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SCHAUDER DECOMPOSITIONS OF NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. Let K be a field with a nontrivial non-Archimedean valuation, under which it is complete. Let E be a non-Archimedean Banach space over K. Some of the main results are:

(1) (Theorem 13) If $(P_n) \subset L(E, E)$ is a Schauder decomposition, then (P_n) is a (UM)-sequence such that $P_n \to 1$ strongly but not uniformly.

(2) (Corollary 14) Let K be spherically complete. If E is a G-space, then E admits no Schauder decompositions.

1. Introduction. Let K be a field with a nontrivial non-Archimedean valuation, under which it is complete. In this paper we deal with Schauder decompositions of Banach spaces over K.

In Archimedean analysis, many authors treat this decomposition (cf. [**1, 4, 5**]). In particular, Dean [**1**] gave the following result.

Theorem [**1**]. *Let E be a Grothendieck space with the Dunford-Pettis property. Then E admits no Schauder decompositions.*

Lotz [**5**] and Leung [**4**] obtained further results along the same lines as Dean's argument.

In non-Archimedean analysis, however, if K is not spherically complete, then the above theorem is not true. In this paper we give an example to indicate it and show the following theorem.

Theorem (Corollary 14). *Let K be spherically complete. If E is a Grothendieck space, then E admits no Schauder decompositions.*

To show this, we need the following theorem, which is also one of our main theorems.

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Theorem (Theorem 13). *If* $(P_n) \subset L(E, E)$ *is a Schauder decomposition, then* (P_n) *is a* (UM)*-sequence such that* $P_n \to 1$ *strongly but not uniformly.*

2. Preliminaries. Throughout, by $E, F, \dots,$ we denote Banach spaces over K, and E' denotes the dual of E. Let $L(E, F)$ be the space of all continuous linear operators from E into F, and let $C(E, F)$ be the subspace of $L(E, F)$ which consists of all compact operators. The identity operator on E is denoted by 1_E or 1 if there is no cause for confusion. By $E \sim F$, we mean that E and F are linearly isometric.

A Banach space E is called a Grothendieck space (G-space) if every sequence $(x'_n) \subset E'$ which converges for weak* topology to zero converges weakly to zero. It is clear that a reflexive space is a G-space. The following are some results on G-spaces.

Proposition 1 (De Grande-de Kimpe [**2**]). *If* $L(E, c_0) = C(E, c_0)$, *then* E *is a* G*-space.*

Combining this proposition with Corollary 5.20 in [**7**] yields the next corollary.

Corollary 2. *If the valuation of* K *is dense and* E *is weakly injective, then* E *is a* G*-space. In particular, if* K *is spherically complete and its valuation is dense, then every dual space is a* G*-space.*

The following proposition is also obtained in De Grande-de Kimpe [**2**], but we give another proof here.

Proposition 3. *Let* K *be spherically complete. If* E *contains a subspace of countable type which is complemented, then* E *is not a* G*-space.*

(Recall that every Banach space contains a subspace of countable type.)

Proof. We may assume that $E = D \oplus c_0$, where D is a closed subspace of E. Then there exists a linear isometry

$$S : D' \times l^\infty \to E'$$

defined by $\langle (x,y), S(d_1,d_2)\rangle = \langle x, d_1\rangle + \langle y, d_2\rangle$ ($d_1 \in D'$, $d_2 \in l^\infty$, $x \in D$, $y \in c_0$). (See [**7**, p. 61].) Put $e_n = (0,0,\dots,0,1_{n-\text{th}},0,\dots) \in l^\infty$ $(n \geq 1)$ and $S(0,e_n) = x'_n \in E'$. Then, for all $(x,y) \in E$ $(x \in D,\ y \in c_0)$, $\langle (x,y), x'_n\rangle = y_n \to 0$ where y_n is the n-th coordinate of y. Therefore, $(x'_n) \in E'$ is weak* convergent to zero. While, since K is spherically complete, there exists $x'' \in (l^\infty)'$ $(\sim c''_0)$ such that $\langle e_n, x''\rangle \not\to 0$ (see [**3**]). Now define $y'' \in (D' \times l^\infty)'$ by $\langle (z',y), y''\rangle = \langle y, x''\rangle$ $(z' \in D',\ y \in l^\infty)$ and put $z'' = y''S^{-1}$. Then $z'' \in E''$ and $\langle x'_n, z''\rangle = \langle e_n, x''\rangle \not\to 0$. Hence $(x'_n) \subset E'$ does not converge weakly to zero. So E is not a G-space. □

Corollary 4. *Let K be spherically complete. If E has a base, then E is not a G-space. In particular, if the valuation of K is discrete, then every Banach space is not a G-space.*

Proof. Combining Proposition 3 with Corollary 3.18 in [**7**], we can show this corollary. □

Corollary 5. (1) *The valuation of K is dense if and only if l^∞ is a G-space.*

(2) *K is not spherically complete if and only if c_0 is a G-space.*

Proof. (1) follows from Proposition 1, Corollary 4 and Corollary 5.19 in [**7**]. (2) follows from Corollary 4 and from the fact that if K is not spherically complete, then c_0 is reflexive. □

A Banach space E is said to have the Dunford-Pettis property (D-P property) if $\lim_n \langle x_n, x'_n\rangle = 0$ whenever $(x_n) \subset E$ tends weakly to zero and $(x'_n) \subset E'$ tends weakly to zero. If K is spherically complete, then every weakly convergent sequence in a Banach space is norm-convergent (see [**6**, p. 70]). Hence, the following lemma holds.

Lemma 6. *If K is spherically complete, then every Banach space has the* D-P *property.*

Theorem 7. *c_0 and l^∞ have the* D-P *property.*

Proof. Suppose that $(x_n) \subset c_0$ tends weakly to zero. Since $c_0' \sim l^\infty$, by the same argument as used in proving Theorem 6 in [**6**, p. 70], we have $\lim_n ||x_n|| = 0$. Further, suppose that $(x_n') \subset c_0'$ tends weakly to zero. Then by the Banach-Steinhaus theorem, $\sup_n ||x_n'|| < \infty$. Therefore, $|\langle x_n, x_n' \rangle| \leq ||x_n'||\,||x_n|| \to 0$ $(n \to \infty)$. So c_0 has the D-P property. We now show that l^∞ has the D-P property. We may assume that K is not spherically complete. Then c_0 and l^∞ are reflexive and $(l^\infty)' \sim c_0$ (see [**7**, p. 111]). Hence, the proof is the same as the proof of c_0. □

Definition (Lotz [**5**]). A sequence $(P_n) \subset L(E, E)$ is said to be a (*weak*) *Schauder decomposition* if the following conditions hold:

(1) $P_m P_n = P_{\min(n,m)}$ for all n, m.

(2) $(P_n x)$ converges (weakly) to x for every $x \in E$.

(3) $P_n \neq P_m$ for $n \neq m$.

Remark. Put $Q_1 = P_1$, $Q_i = P_i - P_{i-1}$ $(i \geq 2)$. Then we see that Q_i is a projection and $E = \oplus Q_i(E)$.

Definition (Lotz [**5**]). A sequence $(S_n) \subset L(E, E)$ is said to be a (UM)-*sequence* if the following conditions hold:

(1) $\sup_n ||S_n|| < \infty$.

(2) $\lim_n ||S_m(S_n - 1)|| = 0$ for all m.

(3) $\lim_n ||(S_n - 1)S_m|| = 0$ for all m.

Example 1. (1) Every (weak) Schauder decomposition (P_n) on E is a (UM)-sequence.

(2) Let (α_n) be a sequence in K such that $\alpha_n \to 0$ $(n \to \infty)$. For every $n \in N$, define a linear operator

$$S_n : l^\infty \to l^\infty$$

by $S_n(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_2, \ldots, x_n, \alpha_{n+1}x_{n+1}, \alpha_{n+2}x_{n+2}, \ldots)$. Then (S_n) is a (UM)-sequence on l^∞.

(3) Let $P \in L(E,E)$ be a projection and λ be an element of K with $||P|| < |\lambda|$. Then $1 - \lambda^{-1}P$ is a bijection, and we have the following expansion:

$$(1 - \lambda^{-1}P)^{-1} = 1 + \lambda^{-1}P + \cdots + (\lambda^{-1}P)^n + \cdots .$$

Hence, for every $x \in E$ there exists $y \in E$ such that

$$x = y + (\lambda^{-1}P)y + \cdots + (\lambda^{-1}P)^n y + \cdots .$$

For every n, define a linear operator

$$S_n : E \to E$$

by $S_n(x) = y + (\lambda^{-1}P)y + \cdots + (\lambda^{-1}P)^n y$. Then (S_n) is a (UM)-sequence on E.

We observe that if (S_n) is a (UM)-sequence on E, then (S'_n) is a (UM)-sequence on E'.

3. Results. The following lemma is similar to the lemma of Lotz [**5**].

Lemma 8. *Let K be spherically complete and let (S_n) be a* (UM)*-sequence on E. Put $X = \overline{\cup S_n(E)}$ and $Y = \cap S_n^{-1}\{0\}$. Then the following assertions hold*:

(1) *$X = \{x \in E \mid \lim_n ||S_n x - x|| = 0\}$ and X is a linear subspace of E and weak closure of $\cup S_n(E)$.*

(2) *If, for every $x \in E$, the sequence $(S_n x)$ has a weak cluster point, then (S_n) converges strongly to a projection P with X as range and Y as kernel.*

Proof. Since K is spherically complete, X is weakly closed. Therefore, the proof is the same as the proof of Lemma 1 in [**5**]. □

Lemma 9. *Let K be spherically complete, and let E be a G-space. If a sequence $(T_n) \subset L(E,E)$ converges strongly to* 1*, then the sequence $(T'_n) \subset L(E',E')$ also does.*

Proof. For every $x \in E$ and for every $x' \in E'$,

$$|\langle x, (T_n - 1)'(x') \rangle| \leq ||x'|| \, ||(T_n - 1)(x)|| \to 0 \qquad (n \to \infty).$$

Hence, $(T_n - 1)'(x') \to 0$ weak*. Since E is a G-space and K is spherically complete, $(T_n - 1)'(x') \to 0$ strongly. □

The following theorem is analogous to Leung's theorem (see [**4**, p. 24).

Theorem 10. *Let K be spherically complete, and let E be a G-space. Then every strongly convergent* (UM)*-sequence on E converges uniformly.*

Proof. Let (S_n) be a strongly convergent (UM)-sequence on E. By Lemma 8, we may assume that $S_n \to 1$ strongly. Assume that $||S_n - 1|| \not\to 0$ $(n \to \infty)$. If, for some m, S'_m is topological isomorphism from E' onto a closed subspace of E', then

$$||S_n - 1|| = ||(S_n - 1)'|| \leq ||{S'_m}^{-1}|| \, ||(S_n - 1)S_m|| \to 0 \qquad (n \to \infty).$$

This is a contradiction. Hence, for all n, S'_n is not a topological isomorphism, and so there exists $x'_n \in E'$ such that $||S'_n(x'_n)|| \leq ||x'_n||/n$ and $|\pi| < ||x'_n|| \leq 1$, where π is an element of K with $|\pi| < 1$. Putting $y'_n = (1 - S_n)'x'_n$, we obtain that

$$|\langle x, y'_n \rangle| \leq ||x'_n|| \, ||(1 - S_n)(x)|| \to 0 \qquad (n \to \infty)$$

for every $x \in E$. Hence, $y'_n \to 0$ weak*. Since E is a G-space, $y'_n \to 0$ weakly. Further, since $||x'_n - y'_n|| \to 0$ $(n \to \infty)$, there exists a positive integer n_0 such that, for all $n > n_0$, $|\pi| < ||y'_n|| \leq 1$. Hence, there is $x_n \in E$ such that $|\pi| \leq |\langle x_n, y'_n \rangle| \leq 1$ and $|\pi| < ||x_n|| \leq 1$. Then, by Lemma 9,

$$|\langle (1 - S_n)x_n, x' \rangle| \leq ||(1 - S_n)'x'|| \, ||x|| \to 0 \qquad (n \to \infty).$$

It follows that $(1 - S_n)x_n \to 0$ weakly. Since K is spherically complete,

$$|\langle x_n, (1 - S_n)'y'_n \rangle| \leq ||(1 - S_n)(x_n)|| \, ||y'_n|| \to 0 \qquad (n \to \infty).$$

While we have

$$\begin{aligned}|\langle x_n, S_n' y_n'\rangle| &\le ||S_n' y_n'|| = ||(1-S_n)' S_n' x_n'|| \\ &\le ||S_n' x_n'|| \max(1, ||S_n||) \to 0 \qquad (n \to \infty).\end{aligned}$$

Hence, $|\langle x_n, y_n'\rangle| \to 0$ $(n \to \infty)$. This contradicts to $|\pi| \le |\langle x_n, y_n'\rangle|$ for all n, and the proof is complete. □

Corollary 11. *Let K be spherically complete. Let E be a G-space, and let (S_n) be a* (UM)*-sequence on E. Then the following are equivalent*:

(1) $S_n \to 1$ *weakly.*

(2) $S_n \to 1$ *strongly.*

(3) $S_n \to 1$ *uniformly.*

(4) $S_n' \to 1_{E'}$ *weak*.*

(5) $S_n' \to 1_{E'}$ *weakly.*

(6) $S_n' \to 1_{E'}$ *strongly.*

(7) $S_n' \to 1_{E'}$ *uniformly.*

Proof. It is clear that (3) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4). Since K is spherically complete, (1) $\Rightarrow$ (2) holds. By Theorem 10, (2) $\Rightarrow$ (3) is true. Finally, by definition, we see that (1) and (4) are equivalent. □

By induction, for every positive integer k we define the conjugate space $E^{(k)}$ and the conjugate operator $T^{(k)} \in L(E^{(k)}, E^{(k)})$ as follows:

$$\begin{aligned} E^{(1)} &= E', \qquad E^{(k+1)} = (E^{(k)})', \\ T^{(1)} &= T', \qquad T^{(k+1)} = (T^{(k)})'. \end{aligned}$$

Then, combining Corollaries 2 and 11, we can obtain the following corollary.

Corollary 12. *Let (S_n) be a* (UM)*-sequence on E. If K is spherically complete and its valuation is dense, then for each k the following are equivalent*:

(1) $S'_n \to 1_{E'}$ *weakly.*

(2) $S'_n \to 1_{E'}$ *strongly.*

(3) $S'_n \to 1_{E'}$ *uniformly.*

(4) $S_n^{(k)} \to 1_{E^{(k)}}$ *weak*.*

(5) $S_n^{(k)} \to 1_{E^{(k)}}$ *weakly.*

(6) $S_n^{(k)} \to 1_{E^{(k)}}$ *strongly.*

(7) $S_n^{(k)} \to 1_{E^{(k)}}$ *uniformly.*

If K is not spherically complete, then Corollary 11 is not true. This is shown by the following example.

Example 2. Let K not be spherically complete. (Recall that c_0 and l^∞ are reflexive and are G-spaces.) For each $n \geq 1$, consider the linear operator

$$S_n : l^\infty \to l^\infty : (x_1, x_2, \dots, x_n, \dots) \to (x_1, x_2, \dots, x_n, 0, \dots).$$

Then (S_n) is a (UM)-sequence on l^∞ which does not converge to 1 strongly. However, since K is not spherically complete, $(l^\infty)' \sim c_0$, and we see that $S_n \to 1$ weakly. Further, let T_n be the restriction of S_n to c_0. Then (T_n) is a Schauder decomposition on c_0, and $T_n \to 1$ strongly but not uniformly.

Moreover, in Corollary 11, we also need the condition that E is a G-space. This is indicated by the next example.

Example 3. Let K be spherically complete. Then c_0 is not a G-space. Let (T_n) be a (UM)-sequence on c_0 in Example 2. Then (T_n) converges to 1 strongly. On the other hand, since $c_0' \sim l^\infty$, we obtain

$$T'_n : l^\infty \to l^\infty : (y_1, y_2, \dots, y_n, \dots) \to (y_1, y_2, \dots, y_n, 0, \dots).$$

This implies that (T'_n) does not converge to 1 uniformly.

Theorem 13. *If $(P_n) \subset L(E, E)$ is a Schauder decomposition, then (P_n) is a* (UM)*-sequence such that $P_n \to 1$ strongly but not uniformly.*

Proof. It follows from the definition of the Schauder decomposition that (P_n) is a (UM)-sequence which converges to 1 strongly. We now show that it is not uniform. By definition, we have $P_n(E) \subsetneqq P_{n+1}(E)$. Then there exists $y \in E$ such that $P_{n+1}(y) \in P_{n+1}(E) \backslash P_n(E)$. Put $x_{n+1} = P_{n+1}(y) - P_n(y)$. Then $P_n(x_{n+1}) = 0$ and

$$||P_n - 1|| \geq \frac{||P_n(x_{n+1}) - x_{n+1}||}{||x_{n+1}||} = 1.$$

This completes the proof. □

From the preceding results, the following corollaries are readily deduced.

Corollary 14. *Let K be spherically complete. If E is a G-space, then E admits no Schauder decompositions.*

Corollary 15. *Let K be spherically complete and its valuation dense. Then every dual space and every weakly injective Banach space admit no Schauder decompositions. In particular, l^∞ and l^∞/c_0 admit no Schauder decompositions.*

Corollary 16. *Let the valuation of K be dense. Let $E_1, E_2, \ldots,$ be an infinite sequence of Banach spaces. Then K is not spherically complete if and only if $(\times E_n)'$ and $\oplus E_n'$ are linearly isometric.*

Proof. "If part" follows from Corollary 15 and "only if part" follows from Theorem 4.22 in [**7**]. □

In Corollary 14, we need the condition that K is spherically complete. This is induced by Example 2. And this example also leads that in non-Archimedean Banach space, Dean's theorem does not hold, for c_0 is a G-space and has the D-P property.

Combining Corollaries 11 and 14, we obtain the following:

Corollary 17. *Let K be spherically complete. If E is a G-space, then E admits no weak Schauder decompositions.*

If K is not spherically complete, then the sequence (S_n) on l^∞ in Example 2 is a weak Schauder decomposition on l^∞ but not a Schauder decomposition. Hence, in Corollary 16, spherical completeness of K is necessary. In general, the following proposition holds.

Proposition 18. *Let K not be spherically complete. Then the Banach space $l^\infty \oplus E$ has a weak Schauder decomposition which is not a Schauder decomposition.*

Combining Theorem 10 with Theorem 4.39 in [**7**], the next corollary follows.

Corollary 19. *Let K be spherically complete. If E is a G-space, then there is not a* (UM)*-sequence (S_n) on E such that for each n, S_n is of finite rank and $S_n \to 1$ strongly.*

Let K be spherically complete. Then c_0 is not a G-space, and the (UM)-sequence (T_n) in Example 2 converges to 1 strongly and T_n is of finite rank for each n. But it does not converge to 1 uniformly. In general, the following corollary holds.

Corollary 20. *Let K be spherically complete. Suppose that E contains a closed subspace D of countable type which is complemented. Then there exists a* (UM)*-sequence (S_n) on E such that for each n, S_n is of finite rank and $S_n \to P$ strongly but not uniformly, where P is a projection of E onto D.*

Proof. Since K is spherically complete and D is of countable type, D has an orthogonal base $\{e_i\}$ such that $|\pi| < ||e_i|| \leq 1$ (see [**7**, p. 169]). Let E_n be a closed linear hull of $\{e_1, e_2, \dots, e_n\}$. Then $D = \overline{\cup E_n}$, and for all $x \in E$ we can write $P(x) = \sum_{i=1}^{\infty} \alpha_i e_i$ ($\alpha_i \in K$, $\alpha_i \to 0$). Define a linear operator

$$S_n : E \to E_n : x \to \sum_{i=1}^{n} \alpha_i e_i \qquad (n = 1, 2, \dots).$$

Then we can see that (S_n) is the required (UM)-sequence. □

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