\eta^{(\varepsilon)}(C) = \{\nu \in C | 1 - (Q(\varepsilon,\nu))/(\mu(\pi(\varepsilon))) > \eta\}$. Thus,

$$\begin{aligned}\Delta(\varepsilon) &= m(\Omega)^{-1} \int_{\Gamma} |\chi_{\Omega}(\nu)|^2 \left[1 - \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right] d\mu(\nu)\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$I_1 = m(\Omega)^{-1} \int_{I_\eta^{(\varepsilon)}(C)} |\chi_{\Omega}(\nu)|^2 \left[1 - \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right] d\mu(\nu)$$

$$I_2 = m(\Omega)^{-1} \int_{C\setminus I_\eta^{(\varepsilon)}(C)} |\chi_{\Omega}(\nu)|^2 \left[1 - \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right] d\mu(\nu)$$

and

$$I_3 = m(\Omega)^{-1} \int_{\Gamma\setminus C} |\chi_{\Omega}(\nu)|^2 \left[1 - \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\right] d\mu(\nu).$$

Noting that $|| \, |\chi_{\Omega}(\nu)|^2 \, ||_\infty \leq m(\Omega)^2$ and $1 - (Q(\varepsilon,\nu))/(\mu(\pi(\varepsilon))) \leq 1$, we obtain

$$\begin{aligned}I_1 &\leq m(\Omega)^{-1} || \, |\chi_{\Omega}(\nu)|^2 ||_\infty \cdot \int_{I_\eta^{(\varepsilon)}(C)} d\mu(\nu)\\ &\leq m(\Omega)^{-1} \cdot m(\Omega)^2 \cdot \eta \cdot m(\Omega)^{-1}\\ &= \eta,\\ I_2 &\leq m(\Omega)^{-1} \int_{C \;\; I_\eta^{(\varepsilon)}(C)} |\chi_{\Omega}(\nu)|^2 \eta \, d\mu(\nu)\\ &\leq m(\Omega)^{-1} \eta \int_{\Gamma} |\chi_{\Omega}(\nu)|^2 \, d\mu(\nu)\\ &= m(\Omega)^{-1} m(\Omega) \eta\\ &= \eta,\end{aligned}$$

and

$$\begin{aligned} I_3 &\leq m(\Omega)^{-1} \int_{\Gamma \setminus C} |\chi_\Omega(\nu)|^2 \, d\mu(\nu) \\ &\leq m(\Omega)^{-1} \eta m(\Omega) \\ &= \eta. \end{aligned}$$

Our assertion follows immediately. □

Let $N^{\pm}(\tau; U_\Omega^{(\varepsilon)})$ be the number of the eigenvalues of $U_\Omega^{(\varepsilon)}$ which are greater than τ if $\pm$ is $+$ or less than $-\tau$, if $\pm$ is $-$, where τ is a positive number.

Lemma 3. *Under the assumption of Lemma* 2, *if* $\Delta(\varepsilon)^{1/2} < 1/2$, *we have*

(i) $N^+(1 - \Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)} \gtrsim m(\Omega)\mu(\pi(\varepsilon))(1 - 4\Delta(\varepsilon)^{1/2})$

and

(ii) $N^+(\Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)}) \lesssim m(\Omega)\mu(\pi(\varepsilon))(1 + 4\Delta(\varepsilon)^{1/2})$,

where $\Delta(\varepsilon)$ *is defined as* (2.4). *Note that* $a(\varepsilon) \lesssim b(\varepsilon)$ *means that* $\overline{\lim}_{\varepsilon \to \infty}(a(\varepsilon)/b(\varepsilon)) \leq 1$, *etc.*

Proof. Let us define

$$\begin{aligned} S_1 &= \{\lambda(\varepsilon, k); \lambda(\varepsilon, k) < \Delta(\varepsilon)^{1/2}\}, \\ S_2 &= \{\lambda(\varepsilon, k); \Delta(\varepsilon)^{1/2} \leq \lambda(\varepsilon, k) \leq 1 - \Delta(\varepsilon)^{1/2}\}, \end{aligned}$$

and

$$S_3 = \{\lambda(\varepsilon, k); 1 - \Delta(\varepsilon)^{1/2} < \lambda(\varepsilon, k)\}.$$

Let $S_2^\#$ and $S_3^\#$ be the number of eigenvalues in S_2 and S_3, respectively. Subtracting (iii) from (ii) in Lemma 2 we obtain

$$\begin{aligned} \sum_{S_2} \lambda(\varepsilon, k)[1 - \lambda(\varepsilon, k)] &\leq \sum_k \lambda(\varepsilon, k)[1 - \lambda(\varepsilon, k)] \\ &= \Delta(\varepsilon) m(\Omega)\mu(\pi(\varepsilon)). \end{aligned}$$

The terms in the sum on the left are nonnegative and exceed $(1/2)\Delta(\varepsilon)^{1/2}$ since $\Delta(\varepsilon)^{1/2} < 1/2$. Consequently,

$$\frac{1}{2}\Delta(\varepsilon)^{1/2} S_2^{\#} \leq \Delta(\varepsilon) m(\Omega)\mu(\pi(\varepsilon)),$$

so

$$S_2^{\#} \leq 2\Delta(\varepsilon)^{1/2} m(\Omega)\mu(\pi(\varepsilon)).$$

Now we have

$$\sum_{S_1} \lambda^2(\varepsilon, k) \leq \Delta(\varepsilon)^{1/2} \sum_k \lambda(\varepsilon, k) = \Delta(\varepsilon)^{1/2} m(\Omega)\mu(\pi(\varepsilon)),$$
$$\sum_{S_2} \lambda^2(\varepsilon, k) \leq S_2^{\#} \leq 2\Delta(\varepsilon)^{1/2} m(\Omega)\mu(\pi(\varepsilon)),$$

and

$$\sum_{S_3} \lambda^2(\varepsilon, k) \leq S_3^{\#}.$$

Inserting these estimates in (iii) of Lemma 2, we find that since

$$\sum_{S_3} \lambda^2(\varepsilon, k) = m(\Omega)\mu(\pi(\varepsilon))(1 - \Delta(\varepsilon)) - \sum_{S_1} \lambda^2(\varepsilon, k) - \sum_{S_2} \lambda^2(\varepsilon, k),$$

then

$$(*) \qquad \begin{aligned} S_3^{\#} &\geq m(\Omega)\mu(\pi(\varepsilon))(1 - \Delta(\varepsilon) - 3\Delta(\varepsilon)^{1/2}) \\ &\geq m(\Omega)\mu(\pi(\varepsilon))(1 - 4\Delta(\varepsilon)^{1/2}). \end{aligned}$$

Also

$$(1 - \Delta(\varepsilon)^{1/2}) S_3^{\#} \leq \sum_{S_3} \lambda(\varepsilon, k) \leq m(\Omega)\mu(\pi(\varepsilon)),$$

and

$$\begin{aligned} S_3^{\#} &\leq m(\Omega)\mu(\pi(\varepsilon))(1 - \Delta(\varepsilon)^{1/2})^{-1}, \\ &\leq m(\Omega)\mu(\pi(\varepsilon))(1 + 2\Delta(\varepsilon)^{1/2}). \end{aligned}$$

Thus,

$$(**) \qquad S_2^{\#} + S_3^{\#} \leq m(\Omega)\mu(\pi(\varepsilon))(1 + 4\Delta(\varepsilon)^{1/2})$$

It follows from $(*)$ and $(**)$ that (i) and (ii) are obtained immediately. $\square$

Remark 1. The proof of Lemma 3 is reproduced for the reader's convenience from Hirschman's paper [**3**].

Lemma 4. *Assume the sufficient condition* (1.1) *of Theorem* 1 *is satisfied and* $U_\Omega^{(\varepsilon)}$ *is defined as in Lemma* 2*. We have for* $0 < \tau < 1$*,*

$$\lim_{\varepsilon\to\infty} N^+(\tau; U_\Omega^{(\varepsilon)})/\mu(\tau(\varepsilon)) = m(\Omega).$$

Proof. We may choose $\varepsilon_0 \in \mathcal{D}$ so that $\Delta(\varepsilon)^{1/2} < \tau < 1 - \Delta(\varepsilon)^{1/2}$ for all $\varepsilon \geq \varepsilon_0$ since $\lim_{\varepsilon\to\infty} \Delta(\varepsilon) = 0$ by Lemma 2. Now it follows from (i) and (ii) of Lemma 3 that

$$\begin{aligned} m(\Omega)\mu(\pi(\varepsilon))(1 - 4\Delta(\varepsilon)^{1/2}) &\leq N^+(1 - \Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)}) \\ &\leq N^+(\tau; U_\Omega^{(\varepsilon)}) \\ &\leq N^+(\Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)}) \\ &\leq m(\Omega)\mu(\pi(\varepsilon))(1 + 4\Delta(\varepsilon)^{1/2}). \end{aligned}$$

Dividing through $\mu(\pi(\varepsilon))$ and taking the limit as $\varepsilon \to \infty$, the above inequalities become

$$\begin{aligned} m(\Omega) &= \varliminf_{\varepsilon\to\infty} N^+(1 - \Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &\leq \varliminf_{\varepsilon\to\infty} N^+(\tau; U_\Omega^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &\leq \varlimsup_{\varepsilon\to\infty} N^+(\tau; U_\Omega^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &\leq \varlimsup_{\varepsilon\to\infty} N^+(\Delta(\varepsilon)^{1/2}; U_\Omega^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &= m(\Omega). \end{aligned}$$

Thus we obtain

$$\lim_{\varepsilon\to\infty} N^+(\tau; U_\Omega^{(\varepsilon)})/\mu(\pi(\varepsilon)) = m(\Omega).$$

The limit established in Lemma 4 is essentially a proof of a sufficient condition of Theorem 1 when the generating function of such a Toeplitz operator is of characteristic type. A general form shown in (1.2) of Theorem 1 will be seen when an arbitrary real function in $L^1(G)$ is treated. The subsequent Lemmas are proofs of a sufficient condition for real simple functions. □

LEMMA **5.** *Under the assumptions of Lemma* 2*, the quantity* $|D(\varepsilon,x)|^2/\mu(\pi(\varepsilon))$ *satisfies the following two properties*:

(i) $\int_G |D(\varepsilon,x)|^2/\mu(\pi(\varepsilon))\,dm(x) = 1$

(ii) $\lim_{\varepsilon\to\infty}\int_{G\setminus W}|D(\varepsilon,x)|^2/\mu(\pi(\varepsilon))\,dm(x) = 0$ *for each neighborhood* W *of the identity* 0.

Proof. It follows from the Plancherel theorem that

$$\int_G |D(\varepsilon,x)|^2/\mu(\pi(\varepsilon))\,dm(x) = \int_\Gamma |\chi_{\pi(\varepsilon)}(\gamma)|^2/\mu(\pi(\varepsilon))\,d\mu(\cdot) = 1.$$

Thus (i) is proved. Given a neighborhood W of 0, choose a compact neighborhood C of 0 such that $C-C\subset W$, let χ_C be the characteristic function on C and put $g(x) = \chi_C * \tilde{\chi}_C(x)/\mu(C)$, where $\tilde{\chi}_C(x)$ is defined by $\tilde{\chi}_C(x) = \chi_C(-x)$. Clearly $g(x)$ is a continuous function with support contained in $W-W$ and bounded by 1. By construction $g^\wedge(\nu) = |\chi_C(\nu)|^2/\mu(C) \in L^1(\Gamma)$, and applying the Fourier inversion theorem to g, we see that

$$\int_\Gamma g^\wedge(\nu)\,d\mu(\nu) = g(0) = 1.$$

Moreover, applying Fubini's theorem and the inversion theorem, we obtain

$$\begin{aligned}
&\int_G g(x)\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)\\
&\quad= \int_G \frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\int_\Gamma g^\wedge(\nu)(x,\nu)\,d\mu(\nu)\,dm(x)\\
&\quad= \int_\Gamma g^\wedge(\nu)\int_G \frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}(x,\nu)\,dm(x)\,d\mu(\nu)\\
&\quad= \int_\Gamma g^\wedge(\nu)\frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\,d\mu(\nu).
\end{aligned}\tag{5.1}$$

Since the condition (1.1) of Theorem 1 holds, the same argument used to prove (2.4) shows that

$$\lim_{\varepsilon\to\infty}\int_\Gamma g^\wedge(\nu)\frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\,d\mu(\nu)=\int_\Gamma g^\wedge(\nu)\,d\mu(\nu)=g(0)=1. \tag{5.2}$$

On the other hand, since the support of $g(x)$ is contained in W, we have

$$\begin{aligned}\int_G g(x)\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)&=\int_W g(x)\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)\\&=\int_W \frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)-\int_{G\backslash W}\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)\\&\le\int_G \frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)-\int_{G\backslash W}\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x)\\&=1-\int_{G\backslash W}\frac{|D(\varepsilon,x)|^2}{\mu(\pi(\varepsilon))}\,dm(x).\end{aligned} \tag{5.3}$$

Now (ii) follows immediately from (5.1), (5.2) and (5.3). The properties (i) and (ii) in Lemma 5 imply that $|D(\varepsilon,x)|^2/\mu(\pi(\varepsilon))$, $\varepsilon\in D$, is an approximate identity in $L^1(G)$. Namely, for $f\in L^1(G)||f*|D(\varepsilon,\cdot)|^2/\mu(\pi(\varepsilon))-f||_1\to 0$ as $\varepsilon\to\infty$. The proof is routine and will be omitted (see [**2**]). □

Lemma 6. *Let Ω_s and Ω_t be two disjoint Borel sets of G such that $m(\Omega_s)<\infty$. Then, under the assumption of Lemma 4,*

$$N^+(\tau;M_{\Omega_t}U^{(\varepsilon)}_{\Omega_s}M_{\Omega_t})=o(\mu(\pi(\varepsilon))) \tag{6.1}$$

as $\varepsilon\to\infty$, for all $\tau>0$.

Proof. Clearly, $M_{\Omega_t}U^{(\varepsilon)}_{\Omega_s}M_{\Omega_t}$ is a nonnegative integral operator with kernel $\chi_{\Omega_t}(x)\int_G D(\varepsilon,x-z)\chi_{\Omega_s}(z)D(\varepsilon,z-y)\,dm(x)\chi_{\Omega_t}(y)$. Let $\mu(\varepsilon,k)$, $k=1,2,\dots$, be the set of eigenvalues of $M_{\Omega_t}U^{(\varepsilon)}_{\Omega_s}M_{\Omega_t}$. Applying Corollary 3 of Mercer's theorem, and noting that $\int_G\chi_{\Omega_t}(x)\chi_{\Omega_s}(x)\,dm(x)=$

0, we obtain

$$\begin{aligned}\sum_k \mu(\varepsilon,k) &\le \int_G \chi_{\Omega_t}(x)\int_G D(\varepsilon,x-z)\chi_{\Omega_s}(z)D(\varepsilon,z-x)\,dm(z)\chi_{\Omega_t}(x)\,dm(x)\\ &= \int_G \chi_{\Omega_t}(x)\int_G \chi_{\Omega_s}(z)|D(\varepsilon,x-z)|^2\,dm(z)\,dm(x)\\ &= \mu(\pi(\varepsilon))\int_G [\chi_{\Omega_t}(x)][\chi_{\Omega_s} * |D(\varepsilon,\cdot)|^2/\mu(\pi(\varepsilon))(x)]\,dm(x).\end{aligned}$$

Since, as $\varepsilon \to \infty$, $[\chi_{\Omega_s} * |D(\varepsilon,\cdot)|^2/\mu(\pi(\varepsilon))](x) \to \chi_{\Omega_s}(x)$ in $L^1(G)$, see the comments just preceding Lemma 6, it follows that

$$\begin{aligned}\lim_{\varepsilon\to\infty}\int_G [\chi_{\Omega_t}(x)][\chi_{\Omega_s} * |D(\varepsilon,\cdot)|^2/\mu(\pi(\varepsilon))](x)\,dm(x)\\ = \int_G \chi_{\Omega_t}(x)\chi_{\Omega_s}(x)\,dm(x) = 0.\end{aligned}$$

This shows that $\sum_k \mu(\varepsilon,k) = o(\mu(\pi(\varepsilon)))$ as $\varepsilon \to \infty$. Our assertion now follows from the formula

$$N^+(\tau; M_{\Omega_t}U_{\Omega_s}^{(\varepsilon)}M_{\Omega_t}) \le \tau^{-1}\sum_k \mu(\varepsilon,k). \qquad \square$$

We next recall some well-known results from operator theory, see [**7**], which we will need. Let A_j, $j = 1,2,\dots,n$, be compact self-adjoint operators on $L^1(G)$ and let $\tau_j > 0$, $j = 1,2,\dots,n$. Then we have

Lemma 7.

$$N^{\pm}\Big(\sum_{j=1}^k \tau_j; \sum_{j=1}^n A_j\Big) \le \sum_{j=1}^n N^{\pm}(\tau_j; A_j) \tag{7.1}$$

and

(7.2)

$$N^+(\tau;A) \le N^+(\tau;B), \qquad N^-(\tau;B) \le N^-(\tau;A) \qquad \text{if } A \le B,$$

i.e. $B - A$ *is nonnegative.*

Proof. The results follow from the minimax characterization of the eigenvalues of such operators. See [**4, 7**] for details in their proofs.

Let $f(x) = \sum_{r=1}^{n} a_r \chi_{\Omega_r}$, where $\{\Omega_r\}_{r=1}^{n}$ is a disjoint family of Borel sets of G with finite measures such that $G = \cup_{r=1}^{n} \Omega_r$ and where a_r, $r = 1, 2, \ldots, n$ are reals. Let I be the identity operator on $L^2(G)$. Clearly, we have $I = \sum_{r=1}^{n} M_{\Omega_r}$. Also, by a linearity of Toeplitz operators with respect to their generating function we have $U_f^{(\varepsilon)} = \sum_{r=1}^{n} a_r U_{\Omega_r}^{(\varepsilon)}$. Therefore,

$$
\begin{aligned}
U_f^{(\varepsilon)} &= \Big(\sum_{s=1}^{n} M_{\Omega_s}\Big)\Big(\sum_{r=1}^{n} a_r U_{\Omega_r}^{(\varepsilon)}\Big)\Big(\sum_{t=1}^{n} M_{\Omega_t}\Big) \\
&= \sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r} \\
&\quad + \sum_{r=1}^{n} a_r \Bigg(\sum_{\substack{st=1 \\ s\neq r \\ \text{or } t\neq r}}^{n} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s} \Bigg) \qquad (8.1) \\
&= \sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r} + W
\end{aligned}
$$

where W denotes the second parts of summation on the last expression. We shall next show that the asymptotic eigenvalue distribution of W is negligible so that both $U_f^{(\varepsilon)}$ and $\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}$ have the same distribution, as ε tends to infinity. □

The above assertions are due to Lemma 6, Lemma 7, and the following Lemma 8 and Lemma 9.

Lemma 8. *For any three sets Ω_r, Ω_s and Ω_t from the above family $\{\Omega_1\}_{i=1}^{n}$ with $\Omega_s \neq \Omega_r$ or $\Omega_t \neq \Omega_r$, then with the same assumption as in Lemma 2, we have*

$$N^{\pm}(\tau; M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}) = o(\mu(\pi(\varepsilon))). \qquad (8.2)$$

Proof. Recall that $P^{(\varepsilon)}$ is a projection on $L^2(G)$. For $\eta > 0$, we have

$$(\eta^{-1} M_{\Omega_s} P^{(\varepsilon)} M_{\Omega_r} \pm \eta M_{\Omega_t} P^{(\varepsilon)} M_{\Omega_r}) \cdot (\eta^{-1} M_{\Omega_s} P^{(\varepsilon)} M_{\Omega_r} \pm \eta M_{\Omega_t} P^{(\varepsilon)} M_{\Omega_r})^* \geq 0$$

or

$$\eta^{-2} M_{\Omega_s} P^{(\varepsilon)} M_{\Omega_r} P^{(\varepsilon)} M_{\Omega_s} + \eta^2 M_{\Omega_t} P^{(\varepsilon)} M_{\Omega_r} P^{(\varepsilon)} M_{\Omega_t} \geq \pm (M_{\Omega_s} P^{(\varepsilon)} M_{\Omega_r} P^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} P^{(\varepsilon)} M_{\Omega_r} P^{(\varepsilon)} M_{\Omega_s})$$

which can be rewritten as

$$\eta^{-2} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s} + \eta^2 M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} \geq \pm (M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}).$$

Applying (7.1) and (7.2), we obtain

$$\begin{aligned} & N^{\pm}(\tau; M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}) \\ & \quad \leq N^{\pm}(\tau; \eta^{-2} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s} + \eta^2 M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t}) \\ & \quad \leq N^{\pm}\left(\frac{\tau}{2}; \eta^{-2} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}\right) + N^{\pm}\left(\frac{\tau}{2}; \eta^2 M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}\right). \end{aligned} \tag{8.3}$$

We shall now estimate the two quantities above for their sizes of numbers' of eigenvalues greater than a given $\tau/2$. Their estimates can be obtained by considering two situations according to the relationships among three sets M_{Ω_r}, M_{Ω_s}, and M_{Ω_t}.

Case 1. The three sets Ω_r, Ω_s and Ω_t are mutually disjoint. In this case, we can immediately apply Lemma 6 to obtain

$$N^{\pm}\left(\frac{\tau}{2}; \eta^{-2} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}\right) = N^{\pm}\left(\frac{\tau}{2}; \eta^2 M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t}\right) = o(\mu(\pi(\varepsilon))).$$

It follows from (8.3) that

$$N^{\pm}(\tau; M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}) = o(\mu(\pi(\varepsilon))).$$

Case 2. If $\Omega_s = \Omega_r$ and $\Omega_t \neq \Omega_r$ or $\Omega_t = \Omega_r$ and $\Omega_s \neq \Omega_r$, say $\Omega_s = \Omega_r$ and $\Omega_t \neq \Omega_r$. Since $\Omega_t \neq \Omega_r$, again by Lemma 5 we have

$$N^{\pm}\left(\frac{\tau}{2}; \eta^2 M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t}\right) = o(\mu(\pi(\varepsilon))).$$

Let us choose η so that $(\tau/2)\cdot\eta^2 > 1$. Since $||M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}|| \leq 1$, then $N^{\pm}(\tau/2; \eta^{-2} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}) = N^{\pm}((\tau/2)\eta^2; M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s}) = 0$, and our claims follow. □

Lemma 9. *Under the same assumption as Lemma* 7, *we have*

$$\text{(9.1)}\qquad \begin{aligned} m\{x|f(x) > \tau\} &\leq \varliminf_{\varepsilon\to\infty} N^{+}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ m\{x|f(x) \geq \tau\} &\geq \varlimsup_{\varepsilon\to\infty} N^{+}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \end{aligned}$$

and

$$\begin{aligned} m\{x|f(x) < -\tau\} &\leq \varliminf_{\varepsilon\to\infty} N^{-}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ m\{x|f(x) \leq -\tau\} &\geq \varlimsup_{\varepsilon\to\infty} N^{-}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \end{aligned}$$

where $f(x) = \sum_{r=1}^{n} a_r \chi_{\Omega_r}(x)$ *is defined previously.*

Proof. Since $W = \sum_{r=1}^{n} a_r (\sum^{n}_{\substack{s,t=1 \\ s\neq r \text{ and/or} \\ t\neq r}} M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_t} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s})$ is a finite sum of those terms $(M_{\Omega_s} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_t} + M_{\Omega_T} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_s})$ in (8.2) multiplied by a corresponding constant factor a_r. It is easy to see that a constant multiple a_r does not affect its estimate; therefore, the finite sum has the same estimates, i.e.,

$$\text{(9.2)}\qquad N^{\pm}(\tau; w) = o(\mu(\pi(\varepsilon))).$$

For finding limiting distribution of eigenvalues of those terms in the sum W, it suffices to estimate each $a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}$. Let Ω_r' be the complement of Ω_r in G, then

$$\begin{aligned} a_r U_{\Omega_r}^{(\varepsilon)} &= a_r (M_{\Omega_r} + M_{\Omega_r'}) U_{\Omega_r}^{(\varepsilon)} (M_{\Omega_r} + M_{\Omega_r'}) \\ &= a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r} + V \end{aligned}$$

where $V = a_r(M_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega'_r} + M_{\Omega'_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}) + a_rM_{\Omega'_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega'_r}$. Applying Lemma 5 and Lemma 8, we obtain

$$N^{\pm}(\tau; a_rM_{\Omega'_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega'_r}) = o(\mu(\pi(\varepsilon)))$$

and $N^{\pm}(\tau; a_r(M_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega'_r} + M_{\Omega'_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r})) = o(\mu(\pi(\varepsilon)))$. It follows that

$$N^{\pm}(\tau; V) = o(\mu(\pi(\varepsilon))). \tag{9.3}$$

Choose a number δ so that $\tau > \delta > 0$, and, applying (8.3), we have

$$\begin{aligned} N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}) &= N^{\pm}(\tau - \delta + \delta; a_rU^{(\varepsilon)}_{\Omega_r} - V) \\ &\le N^{\pm}(\tau - \delta; a_rU^{(\varepsilon)}_{\Omega_r}) + N^{\pm}(\delta; -V) \end{aligned}$$

and

$$N^{\pm}(\tau + \delta; a_rU^{(\varepsilon)}_{\Omega_r}) \le N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}) + N^{\pm}(\delta; V).$$

Combining the last two inequalities, we have

$$\begin{aligned} N^{\pm}(\tau + \delta; a_rU^{(\varepsilon)}_{\Omega_r}) &\le N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}) + N^{\pm}(\delta; V) \\ &\le N^{\pm}(\tau - \delta; a_rU^{(\varepsilon)}_{\Omega_r}) + N^{\pm}(\delta; -V). \end{aligned}$$

Dividing each part of inequalities in the last expression by $\mu(\pi(\varepsilon))$ and letting $\varepsilon \to \infty$, and noting that $\lim_{\varepsilon\to\infty} N^{\pm}(\tau; \pm V)/\mu(\pi(\varepsilon)) = 0$, we have

$$\begin{aligned} &\lim_{\varepsilon\to\infty} N^{\pm}(\tau + \delta; a_rU^{(\varepsilon)}_{\Omega_r})/\mu(\pi(\varepsilon)) \\ &\quad\le \varliminf_{\varepsilon\to\infty} N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r})/\mu(\pi(\varepsilon)) \\ &\quad\le \varlimsup_{\varepsilon\to\infty} N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r})/\mu(\pi(\varepsilon)) \\ &\quad\le \lim_{\varepsilon\to\infty} N^{\pm}(\tau - \delta; a_rU^{(\varepsilon)}_{\Omega_r})/\mu(\pi(\varepsilon)). \end{aligned} \tag{9.4}$$

If $|a_r| \le \tau$, then $N^{\pm}(\tau; a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}) = 0$ since $||a_rM_{\Omega_r}U^{(\varepsilon)}_{\Omega_r}M_{\Omega_r}|| \le |a_r|$. Suppose $\tau < |a_r|$; we choose a δ so that $0 < \delta < \tau$ and $\tau + \delta < |a_r|$.

By Lemma 4, (9.4), and by noting that $N^+(\tau; A) = N^-(\tau; -A)$ for any self-adjoint compact operator A; thus, we have

(9.5)

$$\lim_{\varepsilon\to\infty} N^+(\tau; a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)) = \begin{cases} m(\Omega) & \text{if } 0 < \tau < a_r, \\ 0 & \text{if } a_r \le \tau \text{ or } a_r < 0. \end{cases}$$

and

(9.6)

$$\lim_{\varepsilon\to\infty} N^-(\tau; a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)) = \begin{cases} m(\Omega) & \text{if } a_r < -\tau < 0, \\ 0 & \text{if } a_r \ge -\tau \text{ or } a_r > 0. \end{cases}$$

Since $\sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r}$ is a sum of operators whose product is zero between each other, it follows that a number is an eigenvalue of the sum if and only if it is an eigenvalue of one of those terms in the sum. Thus,

$$\begin{aligned}(9.7)\quad &\lim_{\varepsilon\to\infty} N^+\left(\tau; \sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)\right) \\ &\le \sum_{r=1}^n \lim_{\varepsilon\to\infty} N^+(\tau; a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)) = \sum_{a_r<\tau} m(\Omega_r)\end{aligned}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to\infty} N^-\left(\tau; \sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)\right) \\ (9.8)\quad &= \sum_{r=1}^m \lim_{\varepsilon\to\infty} N^-(\tau; a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r})/\mu(\pi(\varepsilon)) \\ &= \sum_{a_r<-\tau} m(\Omega_r).\end{aligned}$$

Now since $U^{(\varepsilon)}_f = \sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r} + W$, then, for $\tau > \delta > 0$,

$$\begin{aligned}N^\pm(\tau; U^{(\varepsilon)}_f) &= N^\pm\left(\tau - \delta + \delta; \sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r} + W\right) \\ &\le N^\pm\left(\tau - \delta; \sum_{r=1}^n a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r}\right) + N^\pm(\delta; W) \\ &= \sum_{r=1}^n N^\pm(\tau - \delta; a_r M_{\Omega_r} U^{(\varepsilon)}_{\Omega_r} M_{\Omega_r}) + o(\mu(\pi(\varepsilon)))\end{aligned}$$

and

$$
\begin{aligned}
N^{\pm}\left(\tau+\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right) &= N^{\pm}(\tau+\delta; U_f^{(\varepsilon)} - W)\\
&= N^{\pm}(\tau; U_f^{(\varepsilon)}) + N^{\pm}(\delta; -W)\\
&\le N^{\pm}(\tau; U_f^{(\varepsilon)}) + o(\mu(\pi(\varepsilon))).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
N^{\pm}&\left(\tau+\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right) + o(\mu(\pi(\varepsilon)))\\
&\le N^{\pm}(\tau; U_f^{(\varepsilon)})\\
&\le N^{\pm}\left(\tau-\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right) + o(\mu(\pi(\varepsilon))).
\end{aligned}
$$

Dividing through the above inequalities $\mu(\pi(\varepsilon))$ we obtain from (9.7) and (9.8)

$$
\begin{aligned}
\sum_{\tau+\delta<a_r} m(\Omega_r) &= \lim_{\varepsilon\to\infty} N^{+}(\tau+\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r})/\mu(\pi(\varepsilon))\\
&\le \varliminf_{\varepsilon\to\infty} N^{+}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\
&\le \varlimsup_{\varepsilon\to\infty} N^{+}(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\
&\le \lim_{\varepsilon\to\infty} N^{+}\left(\tau-\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right)/\mu(\pi(\varepsilon))\\
&= \sum_{\tau+\delta<a_r} m(\Omega_r)
\end{aligned}
$$

and

$$\begin{aligned}\sum_{a_r<-(\tau+\delta)} m(\Omega_r) &= \lim_{\varepsilon\to\infty} N^-\left(\tau+\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right)/\mu(\pi(\varepsilon))\\ &\le \varliminf_{\varepsilon\to\infty} N^-(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ &\le \varlimsup_{\varepsilon\to\infty} N^-(\tau-\delta;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ &\le \lim_{\varepsilon\to\infty} N^-\left(\tau-\delta;\sum_{r=1}^{n} a_r M_{\Omega_r} U_{\Omega_r}^{(\varepsilon)} M_{\Omega_r}\right)/\mu(\pi(\varepsilon))\\ &= \sum_{a_r<-\tau+\delta} m(\Omega_r).\end{aligned}$$

Since δ is arbitrary, we have from above

$$\begin{aligned} m\{x|f(x)>\tau\} &= \sum_{\tau<a_r} m(\Omega_r) \le \varliminf_{\varepsilon\to\infty} N^+(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ m|\{x|f(x)\ge\tau\} &= \sum_{\tau\le a_r} m(\Omega_r) \ge \varlimsup_{\varepsilon\to\infty} N^+(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\end{aligned} \tag{9.9}$$

and

$$\begin{aligned} m\{x|f(x)<-\tau\} &= \sum_{a_r<-\tau} m(\Omega_r) \le \varliminf_{\varepsilon\to\infty} N^-(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ m\{x|f(x)\le-\tau\} &= \sum_{a_r\le-\tau} m(\Omega_r) \ge \varlimsup_{\varepsilon\to\infty} N^-(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)).\end{aligned}$$

Thus, Lemma 9 is proved. □

Lemma 10. *Under the same assumption in Lemma* 9, *we have*

$$m\{x|f(x)\ge\tau\} = \lim_{\varepsilon\to\infty} N^+(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))$$

and

$$m\{x|f(x)\le-\tau\} = \lim_{\varepsilon\to\infty} N^-(\tau;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)),$$

provided $m\{x|f(x)=\tau\} = m\{x|f(x)=-\tau\} = 0$.

Proof. Follows immediately from Lemma 9. □

Lemma 11. *Let $f_n(x)$, $n = 1, 2, \ldots$, be a monotone sequence of real integrable functions converging to f in $L^1(G)$. If the results of Lemma 9 hold for each f_n, then it holds for f.*

Proof. We first assume that $\{f_n\}$ is an increasing sequence. From this, it is easy to see that $U^{(\varepsilon)}_{f_n} \le U^{(\varepsilon)}_f$ for each n. Thus, for $\tau > 0$, and for each n, we have

$$N^+(\tau; U^{(\varepsilon)}_{f_n}) \le N^+(\tau; U^{(\varepsilon)}_f). \tag{11.1}$$

For $\tau > \delta > 0$, we have

$$\begin{aligned} N^+(\tau; U^{(\varepsilon)}_f) &= N^+(\tau - \delta + \delta; U^{(\varepsilon)}_{f_n} + U^{(\varepsilon)}_{f-f_n}) \\ &\le N^+(\tau - \delta; U^{(\varepsilon)}_{f_n}) + N^+(\delta; U^{(\varepsilon)}_{f-f_n}). \end{aligned}$$

Hence,

$$N^+(\tau; U^{(\varepsilon)}_{f_n}) \le N^+(\tau; U^{(\varepsilon)}_f) \le N^+(\tau - \delta; U^{(\varepsilon)}_{f_n}) + N^+(\delta; U^{(\varepsilon)}_{f-f_n}). \tag{11.2}$$

Let $\lambda^{(n)}(\varepsilon, k)$, $k = 1, 2, \ldots$, be the necessarily nonnegative eigenvalues of $U^{(\varepsilon)}_{f-f_n}$. Applying Mercer's theorem, we have

$$\begin{aligned} N^+(\delta; U^{(\varepsilon)}_{f-f_n}) &\le \delta^{-1} \sum_{\lambda^{(n)}(\varepsilon,k) \ge \delta} \lambda^{(n)}(\varepsilon, k) \\ &\le \delta^{-1} \sum_k \lambda^{(n)}(\tau, k) \\ &= \delta^{-1} \int_G (f(z) - f_n(z)) \int_G D(\varepsilon, z-x) D(\varepsilon, x-z)\, dm(x)\, dm(z) \\ &= \delta^{-1} \int_G |f(z) - f_n(z)|\, dm(z) \cdot D(\varepsilon, 0) \\ &= \delta^{-1} ||f - f_n||_1 \mu(\pi(\varepsilon)). \end{aligned}$$

It follows from (11.2) that

$$\begin{aligned} N^+(\tau; U^{(\varepsilon)}_{f_n}) &\le N^+(\tau; U^{(\varepsilon)}_f) \\ &\le N^+(\tau - \delta; U^{(\varepsilon)}_{f_n}) + \delta^{-1} ||f - f_n||_1 \cdot \mu(\pi(\varepsilon)). \end{aligned}$$

Dividing each part of the last expression by $\mu(\pi(\varepsilon))$ and by the fact that (9.1) holds for each f_n, we have

$$\begin{aligned} m\{x|f_n(x) > \tau\} &\leq \underline{\lim}_{\varepsilon\to\infty} N^+(\tau; U_{f_n}^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &\leq \underline{\lim}_{\varepsilon\to\infty} N^+(\tau; U_{f}^{(\varepsilon)})/\mu(\pi(\varepsilon)) \end{aligned}$$

and

$$\begin{aligned} \overline{\lim}_{\varepsilon\to\infty} N^+(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) &\leq \overline{\lim}_{\varepsilon\to\infty} N^+(\tau-\delta; U_{f_n}^{(\varepsilon)})/\mu(\pi(\varepsilon)) \\ &\quad + \delta^{-1}\cdot ||f-f_n||_1 \\ &\leq m\{x|f_n(x)\geq \tau-\delta\} + \delta^{-1}\cdot ||f-f_n||_1. \end{aligned}$$

Since δ is arbitrary and $\{f_n\}$ is monotone increasing by assumption, we obtain from the last two inequalities,

$$m\{x|f(x) > \tau\} \leq \underline{\lim}_{\varepsilon\to\infty} N^+(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \tag{11.3}$$

and

$$m\{x|f(x) \geq \tau\} \geq \overline{\lim}_{\varepsilon\to\infty} N^+(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)).$$

A similar argument also shows that

$$m\{x|f(x) < -\tau\} \leq \underline{\lim}_{\varepsilon\to\infty} N^-(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \tag{11.4}$$

and

$$m\{x|f(x) \leq -\tau\} \geq \overline{\lim}_{\varepsilon\to\infty} N^-(\tau; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))$$

(11.3) and (11.4) proved Theorem 1 for the case when $\{f_n\}$ is an increasing sequence converging to f in $L^1(G)$. Suppose $\{f_n\}$ is a decreasing sequence converging to f in $L^1(G)$, then $\{-f_n\}$ becomes an increasing sequence converging to $-f$. Consequently, this case follows from previously treated cases.

Lemma 12. *Let f be a real-valued function in $L^1(G)$. Then formulas* (11.3) *and* (11.4) *hold for f if the sufficient condition of Theorem* 1 *is satisfied.*

Proof. Lemma 11 shows that the results hold for a real simple function. Since the set of real functions of $L^1(G)$ is the smallest monotone class containing all real simple functions in $L^1(G)$, our assertion now follows immediately from Lemma 11. □

With the following lemma, we will conclude the proof of the sufficient condition in Theorem 1.

Lemma 13. *Under the same assumptions as those in Lemma* 12, *we have*

$$\lim_{\varepsilon\to\infty} N^{\pm}(I; U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) = m\{x|f(x)\in I\}.$$

Provide $m\{x|f(x)=a\}=m\{x|f(x)=b\}=0$, *where* I *denotes any of the intervals* $(a,b),[a,b),(a,b]$, *or* $[a,b]$ *such that* $0\notin I$.

Proof. Suppose $0<a<b$ and choose δ so that $0<\delta<a$. Then

$$N^+(a;U_f^{(\varepsilon)}) - N^+(b-\delta;U_f^{(\varepsilon)}) \le N((a,b);U_f^{(\varepsilon)})$$

and

$$N([a,b];U_f^{(\varepsilon)}) \le N^+(a-\delta;U_f^{(\varepsilon)}) - N^+(b;U_f^{(\varepsilon)}).$$

Dividing each part of the last inequalities by $\mu(\pi(\varepsilon))$ and letting $\varepsilon\to\infty$, we obtain

$$\begin{aligned}\varliminf_{\varepsilon\to\infty} N^+(a;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) - \varlimsup_{\varepsilon\to\infty} N^+(b-\delta;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ \le \varliminf_{\varepsilon\to\infty} N((a,b);U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\end{aligned}$$

and

$$\begin{aligned}&\varlimsup_{\varepsilon\to\infty} N([a,b];U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))\\ &\quad\le \varlimsup_{\varepsilon\to\infty} N^+(a-\delta;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) - \varliminf_{\varepsilon\to\infty} N^+(b;U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)).\end{aligned}$$

Thus, from the above inequalities and Lemma 12, we obtain

$$m\{x|f(x)>a\} - m\{x|f(x)\ge b-\delta\} \le \varliminf_{\varepsilon\to\infty} N((a,b);U_f^{(\varepsilon)})/\mu(\pi(\varepsilon))$$

and

$$\varlimsup_{\varepsilon\to\infty} N([a,b];U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \le m\{x|f(x)\ge a-\delta\} - m\{x|f(x)>b\}.$$

Since δ is arbitrary, we find from the last two expressions that

$$m\{x|a<f(x)<b\} \le \varliminf_{\varepsilon\to\infty} N((a,b);U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \tag{13.1}$$

and

$$\varlimsup_{\varepsilon\to\infty} N([a,b];U_f^{(\varepsilon)})/\mu(\pi(\varepsilon)) \le m\{x|a\le f(x)\le b\}.$$

Carrying out a similar argument, as in the case $0<a<b$, we obtain the same result (13.1) for the case $a<b<0$. Note that

$$N((a,b);U_f^{(\varepsilon)}) \le N([a,b);U_f^{(\varepsilon)}) \le ([a,b];U_f^{(\varepsilon)})$$

and

$$N((a,b);U_f^{(\varepsilon)}) \le N((a,b];U_f^{(\varepsilon)}) \le N([a,b];U_f^{(\varepsilon)}).$$

Assertion now follows immediately from formula (13.1). Therefore, a proof of sufficient conditions for Theorem 1 is established. □

To complete a proof of Theorem 1, we shall prove its necessary condition from the following lemma:

Lemma 14. *The condition* (1.1) *of Theorem* 1 *is necessary.*

Proof. For a real valued function $f\in L^1(G)$, we define a measure $\Lambda(f;d\lambda)$ by setting $\Lambda(f,B)=m\{x|f(x)\in B\}$, where B is any Borel set of the real line. Similarly, for each index $\varepsilon\in\mathcal{D}$, we define a measure $\Lambda^{(\varepsilon)}(f;d\lambda)$ by

$$\Lambda^{(\varepsilon)}(f;B) = \{\lambda(\varepsilon,k)\in B\}\#/\mu(\pi(\varepsilon))$$

where $\lambda(\varepsilon,k)$, $k=1,2,\dots$, denotes the set of eigenvalues of $U_f^{(\varepsilon)}$. Let us take $f(x)$ to be the characteristic function χ_Ω of a Borel set Ω of finite measure in G. If Theorem 1 is true, then

$$\lim_{\varepsilon\to\infty}\int_{-\infty}^{\infty}\lambda^2\Lambda^{(\varepsilon)}(\chi_\Omega;d\lambda) = \int_{-\infty}^{\infty}\lambda^2\Lambda(\chi_\Omega;d\lambda). \tag{14.1}$$

We shall postpone the proof of (14.1) until later. Let us assume the formula (14.1) holds for the moment. By the definition of $\Lambda(\chi_\Omega; d\lambda)$, it is clear that

$$\int_{-\infty}^{\infty} \lambda^2 \Lambda(\chi_\Omega; d\lambda) = \int_G \chi_\Omega^2(x)\, dm(x) = m(\Omega).$$

Thus, we must have

$$\lim_{\varepsilon \to \infty} \int_{-\infty}^{\infty} \lambda^2 \Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda) = m(\Omega).$$

Also

$$\int_{-\infty}^{\infty} \lambda^2 \Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda) = \mu(\pi(\varepsilon))^{-1} \sum_k \lambda^2(\varepsilon, k)$$

where $\lambda(\varepsilon, k)$, $k = 1, 2, \ldots$, are eigenvalues of $U_\Omega^{(\varepsilon)}$. Consequently,

$$\lim_{\varepsilon \to \infty} \mu(\pi(\varepsilon))^{-1} \sum_k \lambda^2(\varepsilon, k) = m(\Omega). \tag{14.2}$$

If the condition (1.1) is not satisfied, then there would exist a compact set C and an $\eta > 0$ such that

$$\overline{\lim_{\varepsilon \to \infty}}\, \mu(I^{(\varepsilon)}) > 0, \qquad \text{where } I^{(\varepsilon)} = \left\{ \nu \in C \,|\, 1 - \frac{Q(\varepsilon, \nu)}{\mu(\pi(\varepsilon))} > \eta \right\}. \tag{14.3}$$

We choose, as we may, an Ω such that $|\chi_\Omega(\nu)|^2 \geq a$ for $\nu \in C$ and for some positive number a. It follows from (2.2), (2.3) and (14.3) that

$$\begin{aligned} m(\Omega) &= \lim_{\varepsilon\to\infty} \int_\Gamma |\chi_\Omega(x)|^2 \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\, d\mu(\nu) \\ &= \lim_{\varepsilon\to\infty} \bigg(\int_{\Gamma\setminus I^{(\varepsilon)}} |\chi_\Omega(\nu)|^2 \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\, d\mu(\nu) \\ &\qquad\qquad + \int_{I^{(\varepsilon)}} |\chi_\Omega(x)|^2 \frac{Q(\varepsilon,\nu)}{\mu(\pi(\varepsilon))}\, d\mu(\nu) \bigg) \\ &\le \lim_{\varepsilon\to\infty} \bigg(\int_{\Gamma\setminus I^{(\varepsilon)}} |\chi_\Omega(\nu)|^2\, d\mu(\nu) \\ &\qquad\qquad + \int_{I^{(\varepsilon)}} |\chi_\Omega(\nu)|^2 (1-\eta)\, d\mu(\nu) \bigg) \\ &= \lim_{\varepsilon\to\infty} \bigg(\int_\Gamma |\chi_\Omega(\nu)|^2\, d\mu(\nu) - \eta \int_{I^{(\varepsilon)}} |\chi_\Omega(\nu)|^2\, d\mu(\nu) \bigg) \\ &= m(\Omega) - \overline{\lim_{\varepsilon\to\infty}}\, \eta \int_{I^{(\varepsilon)}} |\chi_\Omega(\nu)|^2\, d\mu(\nu) \\ &\le m(\Omega) - \eta a \overline{\lim_{\varepsilon\to\infty}}\, \mu(I^{(\varepsilon)}) < m(\Omega). \end{aligned}$$

This is a contradiction. We return to the proof of formula (14.1). For each $\varepsilon \in \mathcal{D}$, the measure $\Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda)$ clearly satisfies the following properties:

(i) $\operatorname{supp} \Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda) \subset [0,1]$.

(ii) $\Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda)$ converges weakly to $\Lambda(\chi_\Omega; d\lambda) = \delta_1(d\lambda) m(\Omega)$ on $[\eta, 1]$, where η is any positive number and where $\delta_1(d\lambda)$ denotes the Dirac measure with mass at $\lambda = 1$,

(iii) $\int_0^\infty \lambda \Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda) \le m(\Omega)$.

From the properties above, we easily conclude that

$$\lim_{\varepsilon\to\infty} \int_{-\infty}^{\infty} \lambda^2 \Lambda^{(\varepsilon)}(\chi_\Omega; d\lambda) = \int_{-\infty}^{\infty} \lambda^2 \Lambda(\chi_\Omega; d\lambda). \qquad \square$$

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