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A LITTLE WEDDERBURN PRINCIPAL THEOREM

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ABSTRACT. We point out that, even if $A/\operatorname{rad} A$ is inseparable, the finite-dimensional algebra (or, more generally, left Artinian ring) A admits a canonical ideal direct sum decomposition $A = A^0 \oplus A^{\#}$ such that $\operatorname{rad} A^0 = (0)$, $\operatorname{rad} A^{\#} = \operatorname{rad} A$, $\operatorname{rad} A^{\#}$ is an essential ideal of $A^{\#}$, and $A^{\#}$ is unital if and only if A is unital. This is a consequence of a general and maximally elementary splitting of an arbitrary ring relative to a suitable nil ideal. This process of *essentializing* a nil ideal is useful in the study of categories of ideal and radical embeddings.

1. Background. We begin with an associative algebra A, not necessarily unital, which is finite-dimensional over the field k. Let N denote the nilpotent radical $\operatorname{rad} A$ of A. Suppose we wish to explore the interactions of N with other parts of A (the third main problem in the study of algebras). How might the standard theory guide us?

The celebrated Wedderburn principal theorem [**1**, Theorem 3.23; **2**, Theorem 72.19; **5**, Theorem 11.6] assures us that if the semi-simple k-algebra A/N is separable, in particular, if the scalar field k is perfect, then A contains at least one separable subalgebra (*Wedderburn factor*) S such that

$$A = S \dotplus N \qquad (k\text{-direct sum}). \tag{1.1}$$

Next, by Wedderburn's comparatively elementary results on semi-simple algebras, we have a canonical decomposition of S into two-sided ideals

$$S = S_0 \oplus S_1 \qquad (S\text{-direct sum})$$

provided we define $S_0 = \operatorname{ann}_S(N) =$ the two-sided annihilator of the S-bimodule N. Thus, S_0 is either zero or semi-simple and $S_1 = \operatorname{ann}_S(S_0)$ is the unique multiplicatively orthogonal complement of S_0 in S.

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Pushing further, we note that if T is another Wedderburn factor in A, and if $T = T_0 \oplus T_1$ as above, then in fact $T_0 = S_0$. This follows easily from the Malcev addendum to the principal theorem, which assures us, still assuming A/N separable, that any two Wedderburn factors are conjugate in the strong sense that $T = (1+x)S(1+x)^{-1}$ for some x in N. (The 1 here is possibly formal.) Thus we discover that S_0 is in fact a two-sided ideal of the full algebra A (not so in general for S_1) and we thereby arrive at the canonical ideal decomposition

$$A = A^0 \oplus A^{\#} \qquad (A\text{-direct sum}) \tag{1.2}$$

with $A^0 = S_0$ and $A^{\#} = \operatorname{ann}_A(A^0) =$ the two-sided annihilator of A^0 in A. Thus, $A^{\#} = (1-u)A(1-u)$, where u is the unity of A^0 and the 1 is possibly formal.

This decomposition of A illuminates interactions of N with other parts of A as follows:

i) the ideal A^0 has zero overlap with N and comprises those nonradical elements of A which interact trivially with N;

ii) in contrast, the ideal $A^{\#}$ contains N as its radical, just as A does, but now N is an *essential* ideal of $A^{\#}$, having nonzero overlap with every proper ideal of $A^{\#}$. Note also that $A^{\#} = (0)$ if and only if $N = (0)$ and $A^{\#}$ is unital (cf. the idempotent $1-u$ above) if and only if A is unital. This means, of course, that in studying A as algebra-with-radical, a good first step is to forget A^0 and focus on $A^{\#}$, thereby *essentializing* the radical.

The point of this note is that the canonical decomposition (1.2) is both more general and more elementary than indicated above: *every finite-dimensional k-algebra A admits a canonical decomposition* (1.2) *even if A/N is inseparable and A fails to admit a full Wedderburn decomposition* (1.1). In fact, the same is true if A is merely a left Artinian ring (Corollary 2.3 below). Both of these assertions are special cases of a very general statement about arbitrary rings having nil ideals of suitable type (Theorem 2.1).

Our proof below is, of course, much more elementary than that of the full Wedderburn principal theorem. In particular, we require no notion of separability, no conjugacy results, no cohomology, and no facts about the structure of simple algebras. Some such machinery would surely be required to lift complete systems of primitive orthogonal idempotents;

our observation here, stated for A/N, is simply the existence of one particular idempotent which can always be lifted, easily, uniquely, and usefully.

Application. The decomposition (1.2) and its generalizations to other rings are useful in studying categories of embeddings, particularly radical embeddings, of a given nilpotent N, that is, pairs (A, α) with $\alpha : N \to A$ a monomorphism such that $\alpha(N) = \operatorname{rad} A$ [**3**, **4**]. The procedure of replacing A with $A^{\#}$ determines an idempotent functor on the category of radical embeddings of N. It follows that, over an arbitrary field of scalars, not necessarily perfect, every radical embedding (A, α) of N is *stably equivalent* to one in which the radical is an essential ideal [**3**, Section 2]. This in turn yields a canonical form for equivalent radical embeddings and leads to a proof of the existence of extreme radical embeddings.

2. Statements. We fix an associative ring R, not necessarily unital, containing a nil ideal N. Although N need not be an R/N-bimodule in any natural way (alas!), it *is* true that the additive group N/N^2 becomes an R/N-bimodule if we define, for all a in R and x in N,

$$(a + N)(x + N^2) = ax + N^2$$

and similarly for R/N acting on the right.

Having this, we denote $\Omega(R; N) = \operatorname{ann}_{R/N}(N/N^2) =$ the two-sided annihilator of the bimodule N/N^2 in the ring R/N. We will be interested in ideals Q of R/N which are contained in $\Omega(R; N)$ and which are unital as rings. Such an ideal Q will be called an $\Omega(R; N)$-*factor*. Note that the ideal Q is unital if and only if it is generated by an idempotent e central in Q (which idempotent is thereby central in R/N).

We write $N^\infty = \cap_{i \geq 1} N^i$.

Theorem. *Let R be a ring, not necessarily unital, containing a nil ideal N with $N^\infty = (0)$. Let Q be an $\Omega(R; N)$-factor of R/N. Then there exist unique ideals $Q^\wedge, Q^+$ of R such that*

i) $R = Q^\wedge \oplus Q^+$,

ii) *Q^+ contains N,*

iii) *the natural map $R \to R/N$ induces an isomorphism $Q^\wedge \to Q$.*

We will prove this in Section 3 below. Meanwhile, observe that if Q_1, Q_2 are $\Omega(R;N)$-factors with unity elements e_1, e_2, respectively, then the join $Q_1 + Q_2$ has unity $e_1 - e_1e_2 + e_2$ and therefore is also an $\Omega(R;N)$-factor. Thus, if a maximal $\Omega(R;N)$-factor Q exists, then it must be unique and an invariant of the pair R, N. Putting $R^0(N) = Q^\wedge$ and $R^\#(N) = Q^+$, we deduce

2.2 Corollary. *Let R be a ring, not necessarily unital, containing a nil ideal N with $N^\infty = (0)$. Suppose R/N contains a maximal $\Omega(R;N)$-factor. Then there exist unique ideals $R^0(N), R^\#(N)$ of R such that*

i) $R = R^0(N) \oplus R^\#(N)$,

ii) *$R^0(N)$ is a unital ring,*

iii) $R^0(N) \cap N = (0)$,

iv) *$R^0(N)$ is maximal with respect to* i), ii), iii),

v) *$R^\#(N)$ contains N,*

vi) *if $\Omega(R;N)$ is a unital ring, then $R^\#(N)$ contains N as an essential ideal.*

Finally, we observe that if $A = R$ is left Artinian and $N = \operatorname{rad} A$, then $\Omega(A;N)$ is (0) or a semisimple ideal in A/N. This gives us the result of the title, as follows.

2.3 Corollary. *Let A be a left Artinian ring, not necessarily unital. Then there exist unique ideals $A^0, A^\#$ in A such that*

i) $A = A^0 \oplus A^\#$,

ii) $\operatorname{rad} A^0 = (0)$,

iii) *A^0 is maximal with respect to* i) *and* ii)*. In particular,* $\operatorname{rad} A = \operatorname{rad} A^\#$ *is an essential ideal of $A^\#$.*

A consequence of 2.3 is the observation that $\mathrm{ann}_A(\mathrm{rad}\, A)$, the two-sided annihilator of $\mathrm{rad}\, A$ in a left Artinian ring A, decomposes uniquely as $A^0 \oplus \mathrm{ann}_{\mathrm{rad}\, A}(\mathrm{rad}\, A)$, an A-direct sum. The second summand here is the self-annihilator of $\mathrm{rad}\, A$.

3. Proof of Theorem 2.1. Let $\pi : R \to R/N$ be the natural epimorphism, and let e be the unity element of the given $\Omega(R; N)$-factor Q.

(3.1) Claim. *We claim there is an idempotent u in R with $\pi(u) = e$.*

Proof of Claim (3.1). We construct u as in the proof [**2**, p. 160] of Brauer's lemma that every nonnilpotent left ideal in a left Artinian ring contains a nonzero idempotent element. (Curiously enough, Brauer's lemma itself is not quite sufficient to prove 2.1, even if R is Artinian.) Thus, we choose any f in R such that $\pi(f) = e$. Since $\pi(f^2) = e$, we have $f^2 - f = x$ in N. If $x \neq 0$, define f_1 in R by

$$f_1 = f + x - 2fx.$$

Note that $\pi(f_1) = e$ also. One checks that, crucially, $f_1^2 - f_1$ is of the form $ax^2 = x^2 a$. It follows that, since x is nilpotent, repetitions of this procedure will eventually construct an idempotent $u = f_r$ as claimed. □

Note that this part of the proof requires only that N be a nil ideal.

(3.2) Claim. *We claim that $uN = Nu = (0)$.*

Proof of Claim (3.2). To prove this, note that, for n in N, we have $un + N^2 = (u+N)(n+N^2) = e(n+N^2) = 0$, since e is in $Q \subseteq \Omega(R; N)$. Thus, $uN \subseteq N^2$. But u is idempotent. Thus, $uN \subseteq N^i$ for $i = 1, 2, \ldots$, and so $uN \subseteq N^\infty$, which is zero by hypothesis. The claim follows. □

(3.3) Claim. *We claim that u as the unique idempotent of R with $\pi(u) = e$.*

Proof of Claim (3.3). For if v in R has $\pi(v) = e$, then $v = u + n$ with n in N. By (3.2), we have $v^2 = (u+n)^2 = u + n^2$, and this equals $u+n$ if and only if the nilpotent $n = 0$. □

(3.4) Claim. *We claim that u is central in R.*

Proof of Claim (3.4). Let a be in R. Then $\pi(ua - au) = e\pi(a) - \pi(a)e = 0$, so that $ua - au = n$ in N. Multiplying this by u both on the left and on the right and applying (3.2) gives both $ua - uau = 0$ and $uau - au = 0$, whence $ua = au$. □

(3.5) *Note.* The rest of the proof is entirely straightforward. We define $Q^\wedge = uRu$ and $Q^+ = (1-u)R(1-u)$. Here the 1 is possibly formal. Note that $Q^\wedge$ is unique because u is (3.3). Since Q^+ is the orthogonal complement to $Q^\wedge$ with respect to the central idempotent u, the ideal Q^+ is also unique. Statements i), ii) and iii) are immediate. This completes the proof of the theorem. □

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