ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

A CLASS OF STARLIKE FUNCTIONS

ALBERT E. LIVINGSTON

ABSTRACT. We study the class of functions f which are analytic and univalent in the unit disk U, which map U onto a starlike domain and are normalized by $f(0) = 1$ and $f(p) = 0$, $0 < p < 1$. We obtain sharp bounds on the integral means of f and its derivatives centered at zero and p. These lead to sharp bounds on $|f^{(n)}(z)|$ for $n = 0, 1, 2, \dots$.

1. Introduction. The class of functions which are meromorphic and univalent in $U = \{z : |z| < 1\}$ with a simple pole at $z = p$, $0 < p < 1$, and which map U onto the complement of a starlike domain has been studied in a series of papers [**1, 5, 6** and **7**]. The reciprocals of these functions are a subclass of weakly starlike 1-valent functions, which were studied by Hummel [**3, 4**] in a more general setting. The functions in this subclass have the property that they map U onto a starlike domain and are normalized by $f(0) = 1$ and $f(p) = 0$. (Hummel did not require $f(0) = 1$). We will consider several extremal problems in this class. We obtain sharp bounds on the integral means of a function and its derivatives and also sharp bounds on the coefficients of the power series expansions about $z = 0$ and $z = p$.

2. The class $\mathbf{S^*(p)}$. We denote by $S^*(p)$ the class of functions f which are analytic and univalent in U with $f(0) = 1$ and $f(p) = 0$, $0 < p < 1$, and which map U onto a starlike domain. (A starlike domain will always mean a domain starlike with respect to the origin.)

Theorem 1. *A function f with $f(0) = 1$, $f(p) = 0$, $0 < p < 1$, is in $S^*(p)$ if and only if for z in U,*

$$\text{Re}\left[\frac{(z-p)(1-pz)f'(z)}{f(z)}\right] > 0. \tag{2.1}$$

Received by the editors on May 27, 1990.

Proof. Let f be in $S^*(p)$, and let h be defined by

$$h(z) = f\left(\frac{z+p}{1+pz}\right)$$

then $h(0) = f(p) = 0$ and h maps U onto a starlike domain and, therefore, $\operatorname{Re}[zh'(z)/h(z)] > 0$ for z in U. We have

$$\frac{zh'(z)}{h(z)} = \frac{\frac{(1-p^2)z}{(1+pz)^2} f'\left(\frac{z+p}{1+pz}\right)}{f\left(\frac{z+p}{1+pz}\right)}.$$

Thus,

$$\frac{\left(\frac{z-p}{1-pz}\right) h'\left(\frac{z-p}{1-pz}\right)}{h\left(\frac{z-p}{1-pz}\right)} = \frac{(z-p)(1-pz)f'(z)}{(1-p^2)f(z)}.$$

It now follows that (2.1) is satisfied.

Conversely, suppose that f, with $f(0) = 1$ and $f(p) = 0$, satisfies (2.1) and, again, let

$$h(z) = f\left(\frac{z+p}{1+pz}\right).$$

Replacing z by $(z+p)/(1+pz)$ in (2.1), we obtain

$$\operatorname{Re}\left[\frac{\frac{(1-p^2)^2 z}{(1+pz)^2} f'\left(\frac{z+p}{1+pz}\right)}{f\left(\frac{z+p}{1+pz}\right)}\right] > 0 \tag{2.2}$$

for z in U. Inequality (2.2) implies that $\operatorname{Re} zh'(z)/h(z) > 0$ for z in U. Thus, h is univalent in U and maps U onto a starlike domain. Thus, $f(z) = h((z-p)/(1-pz))$ is also univalent and maps U onto a starlike domain. Therefore, f is in $S^*(p)$. □

The next theorem is contained in [**4**] as part of more general considerations. We will give here a fairly simple proof. We let S^* be the class of functions f analytic and univalent in U with $f(0) = 0$ and $f'(0) = 1$ and which map U onto a starlike domain.

Theorem 2. *f is in $S^*(p)$ if and only if there exists g in S^* such that*

$$f(z) = \frac{-(z-p)(1-pz)}{pz} g(z). \tag{2.3}$$

Proof. Suppose f satisfies (2.3). For $0 < r < 1$, let

$$f_r(z) = \frac{-(z-p)(1-pz)}{pz} g(rz).$$

Since g is in S^*, $g(rz)$ maps $|z| = 1$ onto a starlike curve. Since $-(z-p)(1-pz)/(pz)$ is negative on $|z| = 1$, $f_r(z)$ also maps $|z| = 1$ onto a starlike curve. Thus, f_r maps U onto a starlike domain. Letting r tend to 1, it follows that f maps U onto a starlike domain. Therefore, f is in $S^*(p)$.

Conversely, suppose f is in $S^*(p)$, and let

$$h(z) = \frac{1}{(1-p^2)f'(p)} f\left(\frac{z+p}{1+pz}\right),$$

then h is in S^*. For $0 < r < 1$, let $h_r(z) = h(rz)$, then h_r is analytic for $|z| \leq 1$ and maps $|z| = 1$ onto a starlike curve. Define g_r by

$$g_r(z) = \left(\frac{-pz}{(z-p)(1-pz)}\right)(1-p^2)f'(p)h_r\left(\frac{z-p}{1-pz}\right).$$

As above, it follows that g_r is analytic for $|z| \leq 1$ and maps $|z| = 1$ onto a starlike curve. Thus, g_r maps U onto a starlike domain. Letting r tend to 1, it follows that

$$g(z) = \frac{-pz}{(z-p)(1-pz)} f(z)$$

maps U onto a starlike domain and, hence, is a member of S^*. □

The extreme points of the closed convex hull of $S^*(p)$ are now easily obtained from known results. Let P be the set of probability measures on $X = \{z : |z| = 1\}$.

Theorem 3. *The closed convex hull of* $S^*(p)$ *is the class*

$$G = \left\{ \int_X \frac{-(z-p)(1-pz)}{p(1-xz)^2}\, d\mu(x),\ \mu \in P \right\}$$

and the extreme points of G *are the functions*

$$f(z) = \frac{-(z-p)(1-pz)}{p(1-xz)^2}, \qquad |x| = 1$$

which are members of $S^*(p)$.

Proof. This follows immediately from Theorem 2 and known results about the closed convex hull of S^* and its extreme points [**2**], and from the fact that J defined by $J(g) = -(z-p)(1-pz)g(z)/(pz)$ defines a linear homeomorphism between S^* and $S^*(p)$. □

3. Integral means. Let ϕ be nonnegative and Lebesque integrable on $[-a, a]$, and let ϕ^* be its symmetrically decreasing rearrangement as defined in [**2**]. We will make use of the following [**2**].

Lemma 1. *Let* $\phi_1, \phi_2, \dots, \phi_n$ *be nonnegative and integrable on* $[-a, a]$, *then*

$$\int_{-a}^{a} \phi_1(x)\phi_2(x)\cdots\phi_n(x)\, dx \le \int_{-a}^{a} \phi_1^*(x)\phi_2^*(x)\cdots\phi_n^*(x)\, dx$$

Theorem 4. *Let* f *be in* $S^*(p)$, *then for* λ *real and* $0 < r < 1$,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda}\, d\theta \le \int_{-\pi}^{\pi} |F(re^{i\theta})|^{\lambda}\, d\theta$$

where $F(z) = -(z-p)(1-pz)/p(1+z)^2$ *is a member of* $S^*(p)$.

Proof. Suppose first that $\lambda > 0$. Since f is in $S^*(p)$, there exists g in S^* so that $f(z)$ has the representation (2.3). Since g is in S^* there exists $m(t)$ increasing on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that

$$g(z) = z \exp\left[\int_{-\pi}^{\pi} -2\log(1 - e^{-it}z)\, dm(t) \right].$$

Thus,

$$f(z) = \frac{-(z-p)(1-pz)}{p} \exp\left[\int_{-\pi}^{\pi} -2\log(1-e^{-it}z)\, dm(t)\right].$$

Making use of the continuous form of the arithmetic geometric mean inequality [**8**], we have

$$\begin{aligned} |f(re^{i\theta})|^{\lambda} &= \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{p^{\lambda}} \\ &\quad \cdot \exp\left[\int_{-\pi}^{\pi} \log|1-e^{-it}re^{i\theta}|^{-2\lambda}\, dm(t)\right] \\ &\le \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{p^{\lambda}} \int_{-\pi}^{\pi} \frac{1}{|1-re^{i(\theta-t)}|^{2\lambda}}\, dm(t). \end{aligned}$$

Thus,

$$\begin{aligned} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda}\, d\theta &\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{p^{\lambda}|1-re^{i(\theta-t)}|^{2\lambda}}\, d\theta\, dm(t) \\ &\le \int_{-\pi}^{\pi} \sup_{-\pi\le t\le\pi} \int_{-\pi}^{\pi} \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{p^{\lambda}|1-re^{i(\theta-t)}|^{2\lambda}}\, d\theta\, dm(t) \qquad (3.1) \\ &= \frac{1}{p^{\lambda}} \sup_{-\pi\le t\le\pi} \int_{-\pi}^{\pi} \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{|1-re^{i(\theta-t)}|^{2\lambda}}\, d\theta. \end{aligned}$$

Now making use of Lemma 1, we obtain for any t,

$$\int_{-\pi}^{\pi} \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{|1-re^{i(\theta-t)}|^{2\lambda}}\, d\theta \le \int_{-\pi}^{\pi} \frac{|re^{i\theta}+p|^{\lambda}|1+pre^{i\theta}|^{\lambda}}{|1-re^{i\theta}|^{2\lambda}}\, d\theta. \qquad (3.2)$$

Combining (3.1) and (3.2) and making an appropriate change of variables, we get

$$\begin{aligned} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda}\, d\theta &\le \int_{-\pi}^{\pi} \frac{|re^{i\theta}+p|^{\lambda}|1+pre^{i\theta}|^{\lambda}}{p^{\lambda}|1-re^{i\theta}|^{2\lambda}}\, d\theta \\ &= \int_{0}^{2\pi} \frac{|-re^{i\varnothing}+p|^{\lambda}|1-pre^{i\varnothing}|^{\lambda}}{p^{\lambda}|1+re^{i\varnothing}|^{2\lambda}}\, d\varnothing \\ &= \int_{-\pi}^{\pi} \frac{|re^{i\theta}-p|^{\lambda}|1-pre^{i\theta}|^{\lambda}}{p^{\lambda}|1+re^{i\theta}|^{2\lambda}}\, d\theta \\ &= \int_{-\pi}^{\pi} |F(re^{i\theta})|^{\lambda}\, d\theta. \end{aligned}$$

Now suppose $\lambda < 0$, then

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda}\, d\theta = \int_{-\pi}^{\pi} \left| \frac{1}{f(re^{i\theta})} \right|^{-\lambda}.$$

But $1/f(z)$ is a member of $\Lambda^*(p)$, the class of meromorphic starlike functions with pole at p [**6**]. According to Theorem 5 of [**6**], for $r \neq p$,

$$\int_{-\pi}^{\pi} \left| \frac{1}{f(re^{i\theta})} \right|^{-\lambda} d\theta \leq \int_{-\pi}^{\pi} \left| \frac{1}{F(re^{i\theta})} \right|^{-\lambda} d\theta,$$

and the Theorem follows for $\lambda < 0$ and $r \neq p$. The case $r = p$ is obtained by taking limits. □

Note that the first part of the proof implies that Theorem 5 in [**6**] while stated only for positive powers is also true for negative powers. Also note that, for $\lambda \geq 1$, Theorem 4 could be obtained using extreme points.

We will next consider integral means of derivatives. We will restrict ourselves to powers of $\lambda \geq 1$ and therefore will only need to consider extreme points. For this purpose, we need the following Lemma.

Lemma 2. *Let* $F(z,x) = -(z-p)(1-pz)/p(1-xz)^2$, $|x| = 1$, *then for* $n \geq 1$, $F^{(n)}(z,x) = -(A_n(x) + B_n(x)z)/p(1-xz)^{n+2}$ *where*

$$A_n(x) = -p(n+1)!x^n + n(n!)(1+p^2)x^{n-1} - (n-1)n!px^{n-2} \tag{3.3}$$

and

$$B_n(x) = (1+p^2)n!x^n - 2pn!x^{n-1}. \tag{3.4}$$

In particular, $A_n(-1) = (-1)^{n+1}|A_n(-1)|$ *and* $B_n(-1) = (-1)^n|B_n(-1)|$. *Also*

$$|A_n(x)| \leq |A_n(-1)|; \qquad |B_n(x)| \leq |B_n(-1)| \tag{3.5}$$

and

$$1 \leq \left| \frac{A_n(x)}{B_n(x)} \right| \leq \left| \frac{A_n(-1)}{B_n(-1)} \right|. \tag{3.6}$$

Proof. We have

$$F^{(1)}(z,x) = \frac{-(1-2xp+p^2) - ((1+p^2)x - 2p)z}{p(1-xz)^3}.$$

Thus, $A_1(x) = 1 - 2xp + p^2$ and $B_1(x) = (1+p^2)x - 2p$. Thus, (3.3) and (3.4) are satisfied for $n = 1$ and (3.5) is easily checked. Moreover, for any x,

$$\left|\frac{A_1(x)}{B_1(x)}\right| = \left|\frac{(1+p^2) - 2xp}{(1+p^2)x - 2p}\right| = \left|\frac{x((1+p^2)\bar{x} - 2p)}{(1+p^2)x - 2p}\right| = 1.$$

Thus, (3.6) is satisfied for $n = 1$.

We now suppose the Lemma is true for some value of n, then $F^{(n)}(z,x) = -[A_n(x) + B_n(x)z]/p(1-xz)^{n+2}$ and (3.3), (3.4), (3.5) and (3.6) are satisfied. We then have

$$\begin{aligned} F^{(n+1)}(z,x) &= \frac{-[B_n(x) + (n+2)xA_n(x)] - (n+1)xB_n(x)z}{p(1-xz)^{n+3}} \\ &= \frac{-A_{n+1}(x) - B_{n+1}(x)z}{p(1-xz)^{n+3}} \end{aligned}$$

where

(3.7)

$$A_{n+1}(x) = B_n(x) + (n+2)xA_n(x) \quad \text{and} \quad B_{n+1}(x) = (n+1)xB_n(x).$$

Since $A_n(x)$ and $B_n(x)$ satisfy (3.3) and (3.4), we easily obtain from (3.7) that (3.3) and (3.4) hold with n replaced by $(n+1)$. Thus, (3.3) and (3.4) hold for all n. The fact that $A_n(-1) = (-1)^{n+1}|A_n(-1)|$; $B_n(-1) = (-1)^n|B_n(-1)|$ and (3.5) hold for all n are now easily established from (3.3) and (3.4).

Now assume that (3.6) is true for some n. From (3.7) we have

$$\begin{aligned}\left|\frac{A_{n+1}(x)}{B_{n+1}(x)}\right| &= \left|\frac{B_n(x)+(n+2)xA_n(x)}{(n+1)xB_n(x)}\right| \\ &= \left|\frac{1}{(n+1)x}+\left(\frac{n+2}{n+1}\right)\left(\frac{A_n(x)}{B_n(x)}\right)\right| \\ &\le \frac{1}{n+1}+\left(\frac{n+2}{n+1}\right)\left|\frac{A_n(x)}{B_n(x)}\right| \\ &\le \frac{1}{n+1}+\left(\frac{n+2}{n+1}\right)\left|\frac{A_n(-1)}{B_n(-1)}\right| \\ &= \frac{1}{n+1}+\left(\frac{n+2}{n+1}\right)\frac{(-1)^{n+1}A_n(-1)}{(-1)^nB_n(-1)} \\ &= \frac{(-1)^nB_n(-1)+(n+2)(-1)^{n+1}A_n(-1)}{(n+1)(-1)^nB_n(-1)} \\ &= \frac{(-1)^nA_{n+1}(-1)}{(-1)^{n-1}B_{n+1}(-1)} = \frac{|A_{n+1}(-1)|}{|B_{n+1}(-1)|}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{A_{n+1}(x)}{B_{n+1}(x)}\right| &= \left|\frac{1}{(n+1)x}+\left(\frac{n+2}{n+1}\right)\left(\frac{A_n(x)}{B_n(x)}\right)\right| \\ &\ge \left(\frac{n+2}{n+1}\right)\left|\frac{A_n(x)}{B_n(x)}\right|-\frac{1}{n+1} \\ &\ge \frac{n+2}{n+1}-\frac{1}{n+1}=1.\end{aligned}$$

This establishes (3.6) for all n. □

Lemma 3. *With $A_n(x)$ and $B_n(x)$ as in Lemma 2 and $0 \le r < 1$,*

$$\left|\left|\frac{A_n(x)}{B_n(x)}\right|+re^{i\theta}\right| \le \left|\left|\frac{A_n(-1)}{B_n(-1)}\right|+re^{i\theta}\right|.$$

Proof. The inequality of the Lemma is equivalent to

$$\begin{aligned}&\left|\frac{A_n(x)}{B_n(x)}\right|^2+2r\left|\frac{A_n(x)}{B_n(x)}\right|\cos\theta+r^2 \\ &\qquad\le \left|\frac{A_n(-1)}{B_n(-1)}\right|^2+2r\left|\frac{A_n(-1)}{B_n(-1)}\right|\cos\theta+r^2.\end{aligned}$$

This in turn is equivalent to

$$0 \le \left|\frac{A_n(-1)}{B_n(-1)}\right|^2 - \left|\frac{A_n(x)}{B_n(x)}\right|^2 + 2r\cos\theta\left[\left|\frac{A_n(-1)}{B_n(-1)}\right| - \left|\frac{A_n(x)}{B_n(x)}\right|\right]$$

since, by Lemma 2, $|A_n(-1)/B_n(-1)| - |A_n(x)/B_n(x)| \ge 0$. The last inequality is implied by

$$0 \le \left|\frac{A_n(-1)}{B_n(-1)}\right| + \left|\frac{A_n(x)}{B_n(x)}\right| + 2r\cos\theta.$$

Again, by Lemma 2, $|A_n(x)/B_n(x)| \ge 1$ for all x, $|x| = 1$. Thus,

$$\left|\frac{A_n(-1)}{B_n(-1)}\right| + \left|\frac{A_n(x)}{B_n(x)}\right| + 2r\cos\theta \ge 2 + 2r\cos\theta \ge 0.$$

This then proves the Lemma. □

Theorem 5. *Let f be a member of the closed convex hull of $S^*(p)$, then for $n \ge 1$ and $\lambda \ge 1$,*

$$\int_{-\pi}^{\pi} |f^{(n)}(re^{i\theta})|^\lambda \, d\theta \le \int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta})|^\lambda \, d\theta$$

where $F(z) = -(z-p)(1-pz)/p(1+z)^2$ is in $S^*(p)$.

Proof. Since $\lambda \ge 1$ we need only consider extreme points [**2**] $F(z,x) = -(z-p)(1-pz)/p(1-xz)^2$. By Lemma 2,

$$\begin{aligned}\int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta},x)|^\lambda \, d\theta &= \int_{-\pi}^{\pi} \frac{|A_n(x) + B_n(x)re^{i\theta}|^\lambda}{p^\lambda |1 - xre^{i\theta}|^{(n+2)\lambda}} \\ &= \int_{-\pi}^{\pi} \frac{|B_n(x)| \left|\frac{A_n(x)}{B_n(x)} + re^{i\theta}\right|^\lambda}{p^\lambda |1 - xre^{i\theta}|^{(n+2)\lambda}} \, d\theta \qquad (3.8) \\ &\le \frac{|B_n(-1)|}{p^\lambda} \int_{-\pi}^{\pi} \frac{\left|\frac{A_n(x)}{B_n(x)} + re^{i\theta}\right|^\lambda}{|1 - xre^{i\theta}|^{(n+2)\lambda}} \, d\theta.\end{aligned}$$

The symmetrically decreasing rearrangement of $|A_n(x)/B_n(x)+re^{i\theta}|^\lambda$ is $||A_n(x)/B_n(x)|+re^{i\theta}|^\lambda$ and the rearrangement of $1/|1-xre^{i\theta}|^{(n+2)\lambda}$ is $1/|1-re^{i\theta}|^{(n+2)\lambda}$. Thus (3.8) with Lemma 1 gives

$$\int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta},x)|^\lambda \, d\theta \le \frac{|B_n(-1)|^\lambda}{p^\lambda} \int_{-\pi}^{\pi} \frac{\left|\left|\frac{A_n(x)}{B_n(x)}\right| + re^{i\theta}\right|^\lambda}{|1-re^{i\theta}|^{(n+2)\lambda}}\, d\theta. \tag{3.9}$$

Making use of Lemma 3, we obtain from (3.9),

$$\begin{aligned}\int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta},x)|^\lambda \, d\theta &\le \frac{|B_n(-1)|^\lambda}{p^\lambda} \int_{-\pi}^{\pi} \frac{\left|\left|\frac{A_n(-1)}{B_n(-1)}\right| + re^{i\theta}\right|^\lambda}{|1-re^{i\theta}|^{(n+2)\lambda}}\, d\theta \\ &= \int_{-\pi}^{\pi} \frac{||A_n(-1)|+|B_n(-1)|re^{i\theta}|^\lambda}{p^\lambda|1-re^{i\theta}|^{(n+2)\lambda}}\, d\theta \\ &= \int_{-\pi}^{\pi} \frac{|(-1)^{n+1}A_n(-1)+(-1)^n B_n(-1)re^{i\theta}|^\lambda}{p^\lambda|1-re^{i\theta}|^{(n+2)\lambda}}\, d\theta.\end{aligned} \tag{3.10}$$

Making the substitution $\theta = \varnothing + \pi$ in the last integral of (3.10), we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta},x)|^\lambda \, d\theta &\le \int_{-\pi}^{\pi} \frac{|(-1)^{n+1}A_n(-1)+(-1)^{n+1} B_n(-1)re^{i\theta}|^\lambda}{p^\lambda|1+re^{i\theta}|^{(n+2)\lambda}}\, d\theta \\ &= \int_{-\pi}^{\pi} \left|\frac{A_n(-1)+B_n(-1)re^{i\theta}}{-p(1+re^{i\theta})^{n+2}}\right|^\lambda d\theta \\ &= \int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta},-1)|^\lambda \, d\theta \\ &= \int_{-\pi}^{\pi} |F^{(n)}(re^{i\theta})|^\lambda \, d\theta.\end{aligned}$$

Corollary 1. *Let f be a member of the closed, convex hull of $S^*(p)$, then for $n = 0, 1, 2, \ldots,$ and $0 \le r < 1$,*

$$|f^{(n)}(re^{i\theta})| \le \frac{p(n+1)! + [n(1+p^2)+(n-1)p]n! + (1+p)^2 n! r}{p(1-r)^{n+2}}.$$

The inequality is sharp in $S^*(p)$, with equality attained by $F(z) = -(z-p)(1-pz)/p(1+z)^2$ when $\theta = \pi$.

Proof. Raising the inequality in Theorem 4 or Theorem 5 to the $1/\lambda$ and letting λ tend to ∞ gives

$$\max_{\theta} |f^{(n)}(re^{i\theta})| \leq \max_{\theta} |F^{(n)}(re^{i\theta})|,$$

which implies the desired inequality. □

Remark . The cases $n = 0, 1$ of Corollary 1 are also given by Hummel [**4**].

Corollary 2. *Let f be a member of the closed convex hull of $S^*(p)$ and suppose $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for z in U, then for $n \geq 1$,*

$$|a_n| \leq \frac{n(1+p)^2}{p}.$$

The inequality is sharp in $S^*(p)$, equality being attained for all n by $F(z) = -(z-p)(1-pz)/p(1+z)^2$.

Remark . The inequality of Corollary 2 appears as part of Theorem 1 of [**4**].

4. Integral means centered at p. We will consider integral means of the form $\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^{\lambda}\, d\theta$. For this purpose, we need the following lemma.

Lemma 4. *Let*

$$h(z,t) = \frac{(1-p^2-pz)^{\mu/2}}{(1-pe^{-it}-e^{-it}z)^{\mu}} = \sum_{n=0}^{\infty} c_n(t) z^n,$$

for $|z| < 1-p$, where $-\pi \leq t \leq \pi$ and $\mu > 0$, then $|c_n(t)| \leq c_n(0)$ for all n.

Proof. Consider

$$\begin{aligned}\frac{zh'(z,t)}{h(z,t)} &= \frac{-p\mu z}{2(1-p^2-pz)} + \frac{\mu e^{-it}z}{1-pe^{-it}-e^{-it}z} \\ &= \frac{-p\mu z}{2(1-p^2)(1-\frac{pz}{1-p^2})} + \frac{\mu e^{-it}z}{(1-pe^{-it})(1-\frac{e^{-it}z}{1-pe^{-it}})} \\ &= \sum_{n=1}^{\infty} b_n(t)z^n\end{aligned}$$

where

$$\begin{aligned}b_n(t) &= \frac{-\mu p^n}{2(1-p^2)^n} + \frac{\mu e^{-int}}{(1-pe^{-it})^n} \\ &= \mu\left[\left(\frac{e^{-it}}{1-pe^{-it}}\right)^n - \left(\frac{p}{1-p^2}\right)^n + \frac{1}{2}\left(\frac{p}{1-p^2}\right)^n\right] \\ &= \mu\Bigg[\left(\frac{e^{-it}(1-pe^{it})}{(1-p^2)(1-pe^{-it})}\right)\sum_{k=0}^{n-1}\left(\frac{e^{-it}}{1-pe^{it}}\right)^{n-1-k}\left(\frac{p}{1-p^2}\right)^k \\ &\qquad + \frac{1}{2}\left(\frac{p}{1-p^2}\right)^n\Bigg].\end{aligned}$$

Thus, for any n,

(4.1)

$$\begin{aligned}|b_n(t)| &\le \mu\left[\frac{1}{1-p^2}\sum_{k=0}^{\infty}\left(\frac{1}{1-p}\right)^{n-1-k}\left(\frac{p}{1-p^2}\right)^k + \frac{1}{2}\left(\frac{p}{1-p^2}\right)^n\right] \\ &= b_n(0).\end{aligned}$$

Now we have

$$\sum_{n=1}^{\infty} nc_n(t)z^n = \left(\sum_{n=1}^{\infty} b_n(t)z^n\right)\left(\sum_{n=0}^{\infty} c_n(t)z^n\right).$$

Comparing coefficients, we obtain

$$c_1(t) = c_0(t)b_1(t) \quad\text{and}\quad nc_n(t) = \sum_{k=1}^{n} b_k(t)c_{n-k}(t). \tag{4.2}$$

We first note that $c_n(0) > 0$ for all n. We have $c_0(0) = (1-p^2)^{\mu/2}/(1-p)^{\mu} > 0$. Also, $b_n(0) > 0$ for all n. Thus, $c_1(0) = c_0(0)b_1(0) > 0$. Suppose $c_k(0) > 0$ for $k = 0, 1, \dots (n-1)$, then $nc_n(0) = \sum_{k=1}^{\infty} b_k(0)c_{n-k}(0) > 0$.

Next we note that

$$|c_0(t)| = \left|\frac{(1-p^2)^{\mu/2}}{(1-pe^{-it})^{\mu}}\right| \leq \frac{(1-p^2)^{\mu/2}}{(1-p)^{\mu}} = c_0(0),$$

and from (4.1) and (4.2), we obtain

$$|c_1(t)| = |c_0(t)||b_1(t)| \leq c_0(0)b_1(0) = c_1(0).$$

Now suppose $|c_k(t)| \leq c_k(0)$ for $k = 0, 1, \dots (n-1)$, then

$$\begin{aligned} n|c_n(t)| &\leq \sum_{k=1}^{n} |b_k(t)||c_{n-k}(t)| \\ &\leq \sum_{k=1}^{n} b_k(0)c_{n-k}(0) \\ &= nc_n(0). \end{aligned}$$

This completes the proof of the Lemma. □

Theorem 6. *If f is a member of $S^*(p)$ and $\lambda > 0$, then for $0 < r < 1-p$,*

$$\int_{-\pi}^{\pi} |f(p+re^{i\theta})|^{\lambda}\, d\theta \leq \int_{-\pi}^{\pi} |G(p+re^{i\theta})|^{\lambda}\, d\theta$$

where $G(z) = -(z-p)(1-pz)/p(1-z)^2$.

Proof. As in the proof of Theorem 4,

$$f(z) = \frac{-(z-p)(1-pz)}{p} \exp\left[\int_{-\pi}^{\pi} -2\log(1-e^{-it}z)\, dm(t)\right]$$

where $m(t)$ is increasing on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} dm(t) = 1$. Using the continuous form of the arithmetic geometric mean inequality [**8**],

$$|f(p+re^{i\theta})|^{\lambda} \leq \frac{r^{\lambda}|1-p^2-pre^{i\theta}|^{\lambda}}{p^{\lambda}} \int_{-\pi}^{\pi} |1-pe^{-it}-re^{i(\theta-t)}|^{-2\lambda}\, dm(t).$$

Therefore,

$$\int_{-\pi}^{\pi} |f(p+re^{i\theta})|^{\lambda}\, d\theta \le \frac{r^{\lambda}}{p^{\lambda}} \int_{-\pi}^{\pi} I(t)\, dm(t) \tag{4.2}$$

where

$$I(t) = \int_{-\pi}^{\pi} \frac{|1-p^2-pre^{i\theta}|^{\lambda}}{|1-pe^{-it}-re^{i(\theta-t)}|^{2\lambda}}\, d\theta.$$

Making use of Lemma 4, we obtain

$$\begin{aligned} I(t) &= \int_{-\pi}^{\pi} \left| \frac{(1-p^2-pre^{i\theta})^{\lambda/2}}{(1-pe^{-it}-re^{i(\theta-t)})^{\lambda}} \right|^2 d\theta \\ &= 2\pi \sum_{n=0}^{\infty} |c_n(t)|^2 r^{2n} \\ &\le 2\pi \sum_{n=0}^{\infty} |c_n(0)|^2 r^{2n} \\ &= I(0). \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$\begin{aligned} \int_{-\pi}^{\pi} |f(p+re^{i\theta})|^{\lambda}\, d\theta &\le \frac{r^{\lambda}}{p^{\lambda}} \int_{-\pi}^{\pi} I(0)\, dm(t) \\ &= \frac{r^{\lambda}}{p^{\lambda}} I(0) \\ &= \int_{-\pi}^{\pi} |G(p+re^{i\theta})|^{\lambda}\, d\theta. \end{aligned}$$

This completes the proof of the theorem. □

If we let $\Lambda^*(p)$ be the class of functions meromorphic and starlike in U with pole at p [**7**], then we have the following corollary, which was conjectured to be true in [**7**].

Corollary 3. *If f is a member of $\Lambda^*(p)$, then for $\mu < 0$ and $0 < r < 1-p$,*

$$\int_{-\pi}^{\pi} |f(p+re^{i\theta})|^{\mu}\, d\theta \le \int_{-\pi}^{\pi} |H(p+re^{i\theta})|^{\mu}\, d\theta$$

where $H(z) = -p(1-z)^2/(z-p)(1-pz)$.

Proof. If f is a member of $\Lambda^*(p)$, then $1/f$ is a member of $S^*(p)$. Now make use of Theorem 6. □

Theorem 2 in [**7**] immediately gives us the following

Theorem 7. *If f is a member of $S^*(p)$ and $\lambda < 0$, $0 < r < 1-p$, then*

$$\int_{-\pi}^{\pi} |f(p+re^{i\theta})|^{\lambda}\, d\theta \le \int_{-\pi}^{\pi} |F(p+re^{i\theta})|^{\lambda}\, d\theta$$

where $F(z) = -(z-p)(1-pz)/p(1+z)^2$.

Lemma 5. *For $|x| = 1$, let $A_n(x)$ and $B_n(x)$ be as in Lemma 2, and let*

$$h(z) = \left(\frac{A_n(x) + pB_n(x) + B_n(x)z}{(1-xp-xz)^{n+2}}\right)^{\lambda/2} = \sum_{j=0}^{\infty} b_j(x)z^j$$

for $|z| < 1-p$ and $\lambda > 0$, then $|b_j(x)| \le b_j(1)$ for all j.

Proof. Consider

$$\begin{aligned}\frac{zh'(z)}{h(z)} &= \frac{\lambda}{2}\left[\frac{B_n(x)z}{A_n(x)+pB_n(x)+B_n(x)z} + (n+2)\frac{xz}{1-xp-xz}\right] \\ &= \frac{\lambda}{2}\Bigg[\sum_{j=1}^{\infty}(-1)^{j+1}\left(\frac{B_n(x)}{A_n(x)+pB_n(x)}\right)^j z^j \\ &\qquad + (n+2)\sum_{j=1}^{\infty}\left(\frac{x}{1-xp}\right)^j z^j\Bigg] \\ &= \sum_{j=1}^{\infty} c_j(x)z^j\end{aligned}$$

where

(4.4)
$$c_j(x) = \frac{\lambda}{2}\left[(n+2)\left(\frac{x}{1-xp}\right)^j + (-1)^{j+1}\left(\frac{B_n(x)}{A_n(x)+pB_n(x)}\right)^j\right].$$

From (3.3) and (3.4) of Lemma 2, we obtain after straightforward computations

$$\frac{B_n(x)}{A_n(x)+pB_n(x)} = \frac{x[(1+p^2)x-2p]}{[(n-p^2)x-(n-1)p](1-xp)}. \tag{4.5}$$

Combining (4.4) and (4.5), we eventually obtain

$$c_j(x) = \frac{\lambda}{2}\left[(n+2)\left(\frac{x}{1-xp}\right)^j - \left(\frac{\frac{x}{1-xp}}{n\left(\frac{x-p}{1-xp}\right)+p}\left(p-\left(\frac{x-p}{1-xp}\right)\right)\right)^j\right].$$

Let $y=(x-p)/(1-xp)$, then $|y|=1$ and c_j as a function of y is given by

$$\begin{aligned}
c_j(y) &= \frac{\lambda}{2}\left[(n+2)\left(\frac{y+p}{1-p^2}\right)^j - \left(\frac{\frac{y+p}{1-p^2}}{ny+p}(p-y)\right)^j\right]\\
&= \frac{\lambda}{2}\left[\frac{(n+2)(y+p)^j}{(1-p^2)^j} - \frac{(y+p)^j(p-y)^j}{(1-p^2)^j(ny+p)^j}\right]\\
&= \frac{\lambda(y+p)^j}{2(1-p^2)^j(ny+p)^j}[(n+2)(ny+p)^j-(p-y)^j]\\
&= \frac{\lambda(y+p)^j}{2(1-p^2)^j(ny+p)^j}\sum_{k=0}^{j}\binom{j}{k}y^{j-k}p^k((n+2)n^{j-k}\\
&\qquad\qquad + (-1)^{j-k+1}).
\end{aligned}$$

Since $|(y+p)/(ny+p)| \le (1+p)/(n+p)$ for $|y|=1$, we obtain

$$\begin{aligned}
|c_j(y)| &\le \frac{\lambda(1+p)^j}{2(1-p^2)^j(n+p)^j}\sum_{k=0}^{j}\binom{j}{k}p^n((n+2)n^{j-k}+(-1)^{j-k+1})\\
&= c_j(1).
\end{aligned}$$

Since $y=1$ corresponds to $x=1$, then for all j,

$$|c_j(x)| \le c_j(1). \tag{4.6}$$

We now have

$$\sum_{j=1}^{\infty} jb_j(x)z^j = \sum_{j=1}^{\infty} c_j(x)z^j \sum_{j=0}^{\infty} b_j(x).$$

Comparing coefficients, we obtain for $j \geq 1$,

$$jb_j(x) = \sum_{k=0}^{j-1} b_k(x)c_{j-k}(x). \tag{4.7}$$

We first note that

$$\begin{aligned} b_0(x) &= \left(\frac{A_n(x) + pB_n(x)}{(1-xp)^{n+2}}\right)^{\lambda/2} \\ &= \left[\frac{n!x^{n-2}}{(1-px)^n}\left(n\left(\frac{x-p}{1-px}\right) + p\right)\right]^{\lambda/2} \end{aligned}$$

where we have used (3.3) and (3.4). Thus,

$$|b_0(x)| \leq \left(\frac{n!(n+p)}{(1-p)^n}\right)^{\lambda/2} = b_0(1).$$

Suppose we have proven $|b_k(x)| \leq b_k(1)$ for $(k = 0, \dots, j-1)$, then from (4.7) we have

$$\begin{aligned} j|b_j(x)| &\leq \sum_{k=0}^{j-1} |b_k(x)||c_{j-k}(x)| \\ &\leq \sum_{k=0}^{j-1} b_k(1)c_{j-k}(1) \\ &= jb_j(1). \end{aligned}$$

This completes the proof of the lemma. □

Theorem 8. *If f is a member of the closed convex hull of $S^*(p)$, then for $\lambda \geq 1$ and $n = 1, 2, 3, \dots$, and $r < 1-p$,*

$$\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^\lambda \leq \int_{-\pi}^{\pi} |F^{(n)}(p + re^{i\theta})|^\lambda \, d\theta$$

where $F(z) = -(z-p)(1-pz)/p(1-z)^2$ is a member of $S^*(p)$.

Proof. Since $\lambda \geq 1$, we need only prove the theorem for extreme points $F(z,x) = -(z-p)(1-pz)/p(1-xz)^2$ [**2**]. Using Lemma 2 and Lemma 5, we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} |F^{(n)}(p+re^{i\theta},x)|\,d\theta &= \frac{1}{p^\lambda}\int_{-\pi}^{\pi}\left|\left(\frac{A_n(x)+pB_n(x)+B_n(x)re^{i\theta}}{(1-xp-xre^{i\theta})^{n+2}}\right)^{\lambda/2}\right|^2 \\
&= \frac{2\pi}{p^\lambda}\sum_{j=0}^{\infty}|b_j(x)|^2 r^{2j} \\
&\leq \frac{2\pi}{p^\lambda}\sum_{j=0}^{\infty} b_j(1)^2 r^{2j} \\
&= \frac{1}{p^\lambda}\int_{-\pi}^{\pi}\left|\left(\frac{A_n(1)+pB_n(1)+B_n(1)re^{i\theta}}{(1-p-re^{i\theta})^{n+2}}\right)^{\lambda/2}\right|^2 \\
&= \int_{-\pi}^{\pi}|F^{(n)}(p+re^{i\theta},1)|^\lambda\,d\theta \\
&= \int_{-\pi}^{\pi}|F^{(n)}(p+re^{i\theta})|^\lambda\,d\theta. \qquad \square
\end{aligned}$$

Corollary 4. *If f is a member of the closed convex hull of $S^*(p)$, then for $n = 0,1,2,\ldots$ and $0 \leq r < 1-p$,*

$$|f^{(n)}(p+re^{i\theta})| \leq \frac{n!(1-p)^2(n+p+r)}{p(1-p-r)^{n+2}}.$$

Equality is attained by $F(z) = -(z-p)(1-pz)/p(1-z)^2$ when $\theta = 0$.

Proof. Taking the λ-th root of both sides of the inequality in Theorem 6 or Theorem 8 and then letting λ tend to $+\infty$, we get

$$\max_\theta |f^{(n)}(p+re^{i\theta})| \leq \max_\theta |F^{(n)}(p+re^{i\theta})|.$$

Using Lemma 2, we have

$$\begin{aligned} |F^{(n)}(p+re^{i\theta})| &= \left| \frac{A_n(1)+pB_n(1)+B_n(1)re^{i\theta}}{-p(1-p-re^{i\theta})^{n+2}} \right| \\ &= \left| \frac{n!(1-p^2)(n+p+re^{i\theta})}{-p(1-p-re^{i\theta})^{n+2}} \right| \\ &\le \frac{n!(1-p^2)(n+p+r)}{p(1-p-r)^{n+2}}. \qquad \square \end{aligned}$$

Corollary 5. *If f is a member of the closed convex hull of $S^*(p)$ and if $f(z) = \sum_{n=1}^{\infty} b_n(z-p)^n$ for $|z-p| < 1-p$, then for $n = 1, 2, \dots,$*

$$|b_n| \le \frac{n+p}{p(1-p)^n}.$$

Equality is attained for all n by $F(z) = -(z-p)(1-pz)/p(1-z)^2$.

REFERENCES

1. P.J. Eenigenburg and A.E. Livingston, *Meromorphic starlike functions*, Rocky Mountain J. Math. **11** (1981), 441–457.

2. D.J. Hallenbeck and T.H. MacGregor, *Linear problems and convexity techniques in geometric function theory*, Pittman Publishing, Inc., Marshfield, Mass., 1984.

3. J.A. Hummel, *Multivalent starlike functions*, J. Analyse Math. **18** (1967), 133–160.

4. J.A. Hummel, *Extremal properties of weakly starlike p-valent functions*, Trans. Amer. Math. Soc. **130** (1968), 544–551.

5. R.J. Libera and A.E. Livingston, *Weakly starlike meromorphic univalent functions*, Trans. Amer. Math. Soc. **202** (1975), 181–191.

6. A.E. Livingston, *Weakly starlike meromorphic univalent functions* II, Proc. Amer. Math. Soc. **62** (1977), 47–53.

7. A.E. Livingston, *A class of meromorphic starlike functions*, Rocky Mountain J. Math. **13** (1983), 229–240.

8. W. Rudin, *Real and complex analysis*, 2nd ed., McGraw Hill, New York, 1974.

Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, Delaware, 19716