ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

ALMOST REGULAR INTEGER FIBONACCI PENTAGONS

JAMES H. JORDAN AND BLAKE E. PETERSON

Dedicated to the memory of Vern E. Hoggatt, Jr.

Introduction. An *integer pentagon* is a pentagon with integer sides and diagonals. Small examples of *integer pentagons* are scarce but two are shown in Figure 1. The one designated as A appears in Muller [**2**] and in Harborth and Kemnitz [**1**] and is claimed in the latter article to be the *integer pentagon* of smallest diameter (another at least as small exists). The one designated as B may never before have appeared in print. Both of these have the pleasing property of having four distances the same, three other distances the same, and two of the remaining three distances the same as well as having symmetry with respect to the bisector of the base.

Let F_n represent the n-th Fibonacci number and L_n represent the n-th Lucas number. Both of the pentagons of Figure 1 are generated by $F_3 = 2$ in the sense that $8 = 2^3$; $7 = 8 - F_1$; $6 = F_1F_3F_4$; $4 = F_2F_3F_3$; $3 = 8 - F_5$; $10 = F_2F_3F_5$ and $12 = F_3F_3F_4$.

It is the purpose of this paper to display for each F_n, $n \geq 3$, two integer pentagons generated by Fibonacci numbers (one, the type of A, and, the other, the type of B).

The sequences. To generalize, consider Figure 2. The values of $n \geq 3$ are:

$$\begin{aligned}
a_{n-2} &= F_{n-1}F_n^2\\
b_{n-2} &= F_{n-2}F_nF_{n+1}\\
c_{n-2} &= F_n^3\\
d_{n-2} &= F_n^3 + (-1)^n F_{n-2}\\
h_{n-2} &= F_{n+1}F_n^2\\
k_{n-2} &= F_{n-1}F_nF_{n+2}\\
g_{n-2} &= F_n^3 + (-1)^n F_{n+2}.
\end{aligned}$$

Received by the editors on July 18, 1990.

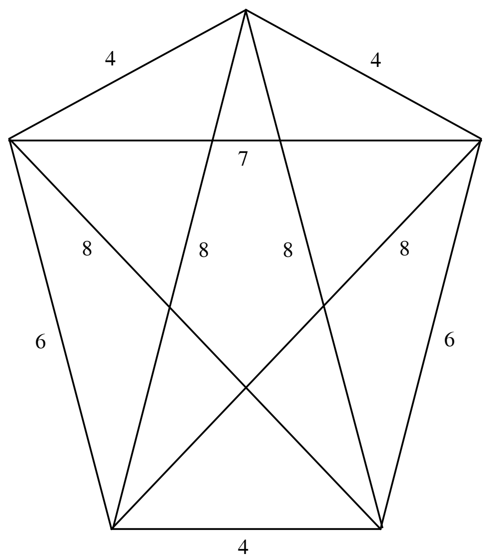

FIGURE 1A. Full House sides, Four of a Kind diagonals.

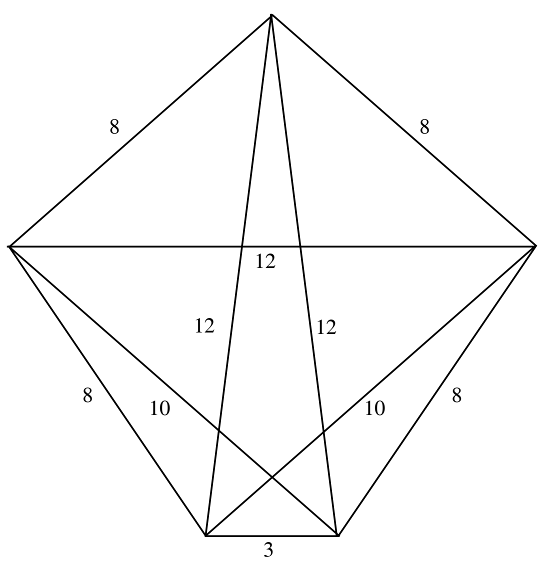

FIGURE 1B. Four of a Kind sides, Full House diagonals.

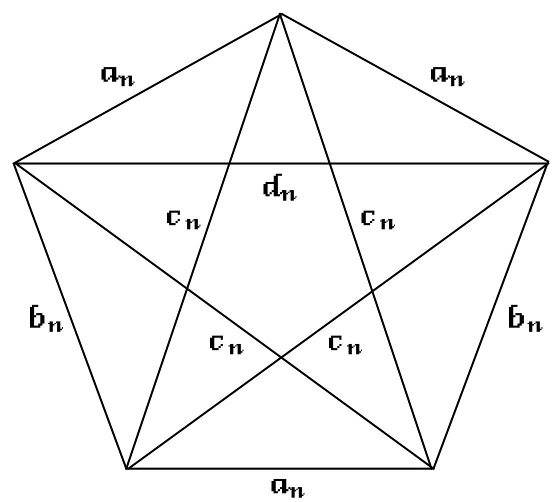

FIGURE 2A. Full House sides, Four of a Kind diagonals.

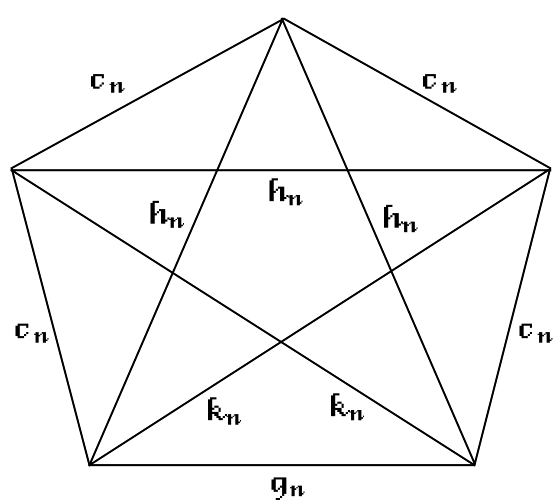

FIGURE 2B. Four of a Kind sides, Full House diagonals.

The verification of the distances requires only elementary concepts and the recurrence definition of Fibonacci numbers. The details are left to the reader.

Regular. As n gets large, the ratios a_n/b_n, c_n/d_n, c_n/g_n, h_n/k_n approach 1, and the ratios c_n/a_n, h_n/c_n, d_n/b_n, k_n/g_n approach $(1+\sqrt{5})/2$. This indicates that both the sequences of pentagons approach regular pentagons (percentage wise).

Circumscribable. Each of the pentagons has its vertices on a circle; hence they are cyclic pentagons. The radii of those circles for type A and type B are respectively:

$$\text{(A)} \quad r_{A,n-2} = \frac{F_n^4}{\sqrt{L_{n-1}F_{n+2}}} \qquad \text{(B)} \quad r_{B,n-2} = \frac{F_n^4}{\sqrt{L_{n+1}F_{n-2}}}.$$

If these circles were centered at the origin then the coordinates of their vertices would be:

(A)

$$(0, r_{A,m}); \ \left(\frac{\pm a_m}{2}; -\frac{\sqrt{4r_{A,m}^2 - a_m^2}}{2}\right); \ \left(\frac{\pm d_m}{2}; -\frac{\sqrt{4r_{A,m}^2 - d_m^2}}{2}\right);$$

(B)

$$(0, r_{B,m}); \ \left(\frac{\pm g_m}{2}; -\frac{\sqrt{4r_{B,m}^2 - g_m^2}}{2}\right); \ \left(\frac{\pm h_m}{2}; -\frac{\sqrt{4r_{B,m}^2 - h_m^2}}{2}\right).$$

Generated. To generate the A sequence, begin with the Fibonacci numbers F_n and F_{n-1} and form the triangle that is similar, by a factor of F_n^2, to the isosceles triangle with two sides of length F_n and one side of length F_{n-1}, $n \geq 3$. From the apex of this triangle build the other two symmetric vertices.

To generate the B sequence, begin with the Fibonacci numbers F_n and F_{n+1} and form two triangles that are similar, by a factor of F_n^2 to the isosceles triangle with two sides of length F_n and one side of length F_{n+1}, $n \geq 3$. Attach these two triangles at a vertex and insert the "rafter" with length the same as the longest side to complete the structure.

REFERENCES

1. H. Harborth and A. Kemnitz, *Diameters of integral point sets*, Colloquia Mathematica Societatis Jano Bolyai **48**, Intuitive Geometry, Siofok, 1985.

2. A. Muller, *Auf Eine Kreis Liegende Punktmengen Ganzzahliger Entfurnungen*, Elemente der Math. **8** (1953), 37–38.

Washington State University, Pullman, WA 99164–2930