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EXTENSIONS OF SOME CLASSES OF OPERATORS AND APPLICATIONS

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ABSTRACT. Let X be a Banach space such that its dual X^* is isometric to an L^1-space, let K denote the unit ball of X^* endowed with the weak* topology and let Y be any Banach space. In this paper we shall study when an unconditionally converging (respectively, completely continuous or weakly completely continuous) operator on the injective tensor product space $X\hat{\otimes}_\varepsilon Y$ of X and Y extends to an unconditionally converging (respectively, completely continuous or weakly completely continuous) operator on the space $C(K,Y)$ of all continuous Y-valued functions defined on K. We will also introduce and study the class of Banach spaces such that every bounded linear operator $u : E \to E^*$ is weakly compact.

Introduction. Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space, and denote by K the unit ball of X^* equipped with the weak*-topology. Let $C(K,Y)$ stand for the space of continuous Y-valued functions on K. In this paper we shall study the behavior of unconditionally converging, completely continuous and weakly completely continuous operators on the injective tensor product spaces $X\hat{\otimes}_\varepsilon Y$. Precisely, we will show that any bounded linear operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ extends to a bounded linear operator $\hat{T} : C(K,Y) \to Z^{**}$ with $||\hat{T}|| = ||T||$. The nature of questions we would like then to address is as follows: Suppose that T is unconditionally converging, completely continuous or weakly completely continuous, does it follow that $\hat{T}$ will also be unconditionally converging, completely continuous or weakly completely continuous, respectively? These questions are motivated by a result of [**14**] where it is shown that when X^* is isometric to an L^1-space, every unconditionally converging operator U on X is weakly compact, in particular U extends to a weakly compact operator $\hat{U}$ on $C(K)$ [**15**]. Thus it follows from [**9**, p. 160], that an operator U on X is weakly compact if and only if U is unconditionally converging if and only if U is completely continuous if

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and only if U is weakly completely continuous. In this paper we shall show that if Y is a Banach space that contains no subspace isomorphic to c_0, then $\hat{T} : C(K,Y) \to Z$ is unconditionally converging as soon as $T : X \hat{\otimes}_\varepsilon Y \to Z$ is unconditionally converging, similarly when Y is weakly sequentially complete (respectively, Y has the Schur property), then $\hat{T}$ is weakly completely continuous (respectively, $\hat{T}$ is completely continuous) whenever T is weakly completely continuous (respectively, T is completely continuous). In case $T : X \hat{\otimes}_\varepsilon Y \to Z$ is weakly compact (respectively, compact), it can be shown that $\hat{T} : C(K,Y) \to Z$ is weakly compact (respectively, compact) for any Banach space Y. Some of the above results extend and strengthen those of [**23, 24**]. The techniques used to prove the above mentioned results allow us to study some stabilities of the class of Banach spaces E such that every bounded linear operator $u : E \to E^*$ is weakly compact.

Notations and definitions. For a series $\sum_n x_n$ in the Banach space E we say that $\sum_n x_n$ is a *weakly unconditionally Cauchy* series in E if it satisfies one of the following equivalent statements:

a) $\sum_n |x^*(x_n)| < \infty$, for every $x^* \in E^*$;

b) $\sup\{||\sum_{n\in\sigma} x_n|| : \sigma \text{ finite subset of } \mathbf{N}\} < \infty$

c) $\sup_n \sup_{\varepsilon_i=\pm1} ||\sum_{i=1}^n \varepsilon_i x_i|| < \infty$.

Let E and F be Banach spaces. A bounded linear operator $T : E \to F$ is said to be *unconditionally converging* if T sends weakly unconditionally Cauchy series in E into unconditionally convergent series in F, and T is said to be *weakly completely continuous* (also called a Dieudonné operator) if T sends weakly Cauchy sequences in E into weakly convergent sequences in F. Finally, T is said to be *completely continuous* (also called a Dunford-Pettis operator) if T sends weakly Cauchy sequences into norm convergent sequences in F. It is immediate that a completely continuous operator is weakly completely continuous which in turn is unconditionally converging.

If Ω is a compact Hausdorff space and Y is a Banach space, then $C(\Omega, Y)$ will denote the Banach space of all continuous Y-valued functions on Ω under the uniform norm. It is well known [**9**] that the dual of $C(\Omega, Y)$ is isometrically isomorphic to the space $M(\Omega, Y^*)$ of all regular Y^*-valued measures on Ω that are of bounded variation. When Y is the scalar field $C(\Omega, Y)$ will be denoted by $C(\Omega)$ and $M(\Omega, Y^*)$ by

$M(\Omega)$. If $\mu \in M(\Omega, Y^*)$ we will denote by $|\mu|$ the variation of μ which is an element of $M(\Omega)$ and for each $y \in Y$ we will denote by $\langle y, \mu \rangle$ the element of $M(\Omega)$ such that for each Borel subset B of Ω we have

$$\langle y, \mu \rangle (B) = \mu(B)(y).$$

If $f \in C(\Omega)$ and $y \in Y$, we let $f \otimes y$ denote the element of $C(\Omega, Y)$ such that

$$f \otimes y(w) = f(w)y, \quad \text{for all } w \in \Omega.$$

It is well known that the set $\{f \otimes y : f \in C(\Omega) \text{ and } y \in Y\}$ is total in $C(\Omega, Y)$, and the duality between $C(\Omega, Y)$ and $M(\Omega, Y^*)$ is as follows: If $\mu \in M(\Omega, Y^*)$, $f \in C(\Omega)$ and $y \in Y$,

$$\langle \mu, f \otimes y \rangle = \int_\Omega f \, d\langle y, \mu \rangle.$$

Finally, if B is a Borel subset of Ω and $y \in Y$, we let $1_B \otimes y$ denote the element of $M(\Omega, Y^*)^*$ such that for $\mu \in M(\Omega, Y^*)$

$$\langle 1_B \otimes y, \mu \rangle = \mu(B)(y).$$

In particular, 1_B will denote the element of $M(\Omega)^*$ such that for each $\lambda \in M(\Omega)$

$$\langle 1_B, \lambda \rangle = \lambda(B).$$

If X and Y are Banach spaces, we denote by $X \otimes_\varepsilon Y$ the algebraic tensor product of X and Y endowed with the norm

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\| = \sup \left\{ \left| \sum_{i=1}^n x^*(x_i) y^*(y_i) \right| \mid ||x^*||, ||y^*|| \le 1 \right\}.$$

The completion $X \hat{\otimes}_\varepsilon Y$ of $X \otimes_\varepsilon Y$ is called the *injective tensor* product of X and Y. If Ω is a compact Hausdorff space, then $C(\Omega, Y)$ is isometrically isomorphic to $C(\Omega) \hat{\otimes}_\varepsilon Y$. All notions, notations and results used and not defined can be found in [**9, 16**].

Let X be a Banach space, and let K denote the unit ball of X^* equipped with the weak* topology. Let Y be another Banach space, then $X \hat{\otimes}_\varepsilon Y$ can be viewed as a subspace of $C(K, Y)$. It was shown in

[**20**] that if X^* is isometric to an L^1-space, then for any Banach space Y there exists a linear isometry

$$S : (X\hat{\otimes}_\varepsilon Y)^* \to M(K, Y^*)$$

such that for each $L \in (X\hat{\otimes}_\varepsilon Y)^*$ one has $S(L) = L$ on $X\hat{\otimes}_\varepsilon Y$. In the sequel we will always refer to this isometry by S. In case Y is the scalar field, then S will be denoted by $s : X^* \to M(K)$ which is a linear isometry such that $s(l) = l$ on X.

It follows from [**20**] that the two mappings S and s are such that if $L \in (X\hat{\otimes}_\varepsilon Y)^*$ and if $\langle y, L\rangle$ is the element of X^* with $\langle y, L\rangle(x) = L(x \otimes y)$ for all $x \in X$, then

$$\langle y, S(L)\rangle = s(\langle y, L\rangle). \tag{1}$$

Definition 1. A Banach space E is said to have *Pelczynski's property* (V) if every unconditionally converging operator on E is weakly compact.

Pełczynski [**17**] showed that if Ω is a compact Hausdorff space, then the space $C(\Omega)$ has property (V). In [**14**] the authors showed that more generally any Banach space X whose dual is isometric to an L_1-space has property (V). This fact was crucial for the proof of the following proposition which was proved in [**20**] and which we shall use later.

Proposition 1. *Let X, Y and Z be Banach spaces such that X^* is isometric to an L_1-space, and let $T : X\hat{\otimes}_\varepsilon Y \to Z$ be an unconditionally converging operator, then the set*

$$\{S(T^*z^*) \mid ||z^*|| \le 1\}$$

is uniformly countably additive in $M(K, Y^*)$.

Proposition 2. *The following properties of a Banach space X are equivalent*:

(i) *X^* is isometric to an L^1-space*;

(ii) *For any Banach spaces Y and Z, every bounded linear operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ extends to a bounded linear operator $\hat{T} : C(K,Y) \to Z^{**}$ with $||\hat{T}|| = ||T||$.*

Proof. To show (i) $\Rightarrow$ (ii), let

$$S : (X\hat{\otimes}_\varepsilon Y)^* \to M(K, Y^*)$$

be the isometry mentioned above. It is enough to define $\hat{T} : C(K,Y) \to Z^{**}$ as the composition $\hat{T} = T^{**} \circ S^* \circ i$ where $i : C(K,Y) \to C(K,Y)^{**}$ denotes the natural embedding of $C(K,Y)$ into $C(K,Y)^{**}$. It is immediate that $\hat{T} = T$ on $\hat{X}\hat{\otimes}_\varepsilon Y$ and that $||\hat{T}|| = ||T||$. Conversely, assume (ii); then there exists a bounded linear operator $u : C(K) \to X^{**}$ such that $u(x) = x$ for all $x \in X$ and $||u|| = 1$. This in particular implies that u^* restricted to X^* is an isometry. Let $R : C(K)^* \to X^*$ be the bounded linear operator such that for each $\mu \in C(K)^*$, $R\mu$ is the restrict of μ to X. Then $P = u^* \circ R$ is a bounded linear projection of $C(K)^*$ onto $u^*(X^*)$ and $||P|| = 1$. An appeal to [**12**] shows that X^* is isometric to an L_1-space. □

In case Y is the scalar field, the above proposition follows immediately from [**15**].

Remark 1. Assume that X^* is isometric to an L^1-space and let Y and Z be any Banach spaces. To every bounded linear operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ one can associate a finitely additive vector measure G defined on the σ-field Σ of Borel subsets of K with values in $\mathcal{L}(Y, Z^{**})$ the space of all bounded linear operators from Y to Z^{**}. The measure $G : \Sigma \to \mathcal{L}(Y, Z^{**})$ is defined as follows:

$$G(B)(y) = T^{**}(S^*(1_B \otimes y))$$

for all $B \in \Sigma$ and all $y \in Y$. For each $z^* \in Z^*$, let G_{z^*} denote the element of $M(K, Y^*)$ such that

$$G_{z^*}(B)(y) = z^*(G(B)y$$

for all $B \in \Sigma$ and $y \in Y$, then it is not hard to check that

$$G_{z^*} = S(T^* z^*)$$

for all $z^* \in Z^*$. We can also define the semi-variation $||G||$ of G as follows. For each $B \in \Sigma$

$$||G||(B) = \sup\{|G_{z^*}|(B) \mid ||z^*|| \leq 1\}.$$

Since S is an isometry, it follows that $||G||(K) = ||T|| = ||\hat{T}||$.

In what follows we shall refer to G as *the measure representing* T. Our next result can be viewed as an extension of a theorem of [**10**], where the author showed that unconditionally converging (respectively, completely continuous) operators on $C(\Omega, Y)$ spaces satisfy conditions similar to conditions (ii) and (iii) of Theorem 3 below.

Theorem 3. *Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space. If $T : X\hat{\otimes}_\varepsilon Y \to Z$ is unconditionally converging (respectively, completely continuous), then*

(i) *$\hat{T}$ takes its values in Z,*

(ii) *there exists a nonnegative regular Borel measure λ on K such that $\lim_{\lambda(B)\to 0} ||G||(B) = 0$, and*

(iii) *for each Borel subset B of K, the operator $G(B) : Y \to Z$ is unconditionally converging (respectively, completely continuous).*

Proof. Let

$$S : (X\hat{\otimes}_\varepsilon Y)^* \to M(K, Y^*)$$

and

$$s : X^* \to M(K)$$

be the two isometries mentioned above. To prove (i), note that since the set $\{f \otimes y : f \in C(K), y \in Y\}$ is total in $C(K, Y)$, it is enough to check that $\hat{T}(f \otimes y) \in Z$ for each $f \in C(K)$ and $y \in Y$. For this, fix $f \in C(K)$ and $y \in Y$, then

$$\hat{T}(f \otimes y) = T^{**}(S^*(f \otimes y)). \tag{2}$$

Let $T_y : X \to Z$ such that

$$T_y(x) = T(x \otimes y) \qquad \text{for all } x \in X.$$

It follows easily from (1) that, for any Borel subset B of K,

$$T^{**}(S^*(1_B \otimes y)) = T_y^{**}(s^*(1_B)). \tag{3}$$

To see this, let z^* be an arbitrary element in Z^*, then

$$\begin{aligned} T^{**}(S^*(1_B \otimes y))(z^*) &= S^*(1_B \otimes y)(T^* z^*) \\ &= S(T^* z^*)(1_B \otimes y) \\ &= \langle y, S(T^* z^*)\rangle(1_B) \end{aligned}$$

but, by (1),

$$\langle y, S(T^* z^*)\rangle = s(\langle y, T^* z^*\rangle).$$

On the other hand,

$$\langle y, T^* z^*\rangle = T_y^* z^*.$$

Hence,

$$\begin{aligned} s(\langle y, T^* z^*\rangle)(1_B) &= s(T_y^* z^*)(1_B) \\ &= s^*(1_B)(T_y^* z^*) \\ &= T_y^{**}(s^*(1_B))(z^*). \end{aligned}$$

Since T is unconditionally converging, it follows that $T_y : X \to Z$ is unconditionally converging. Hence, T_y is weakly compact by [**14**]. Thus, T_y^{**} takes its values in Z. The equations (2) coupled with (3) now imply that $\hat{T}$ takes its values in Z.

To prove (ii), note that by Proposition 1 we have that the set

$$\{S(T^* z^*) \mid ||z^*|| \leq 1\}$$

is uniformly countably additive in $M(K, Y^*)$. The existence of the measure $\lambda \geq 0$ follows from the general vector measure techniques (see [**9**, p. 11]).

Finally, to show (iii), fix B a Borel subset of K. First note that it follows immediately from (3) that $G(B)$ is an operator on Y with values in Z. To show that $G(B) : Y \to Z$ is unconditionally converging, it is enough to show that for any weakly unconditionally Cauchy series $\sum y_n$ in Y, $\lim_{n\to\infty} G(B)y_n = 0$. For this, let $\sum y_n$ be a weakly unconditionally Cauchy series in Y. Then, for each $n \geq 1$, we have

$$||G(B)y_n|| \leq ||T_{y_n}||. \tag{4}$$

To see this, note that for all $y \in Y$,

$$\begin{aligned} G(B)y &= T^{**}(S^*(1_B \otimes y)) \\ &= T_y^{**}(s^*(1_B)). \end{aligned}$$

Thus, we have (4) since s is an isometry and $||1_B|| \leq 1$.

Moreover, since T is unconditionally converging, one can easily show (see [**20**]) that

$$\lim_{n\to\infty} \sup_{||x||\leq 1} ||T(x \otimes y_n)|| = 0.$$

That is,

$$\lim_{n\to\infty} ||T_{y_n}|| = 0.$$

Hence,

$$\lim_{n\to\infty} ||G(B)y_n|| = 0.$$

The case of the completely continuous operators follows in a similar fashion. □

Remark 2. In [**10**] the author asked whether conditions (ii) and (iii) characterize unconditionally converging operators on $C(\Omega, Y)$ spaces. In [**3**] the authors gave an example to show that this is not the case in geneal. Actually more can be said. In [**23**] it was shown that any time the space Y contains a subpsace isomorphic to c_0, one can exhibit a bounded linear operator $U : C(\Delta, Y) \to c_0$, here $\Delta = \{-1, 1\}^{\mathbf{N}}$ is the Cantor group, the operator U is not unconditionally converging, yet its representing measure satisfies conditions (ii) and (iii) of Theorem 3. Similar assertions can be made about completely continuous operators as soon as Y fails to have the Schur property (weakly compact sets are norm compact), see [**23**] for more details. In case the space Y contains no subspace isomorphic to c_0 (respectively, Y has the Schur property), we can offer the following result:

Theorem 4. *Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space and that Y contains no subspace isomorphic to c_0 (respectively, Y has the Schur property). Then every unconditionally converging operator $T : X \hat{\otimes}_\varepsilon Y \to Z$ (respectively, completely continuous) extends to an unconditionally converging operator (respectively, completely continuous) $\hat{T} : C(K, Y) \to Z$.*

Proof. In case the operator T is unconditionally converging and Y contains no subspace isomorphic to c_0, the needed result follows immediately from Theorem 3 and the following argument of [**8**] which we shall include for the sake of completeness. Indeed, it is enough to show that, for any weakly unconditionally Cauchy series $\sum_n \varphi_n$ in $C(K,Y)$, one has $\lim_{n\to\infty} ||\hat{T}\varphi_n|| = 0$. For this, let $\lambda \geq 0$ be a regular Borel measure whose existence is guaranteed by part (ii) of Theorem 3, and let $\sum_n \varphi_n$ be a weakly unconditionally Cauchy series in $C(K,Y)$. Then, for each $k \in K$, the series $\sum_n \varphi_n(k)$ is a weakly unconditionally Cauchy series in Y. By a result of [**2**], the series $\sum_n \varphi_n(k)$ is unconditionally convergent in Y for all $k \in K$. Let $\varepsilon > 0$ be given. By (ii) of Theorem 3, one can find $\delta > 0$ such that

$$||G||(B) < \varepsilon$$

whenever

$$\lambda(B) < \delta.$$

Since $\lim_{n\to\infty} ||\varphi_n(k)|| = 0$ for each $k \in K$, there exists a compact subset K_δ of K such that

$$\lambda(K\backslash K_\delta) < \delta$$

and

$$\lim_{n\to\infty} \sup_{k\in K_\delta} ||\varphi_n(k)|| = 0.$$

To finish the proof, note that

$$\hat{T}\varphi_n = \int_{K_\delta} \varphi_n \, dG + \int_{K\backslash K_\delta} \varphi_n \, dG$$

and therefore, for each $n \geq 1$,

$$||\hat{T}\varphi_n|| \leq \sup_{k\in K_\delta} ||\varphi_n(k)|| \, ||G||(K) + \sup_n ||\varphi_n|| \, ||G||(K\backslash K_\delta).$$

For the case when T is completely continuous, one can proceed in a similar fashion (see [**23**]). □

We turn our attention now to those operators T that are weakly completely continuous. As we pointed out earlier, since every weakly completely continuous operator is unconditionally converging, we can very quickly obtain conditions (i) and (ii) of Theorem 3, we also have a statement similar to condition (iii) of that theorem.

Theorem 5. *Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space, and let $T : X\hat{\otimes}_\varepsilon Y \to Z$ be a weakly completely continuous operator; then*

(i) $\hat{T} : C(K, Y) \to Z$,

(ii) *there exists a nonnegative regular Borel measure λ on K such that $\lim_{\lambda(B)\to 0} ||G||(B) = 0$, and*

(iii) *for each Borel subset B of K, the operator $G(B) : Y \to Z$ is weakly completely continuous.*

Proof. We need only show (iii). For this, let

$$\beta_1(Y) = \{y^{**} \in Y^{**} \mid y^{**} \text{ is the weak}^* \text{ limit of some weakly Cauchy sequence in } Y\}.$$

Fix B a Borel subset of K. To show that $G(B) : Y \to Z$ is weakly completely continuous, it is enough to show (see [**11**, p. 644]) that $G(B)^{**}(\beta_1(Y)) \subset Z$. For this, let (y_n) be a weakly Cauchy sequence in Y and let $y^{**} \in \beta_1(Y)$ be its weak*-limit. Thus,

$$G(B)^{**}y^{**} = \text{weak}^*\text{-limit } G(B)y_n.$$

But for each $n \geq 1$,

$$G(B)y_n = T^{**}(S^*(1_B \otimes y_n)).$$

Since S^* and T^{**} are weak* to weak* continuous, one can easily check that

$$G(B)^{**}y^{**} = T^{**}(S^*(1_B \otimes y^{**})) \tag{5}$$

where $1_B \otimes y^{**}$ is the element of $M(K, Y^*)^*$ such that, for each $\lambda \in M(K, Y^*)$,

$$1_B \otimes y^{**}(\lambda) = y^{**}(\lambda(B)).$$

On the other hand, let $T_{y^{**}} : X \to Z$ be the operator defined by

$$\begin{aligned} T_{y^{**}}(x) &= T^{**}(x \otimes y^{**}) \\ &= \text{weak-limit } T(x \otimes y_n). \end{aligned}$$

It is clear that $T_{y^{**}}(x) \in Z$ for each $x \in X$, since T is weakly completely continuous and the sequence $(x \otimes y_n)$ is obviously weak Cauchy in $X \hat{\otimes}_\varepsilon Y$. Moreover, $T_{y^{**}}$ is linear and bounded because

$$||T_{y^{**}}|| \leq ||T|| \, ||y^{**}||.$$

We claim that $T_{y^{**}}$ is weakly compact. Since X^* is isometric to an L^1-space, it is enough by [**14**] to check that $T_{y^{**}}$ is unconditionally converging. To do that, let $\sum x_n$ be a weakly unconditionally Cauchy series in X. For each $x \in X$, we let T_x be the operator from Y to Z defined by $T_x(y) = T(x \otimes y)$ for all $y \in Y$. With this in mind, we have

$$T_{y^{**}}(x_n) = T^{**}(x_n \otimes y^{**}) = T^{**}_{x_n}(y^{**})$$

for each $n \geq 1$. Thus,

$$||T_{y^{**}}(x_n)|| \leq ||T_{x_n}|| \, ||y^{**}||.$$

Since T is in particular unconditionally converging, one can argue as in [**20**] to show that

$$\lim_{n\to\infty} \sup_{||y|| \leq 1} ||T(x_n \otimes y)|| = 0;$$

thus,

$$\lim_{n\to\infty} ||T_{x_n}|| = 0.$$

Therefore, $T_{y^{**}} : X \to Z$ is weakly compact. Now an easy computation shows that

$$T^{**}_{y^{**}}(s^*(1_B)) = T^{**}(S^*(1_B \otimes y^{**})). \tag{6}$$

It follows from (5) and (6) that

$$G(B)^{**} y^{**} = T^{**}_{y^{**}}(s^*(1_B)).$$

Thus, $G(B)^{**}y^{**} \in Z$. This completes the proof. □

In what follows, we offer some applications to the above theorem. Our first application extends a result of [**4**].

Theorem 6. *Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space and that Y is weakly sequentially complete. Every bounded linear operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ whose representing G is such that $\lim_{\lambda(B)\to 0} ||G||(B) = 0$ for some nonnegative regular Borel measure λ, extends to a weakly completely continuous operator $\hat{T}$ on $C(K, Y)$. In particular, T is weakly completely continuous.*

Proof. Let $T : X\hat{\otimes}_\varepsilon Y \to Z$ be a bounded linear operator with representing measure G, and let $\lambda \geq 0$ be a regular Borel measure on K such that

$$\lim_{\lambda(B)\to 0} ||G||(B) = 0. \tag{7}$$

As pointed out before, the above condition (7) is equivalent to the fact that the set

$$\{S(T^*z^*) \mid ||z^*|| \leq 1\}$$

is uniformly countably additive in $M(K, Y^*)$, in particular this impleis that for each $y \in Y$, the set $\{\langle y, S(T^*z^*)\rangle \mid ||z^*|| \leq 1\}$ is relatively weakly compact in $M(K)$ but

$$\langle y, S(T^*z^*)\rangle = s(T_y^*z^*).$$

Since s is an isometry, it follows that the set $\{T_y^*z^* : ||z^*|| \leq 1\}$ is relatively weakly compact in X^*. Hence, for each $y \in Y$, the operator $T_y : X \to Z$ is weakly compact and therefore $\hat{T}$ is a bounded linear operator from $C(K, Y)$ into Z. By [**4**], $\hat{T}$ is weakly completely continuous. □

Finally, we offer the following theorem which was proved in [**4**] for the case where $X = C(\Omega)$ space.

Theorem 7. *Let X, Y and Z be Banach spaces. Assume that X^* is isometric to an L^1-space and that Y^* and Y^{**} have the Radon-Nikodym* *property. Then an operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ is weakly completely continuous if and only if its representing measure G satisfies the following conditions: for each Borel subset B of K, the operator $G(B) : Y \to Z$ is weakly completely continuous and $\lim_{\lambda(B)\to 0} ||G||(B) = 0$, for some nonnegative regular Borel measure λ on K.*

Proof. Let $T : X\hat{\otimes}_\varepsilon Y \to Z$ be such that $G(B) : Y \to Z$ is weakly completely continuous for each Borel subset B of K and $\lim_{\lambda(B)\to 0} ||G||(B) = 0$ for some $\lambda \geq 0$. As noted in the previous theorem, the above hypotheses imply that $\hat{T}$ takes its values in Z. Since Y^* has the Radon-Nikodym property, then in particular Y contains no subspace isomorphic to l_1. By Rosenthal's theorem [**19**], it follows that for each Borel subset B of K the map $G(B) : Y \to Z$ is a weakly compact operator. Since Y^* and Y^{**} have the Radon-Nikodym property, an appeal to [**8**] shows that $\hat{T}$ is in fact weakly compact. □

Remark 3. One should note at this stage that when we deal with weakly compact (respectively, compact) operators on $X\hat{\otimes}_\varepsilon Y$, where X^* is isometric to an L^1-space, an operator T on $X\hat{\otimes}_\varepsilon Y$ is weakly compact (respectively, compact) if and only if $\hat{T}$ on $C(K, Y)$ is weakly compact (respectively, compact). This follows directly from the fact that T is weakly compact (respectively, compact) if and only if T^{**} is weakly compact (respectively, compact) and $\hat{T} = T^{**} \circ S^* \circ i$. While we cannot easily claim similar results for unconditionally converging, completely continuous or weakly completely continuous operators, it seems natural to conjecture that this would be the case in general.

Let us now turn our attention to a new class of Banach spaces. In [**1**, Theorem 8.3], Aron, Cole and Gamelin isolated a family of Banach spaces that satisfies the following condition:

A Banach space E belongs to this family if any symmetric operator $T : E \to E^*$ is weakly compact, T is said to be symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in E$. Influenced by that, let us give the following definition:

Definition 2. A Banach space E is said to have the *property* (w) if and only if every bounded linear operator $T : E \to E^*$ is weakly compact.

In [**1**], the authors noticed that if Ω is a compact Hausdorff space, then the space $C(\Omega)$ has the property (w). Actually more can be said. But first, let us recall the following definition.

Definition 3. A Banach space E is said to have *Pełczynski's property* (V^*) if a subset C of E is relatively weakly compact whenever

$$\lim_{n\to\infty} \sup_{c\in C} x_n^*(c) = 0$$

for all weakly unconditionally Cauchy series $\sum x_n^*$ in E^*.

In [**17**], Pełczynski showed that if E has property (V^*) then E is weakly sequentially complete. He also showed that if E has property (V), then E^* has property (V^*). It was observed in [**22**] that the converse is not true.

Proposition 8. *Any Banach space E such that E^* has the property (V^*) has the property* (w)*. In particular, any Banach space with the property* (V) *has the property* (w)*.*

Proof. Let $T : E \to E^*$ be a bounded linear operator, and let $C = T(B_E)$ be the image of the unit ball of E by T. Let $\sum_{i=1}^{\infty} u_i$ be a weakly unconditionally Cauchy series in E^{**}. Since E^* is weakly sequentially complete, then T^* is unconditionally converging and therefore $\lim_n ||T^*(u_n)|| = 0$, but

$$\lim_n \sup_{x^*\in C} u_n(x^*) = \lim_n ||T^*(u_n)|| = 0.$$

Hence, C is relatively weakly compact since E^* has (V^*). □

Corollary 9. *Any $\mathcal{L}_\infty$ space has the property* (w)*.*

Proof. Let E be an $\mathcal{L}_\infty$ space, then E^* is an $\mathcal{L}_1$-space, and hence E^* has (V^*). Apply Proposition 8 to conclude. □

The space constructed by Bourgain and Delbaen in [**6**] is an $\mathcal{L}_\infty$ space, so it has (w) but does not have (V) since it has the Schur property.

The spaces X constructed by Pisier [**18**] are such that every bounded linear operator from X to X^* is integral and hence weakly compact so all these spaces X have the property (w).

It is easy to see that any complemented subspace of a Banach space with the property (w) has this property. Since l_1 does not have the property (w), then any space E with the property (w) cannot contain a complemented copy of l_1, and, therefore, E^* cannot contain a copy of c_0 [**2**]. With the help of Proposition 8 and a result of [**22**], one can deduce the following:

Corollary 10. *Let E be a Banach space such that E^* is complemented in a Banach lattice or E^* is isomorphic to a closed subspace of an order continuous Banach lattice. Then the following statements are equivalent*:

(i) *E has the property* (w);

(ii) *E does not contain a complemented copy of l_1*;

(iii) *E^* has the property (V^*).*

If a Banach space E does not contain a copy of l_1 and E^* is weakly sequentially complete, then E has the property (w). To see this, notice that any bounded linear operator $T : E \to E^*$ is weakly precompact (the image of the unit ball of E by T does not contain a copy of l_1) and therefore is weakly compact since E^* is weakly sequentially complete.

We can now offer the following theorem:

Theorem 11. *Let X be the Banach space whose dual is isometric to an L_1-space, and let Y be a Banach space that does not contain a copy of l_1 and assume that Y^* is weakly sequentially complete. Then $X\hat{\otimes}_\varepsilon Y$ has the property* (w).

Proof. Let K be the closed unit ball of X^* equipped with the weak* topology. As pointed out earlier, it was shown in [**20**] that the dual of $X\hat{\otimes}_\varepsilon Y$ is isometrically isomorphic to a closed subspace of the dual of $C(K, Y)$ which is weakly sequentially complete by [**25**]. This shows that every operator from $X\hat{\otimes}_\varepsilon Y$ to its dual is unconditionally converging. We will be done if we can show that for any Banach space Z, any operator $T : X\hat{\otimes}_\varepsilon Y \to Z$ that is unconditionally converging is weakly precompact. By Proposition 1, the set

$$\{S(T^*(z^*)); z^* \in Z^* \text{ and } ||z^*|| \leq 1\}$$

is uniformly countably additive as a subset of $(C(K,Y))^*$ when the latter is identified with $M(K,Y^*)$ the space of all regular Y^*-valued measures on K that are of bounded variation. Let λ be a control measure for this set [**9**], and let $(f_n)_{n\geq 1}$ be a sequence in the unit ball of $X\hat{\otimes}_\varepsilon Y$. Since Y does not contain a copy of l_1, it follows from [**5, 25**] that there is a subsequence $(f_{n_k})_{k\geq 1}$ that is weak Cauchy in $L_1(Y)$, the space of all λ-Bochner integrable functions defined on K with values in E [**9**]. To finish the proof, one has to show that the sequence $(T(f_{n_k}))_{k\geq 1}$ is weak Cauchy in Z. For this, fix $z^* \in Z^*$ with $||z^*|| \leq 1$. Since $S(T^*(z^*))$ is λ-continuous, then it has a weak* λ-derivative [**26**] $g : K \to Y^*$ so that, for every $f \in X\hat{\otimes}_\varepsilon Y$, one has

$$\langle f, S(T^*(z^*))\rangle = \int_K \langle f(t), g(t)\rangle \, d\lambda(t)$$

and

$$||S(T^*(z^*))|| = \int_K ||g(t)|| \, d\lambda(t).$$

Now let $\varepsilon > 0$ and choose $N > 0$ so that

$$\int_{[||g||>N]} ||g(t)|| \, d\lambda(t) < \varepsilon.$$

Let $h = g\,1_{[||g||<N]}$, then $h \in (L_1(Y))^*$. To conclude, notice that

$$\begin{aligned}\langle z^*, T(f_{n_k})\rangle &= \langle T^*(z^*), f_{n_k}\rangle \\ &= \langle S(T^*(z^*)), f_{n_k}\rangle \\ &= \int_K \langle f_{n_k}(t), g(t)\rangle \, d\lambda(t) \\ &= \int_{[||g||>N]} \langle f_{n_k}(t), g(t)\rangle \, d\lambda(t) \\ &\quad + \int_{[||g||\leq N]} \langle f_{n_k}(t), g(t)\rangle \, d\lambda(t). \qquad \square\end{aligned}$$

In the special case where $X = C(K)$, the above theorem can be deduced from an unpublished result of C. Fierro [**13**].

The following question now arises naturally.

Question 1. Does the property (w) pass from any Banach space Y to $C(K, Y)$?

By Proposition 8 and [**21**], the answer is yes if Y is Banach lattice. Actually more can be said as the following proposition shows:

Proposition 12. *Let X be the Banach space whose dual is isometric to an L_1-space, and let Y be a Banach lattice having the property* (w)*; then $X \hat{\otimes}_\varepsilon Y$ has the property* (w)*.*

Proof. The dual of $X \hat{\otimes}_\varepsilon Y$ is isomorphic to a subspace of the dual of a $C(K, Y)$ space [**20**]. Moreover, the space $C(K, Y)$ cannot contain a complemented copy of l_1 by [**21**] and therefore its dual which is a Banach lattice has (V^*). An appeal to Proposition 8 finishes the proof. □

Most of the Banach spaces Y known to us that have (w) are such that their duals have the property (V^*). However, the spaces constructed by Bourgain and Pisier [**7**] have the property (w) but nothing can be said about their duals according to Pisier (private communication).

Question 2. Suppose that a Banach space Y has the property (w); does it follow that Y^* has (V^*) or at least that Y^* is weakly sequentially complete?

Note added in proof. Recently D. Leung has shown that the answer to Question 2 is negative.

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