ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

FREE-BY-FINITE CYCLIC AUTOMORPHISM GROUPS

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ABSTRACT. Motivated by an example of J.L. Dyer, we consider a group G whose automorphism group is isomorphic to the fundamental group of a graph of locally cyclic groups. The conclusion is that G is infinitely generated Abelian and $\mathrm{Aut}\, G$ is free-by-finite cyclic with all torsion elements having order dividing 8, 10, 12 or 30.

1. Introduction. It is a natural problem to attempt to characterize those groups which arise as the full automorphism group of some group, or, at least, to ask whether such groups share any common structural features. While little (if any) evidence exists to suggest any reasonably general answers to this question, some progress has been made (notably in [**3, 5, 12**]) on the less ambitious problem of finding large classes of groups whose structure precludes them from being automorphism groups. The purpose of this note is to record some further examples of this type. (For other work along these lines see, for example, [**4, 7, 10, 11, 14**].)

The result described here is combinatorial in flavor and evolved from an attempt to understand in a more general context an observation of J.L. Dyer [**4**] that $SL_2(\mathbf{Z})$ is not the automorphism group of any finitely generated group. Dyer's demonstration depends on the well-known presentation of $SL_2(\mathbf{Z})$ as an amalgam $\mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$. A fact which makes this example of particular interest is that, while $\mathbf{Z}_2 * \mathbf{Z}_2$ is the only freely decomposable automorphism group [**5**], among amalgams in general, automorphism groups appear to be not at all uncommon [**9**]. Thus, it is reasonable to ask whether Dyer's example is an isolated one or whether it is representative of a class of examples. The gist of the following result is that Dyer's observation applies to the fundamental group of any graph of locally cyclic groups.

Theorem. *Let A be a treed* HNN *group with locally cyclic vertex*

Received by the editors on September 20, 1990.
Mathematics Subject Classification (1985). Primary 20F28, Secondary 20E06.

groups which is neither finite cyclic nor infinite dihedral. If A is isomorphic to the full automorphism group of a group G, then

(a) *G is infinitely generated Abelian and, moreover, is indecomposable and torsion-free or else contains such a group as a characteristic direct factor of index* 2.

(b) *All vertex stabilizers in A are finite of order dividing* $8, 10, 12$, *or* 30.

(c) *A contains a free normal subgroup B such that A/B is cyclic of order dividing* 120.

Corollary 1. *With the exception of the finite cyclic and infinite dihedral groups, the fundamental group of a graph of locally cyclic groups is not the automorphism group of any finitely generated group.*

Corollary 2. *Except for the infinite dihedral group, a nontrivial free-by-finite cyclic group whose center contains no element of order* 2 *is not the automorphism group of any group.*

Corollary 3. *An infinite free-by-finite cyclic group which contains an element of finite order not dividing* $8, 10, 12$, *or* 30 *is not the automorphism group of any group.*

The main tool in the proof of the theorem is the Bass-Serre theory of groups acting on trees. (See [**2, 13**].)

2. Proof of Theorem. Suppose that A and G satisfy the hypotheses of the theorem. In the Bass-Serre terminology, A is the fundamental group of a graph of locally cyclic groups, and so there is a (directed) tree T on which A acts with locally cyclic vertex stabilizers. Since the central quotient $G/Z(G)$ is isomorphic to the group of inner automorphisms of G, T is a G-tree, and the stabilizer in G of any vertex of T is Abelian.

To establish notation, we briefly summarize the main result of the Bass-Serre theory as it applies to G. With the action of G on T is associated a graph of groups consisting of the following: the quotient graph $X = T/G$; for each vertex v of X, a vertex group $G(v)$ which is isomorphic to the stabilizer in G of any preimage of v in T; for each edge e, an edge group $G(e)$ which is a subgroup of the vertex group $G(ie)$ corresponding to the initial vertex of e (and which is isomorphic to the stabilizer in G of any preimage of e in T) and finally; for each edge e of X, a monomorphism f_e from $G(e)$ to the vertex group $G(te)$ corresponding to the terminal vertex of e. If M is a maximal subtree of X, then the fundamental theorem ([**13**, Theorem 13, **2**, I.4.1]) asserts that G has the following presentation:

Generators: The elements of all the vertex groups $G(v)$ together with elements x_e, one for each edge e of X.

Relations: The internal relations for each $G(v)$, plus $x_e = 1$ if e is in M, and $x_e^{-1} g x_e = f_e(g)$ for all edges e of X and $g \in G(e)$.

The vertex groups may be identified as subgroups of G, and under this identification, each vertex stabilizer in the action of G on T is conjugate in G to a unique vertex group. The elements x_e, $e \notin M$, may be identified with elements of G which we shall call "HNN indeterminates." They form a basis for a free subgroup of G.

We suppose first that G is not Abelian.

Observe that in this case, $G/Z(G)$ cannot be free. For if so, then $G = Z(G) \times H$ where H is free of rank at least 2 and $\operatorname{Aut} H$ is isomorphic to a subgroup of $\operatorname{Aut} G$. But H admits automorphisms which invert one element of a free basis and fix the rest, so $\operatorname{Aut} H$ contains $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since every finite subgroup of $\operatorname{Aut} G$ fixes some vertex of T [**13**, Proposition 19] and hence, is cyclic, we have a contradiction.

Since $G/Z(G)$ is not free, some vertex of T has nontrivial stabilizer in $G/Z(G)$ [**13**, Theorem 4] and so some vertex group $H = G(v)$ is not contained in $Z(G)$. Let $h \in H \backslash Z(G)$ and i_h be the corresponding inner automorphism of G. From the presentation of G described above and the fact that the vertex groups are Abelian, it is apparent that G admits an automorphism α which inverts all vertex groups and fixes all HNN indeterminates. Then α and αi_h each have order at most 2 and so each fixes a vertex of T. It follows [**13**, Corollary 1, p. 64] that $\langle \alpha, i_h \rangle$ fixes a vertex of T. Since vertex stabilizers in $\operatorname{Aut} G$ are locally cyclic, α centralizes i_h and so $i_h = i_h^{-1}$ and $\alpha \in \langle i_h \rangle$.

If $\alpha = i_h$, then for any element x of any vertex subgroup of G, $x^{-1} = x^\alpha = x^h$ and so $hxZ(G)$ has order at most 2 in $G/Z(G)$. Thus, h, x, and hx all fix vertices in T whence, by [**13**, Corollary 1, p. 64], $\langle x, h\rangle$ stabilizes a vertex of T. But then, $\langle x, h\rangle$ is Abelian and so $x^\alpha = x$. Therefore, α fixes all generators of G, and so $\alpha = 1$, a contradiction since i_h is nontrivial.

Thus, $\alpha = 1$ and so every vertex subgroup of G is an elementary Abelian 2-group. In particular, this is the structure of $Z(G)$.

We claim that if e is an edge of X *not* in the maximal tree M, then the corresponding edge group $G(e)$ is exactly $Z(G)$. For suppose that e is such an edge and g is an element of $G(e)\backslash Z(G)$. Let $x = x_e$ be the corresponding HNN indeterminate. Since $G(e)$ is Abelian, G admits an automorphism β which maps x to gx and fixes all other generators. Then $|\beta| = |i_g| = 2$ and $\langle \beta, i_g\rangle$ is Abelian so $|\langle \beta, i_g\rangle| \leq 4$. Therefore, $\langle \beta, i_g\rangle$ fixes a vertex of T and consequently is cyclic, whence $\beta = i_g$ and $gxg = g^{-1}xg = x^\beta = gx$. Since $g \neq 1$, this is a contradiction, so the claim is proved.

We conclude from this that if B is the subgroup of G generated by the vertex groups (so $B/Z(G)$ is a free product of $\mathbf{Z}_2$'s), then $G/Z(G)$ is the free product of $B/Z(G)$ with the free subgroup generated by the $x_eZ(G)$'s, $e \notin M$. Since G is not Abelian, $G/Z(G)$ is not cyclic and so has $\mathbf{Z}_2 \times \mathbf{Z}_2$ as a homomorphic image. Thus, if $Z(G)$ is nontrivial, $\operatorname{Hom}(G/Z(G), Z(G))$ contains a copy of $\mathbf{Z}_2 \times \mathbf{Z}_2$. But $\operatorname{Hom}(G/Z(G), Z(G))$ is isomorphic to the centralizer in $\operatorname{Aut} G$ of the chain $1 \trianglelefteq Z(G) \trianglelefteq G$. Since every finite subgroup of A fixes a vertex of T and, hence, is cyclic, it follows that $Z(G)$ must be trivial. Thus, $G = B * F$ where F is the free group generated by $\{x_e : e \notin M\}$ and each nontrivial vertex group of G has order 2.

Suppose that $F \neq 1$ so $x = x_e \neq 1$ for some edge e of X. Since G is not free, there is also at least one nontrivial vertex group, say $G(v) = \langle g\rangle$. Then G admits an automorphism γ which inverts x and fixes all other generators of G (including g). Since $\langle x, g\rangle \cong \langle x\rangle * \langle g\rangle$, $x^g \neq x^{-1}$ and so γ and i_g are distinct commuting elements of order 2 in $\operatorname{Aut} G$, contradicting the fact that all finite subgroups of $\operatorname{Aut} G$ are cyclic. We conclude that $F = 1$ and G is a free product of $\mathbf{Z}_2$'s.

If there are three or more factors in this free product, then by permuting generators of these, we get a subgroup of $\operatorname{Aut} G$ isomorphic to the symmetric group S_3. Again, because all finite subgroups of $\operatorname{Aut} G$ are cyclic, this is absurd. On the other hand, if there are only two factors, then G (and, hence, $\operatorname{Aut} G$) is infinite dihedral, and this case is excluded by hypothesis. This contradiction completes the proof of the Theorem in the case that G is non-Abelian.

We assume for the remainder of the proof that G is Abelian.

Since $\operatorname{Aut} G$ does not contain $\mathbf{Z}_2 \times \mathbf{Z}_2$, the inversion map ε is the unique element of order 2 in $\operatorname{Aut} G$. Also G is either indecomposable or is a direct product $Z \times H$ where $Z \cong \mathbf{Z}_2$ and H is indecomposable. In the latter case, $\operatorname{Hom}(G/Z, Z) = 0$ (else, $\operatorname{Aut} G$ contains an involution distinct from ε) and so $H = H^2 = G^2$ which is characteristic in G. In particular, $\operatorname{Aut} G \cong \operatorname{Aut} H$. Because of the assumed structure of $\operatorname{Aut} G$, it is easily seen that H (or in the former case, G) is neither cyclic nor quasi-cyclic and so indecomposability forces it to be torsion free and infinitely generated. The first conclusion of the Theorem is now proved and,since the remaining statements concern only the structure of $\operatorname{Aut} G$, we will henceforth assume that G is torsion-free.

We note at this point that if α is an element of the stabilizer in $\operatorname{Aut} G$ of some vertex of T, then by [**13**, Corollary 1, p. 66], $\langle \alpha, \varepsilon \rangle$ fixes a vertex of T and so is locally cyclic. It follows that every vertex stabilizer is periodic.

To prove statement (b) of the Theorem, we adapt a technique used by Hallett and Hirsch [**6**] and extended by Dixon and Evans [**3**]. Let α be a torsion element in $\operatorname{Aut} G$, and let $S(\alpha)$ be the ring of endomorphisms generated by α. Let $g(x)$ be an integral polynomial of minimal degree such that $g(\alpha) = 0$. If $f(x) \in \mathbf{Z}[x]$ such that $f(\alpha) = 0$, then $g(x)$ divides $f(x)$ in $\mathbf{Z}[x]$. Therefore, $S(\alpha) \cong \mathbf{Z}[x]/(g(x))$. Also, if α has order m, then $g(x)$ divides $x^m - 1 = \prod_{d|m} \Phi_d(x)$ (where $\Phi_d(x)$ denotes the d^{th} cyclotomic polynomial).

Suppose $\Phi_d(x)|g(x)$, so $d|m$ and $(g(x)) \subseteq (\Phi_d(x))$. Then there is a ring homomorphism $S(\alpha) \to \mathbf{Z}[x]/(\Phi_d(x)) \cong Q(\varepsilon_d)$ (where ε_d is a complex primitive d^{th} root of unity). Since Lemma 4.1 of [**3**] actually holds for every $n > 2$ (not just for prime powers), the argument in Theorem 4.2 of that paper yields that the group $U(S(\alpha))$ of units of $S(\alpha)$ contains a free Abelian subgroup of rank $(\phi(d)/2) - 1$ if $d > 2$. (See [**8**, 2.2.12].) But since the vertex stabilizers in $\operatorname{Aut} G$ are periodic, any torsion-free subgroup of $\operatorname{Aut} G$ is free [**13**, Theorem

4], so $(\phi(d)/2) - 1 \leq 1$. Thus, $\phi(d) \leq 4$, and it is easily verified that $d \in D = \{2, 3, 4, 5, 6, 8, 10, 12\}$.

Now if $m = p^k$ for some prime p, then $\Phi_m(x)$ must divide $g(x)$, for if not, then $g(x)$ divides $(x^m - 1)/\Phi_m(x) = x^{p^{k-1}} - 1$, so $\alpha^{p^{k-1}} = 1$, contradicting the fact that $|\alpha| = m$. Thus, if m is a prime power, then $m = 2, 3, 4, 5$, or 8. It follows that the possible orders of torsion elements in $\operatorname{Aut} G$ are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, or 30. If we can eliminate 20 and 24 as possibilities, we shall have proved statement (b) of the Theorem.

Suppose there exists in $\operatorname{Aut} G$ an element α of order 20. Since inversion is the unique involution in $\operatorname{Aut} G$, $\alpha^{10} + 1 = 0$, and so $g(x)$ divides $x^{10} + 1 = \Phi_4(x)\Phi_{20}(x)$. But 20 is not in the set D defined above, and so $\Phi_{20}(x)$ does not divide $g(x)$. Thus, $g(x) = \Phi_4(x)$ which divides $x^4 - 1$. This implies that $\alpha^4 = 1$, a contradiction.

Similarly, if $|\alpha| = 24$, $g(x)$ divides $x^{12} + 1 = \Phi_8(x)\Phi_{24}(x)$ and, since $24 \notin D$, $g(x) = \Phi_8(x)$. Since $\Phi_8(x)$ divides $x^8 - 1$, we again have a contradiction. Thus, the proof of the second conclusion of the Theorem is complete.

To prove statement (c), we recall that A is the fundamental group of a graph of groups consisting, say, of a graph Y $(= T/A)$, the vertex groups $A(v)$, edge groups $A(e)$, and corresponding monomorphisms f_e from $A(e)$ to $A(te)$. As a consequence of statement (b), we may choose for every vertex v of Y, a monomorphism $h_v : A(v) \to \mathbf{Z}_{120}$. For any edge e of Y and any generator g of $A(e)$, $h_{ie}(g)$ and $h_{te}(f_e(g))$ are conjugate by an automorphism of $\mathbf{Z}_{120}$ and hence, by an element $h(e)$ of the holomorph $\operatorname{Hol}(\mathbf{Z}_{120})$. As in [**2**, I.7.2], there is a homomorphism $j : A \to \operatorname{Hol}(\mathbf{Z}_{120})$ which is injective on each vertex group and such that $j(A) \leq \mathbf{Z}_{120}$ (identifying $\mathbf{Z}_{120}$ as a normal subgroup of its holomorph). By [**2**, I.7.10], the kernel of j is free, and so the proof of statement (c) (and of the Theorem) is complete. □

Remark . The possibility that G is not quite torsion-free but is isomorphic to $\mathbf{Z}_2 \times H$ where H is torsion-free is probably unavoidable, but it is perhaps worth noting that the structure of $\operatorname{Aut} G$ can be pinned down quite precisely in this case. For, as shown above, $H = H^2$ and so the map $h \mapsto h^2$ defines an automorphism of H which extends to a central element θ of infinite order in $\operatorname{Aut} G$. Since all torsion-free subgroups of $\operatorname{Aut} G$ are free, the subgroup B (described in statement (c) of the Theorem) intersects $\langle\theta\rangle$ nontrivially (and so has nontrivial center). Hence, B is cyclic and $\operatorname{Aut} G/\langle\theta\rangle$ is finite. It is then not difficult to see that $\operatorname{Aut} G$ must be a semi-direct product of a finite cyclic normal subgroup (of order 2, 4, 6, 8, 10, 12 or 30) with an infinite cyclic subgroup. So, if this short list of possibilities is excluded in addition to the finite cyclic and infinite dihedral groups, then G is necessarily indecomposable torsion-free Abelian (and $G^n \neq G$ for every integer n).

3. The Corollaries. Corollary 1 is immediate.

To prove Corollaries 2 and 3, suppose that A has a nontrivial free normal subgroup B such that A/B is finite cyclic. By [**13**, Theorem 4] or [**2**, I.8.3], B acts freely on a tree and so, by [**2**, IV.1.5], A acts on a tree with B-free vertex set. Each vertex stabilizer in A is then finite cyclic and so the Theorem applies. If $A = \operatorname{Aut} G$, G admits inversion as a nontrivial automorphism, so Corollary 2 is proved. Corollary 3 follows from the fact that every element of finite order stabilizes some vertex.

4. Some examples. We finish with some examples to confirm that the set $\{8, 10, 12, 30\}$ which occurs in statement (b) of the Theorem and in Corollary 2 cannot be reduced. Following what has become a standard recipe for constructing examples, we exploit a remarkable theorem of Corner [**1**] that any countable reduced torsion-free ring R with 1 is isomorphic to the endomorphism ring of a countable reduced torsion-free Abelian group G (and, hence, the group $U(R)$ of units of R is isomorphic to $\operatorname{Aut} G$).

If $1 \leq n \leq 3$ and ε_{2n} is a primitive complex $2n^{\text{th}}$ root of unity, then $U(\mathbf{Z}[\varepsilon_{2n}]) \cong \mathbf{Z}_{2n}$, and so, if $R = \mathbf{Z}[\varepsilon_{2n}][x, 1/x]$, then $U(R) = U(\mathbf{Z}[\varepsilon_{2n}]) \times \mathbf{Z} \cong \mathbf{Z}_{2n} \times \mathbf{Z}$. If $4 \leq n \leq 6$, then $U(\mathbf{Z}[\varepsilon_{2n}]) \cong \mathbf{Z}_{2n} \times \mathbf{Z}$. (See [**8**, 2.2.12].) Thus, each of the groups $\mathbf{Z}_{2n} \times \mathbf{Z}$, $1 \leq n \leq 6$, is an automorphism group satisfying the hypotheses of the Theorem and of Corollary 3.

Proving that the integer 30 cannot be omitted from the set seems to require a little more effort. For this, let $p(x) = \Phi_2(x)\Phi_6(x)\Phi_{10}(x)$

and $R = \mathbf{Z}[x]/(p(x))$. We shall show that $U(R) \cong \mathbf{Z}_{30} \times \mathbf{Z}$ and so this group also occurs as an automorphism group.

Since $\mathbf{Q}[x]/(p(x)) \cong \mathbf{Q} \oplus \mathbf{Q}(\varepsilon_6) \oplus \mathbf{Q}(\varepsilon_{10})$, it follows that R is isomorphic to a subring of $S = \mathbf{Z} \oplus \mathbf{Z}[\varepsilon_6] \oplus \mathbf{Z}[\varepsilon_{10}]$, and so $U(R) \leq U(S) \cong \mathbf{Z}_2 \times \mathbf{Z}_6 \times \mathbf{Z}_{10} \times \mathbf{Z} \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{30} \times \mathbf{Z}$. The additive groups of R and S are each free Abelian of rank $\phi(2) + \phi(6) + \phi(10) = 7$, and so, by [**8**, 8.9.27], $U(R)$ has the same torsion-free rank as $U(S)$, namely 1.

If $\bar{x} = x + (p(x)) \in R$, then since $x^{15} + 1 = p(x)\Phi_{30}(x)$, we have $\bar{x}^{15} = -1$. It follows that $\bar{x}$ is a unit in R of even order dividing 30. If its order were less than 30, it would have to be 2, 6, or 10 and so $p(x)$ would have to divide either $x^6 - 1$ or $x^{10} - 1$. Since $p(x)$ has degree 7 and since $\Phi_6(x)$ does not divide $x^{10} - 1$, none of these possibilities occurs. Thus, $\bar{x}$ has order 30.

To complete the proof that $U(R) \cong \mathbf{Z}_{30} \times \mathbf{Z}$, it suffices to show that -1 is the unique involution in $U(R)$, so suppose that $f(x) \in \mathbf{Z}[x]$ such that $f(\bar{x})^2 = 1$ in R. Then $p(x)$ divides $f(x)^2 - 1$. We must show that $p(x)$ divides $f(x) \pm 1$. If not, then for some $\varepsilon \in \{1, -1\}$, two of the cyclotomic factors of $p(x)$ divide $f(x) - \varepsilon$ and the other divides $f(x) + \varepsilon$. Since $(f(x)+1) - (f(x)-1) = 2$, there exist integral polynomials $m(x)$ and $n(x)$ such that one of the following holds:

$$m(x)\Phi_2(x) + n(x)\Phi_6(x)\Phi_{10}(x) = 2$$
$$m(x)\Phi_6(x) + n(x)\Phi_{10}(x)\Phi_2(x) = 2, \text{ or}$$
$$m(x)\Phi_{10}(x) + n(x)\Phi_2(x)\Phi_6(x) = 2.$$

Substituting $x = -1$ in any of these relations yields an absurdity. Thus, -1 is the unique involution in $U(R)$ as required.

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