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THE COINCIDENCE NIELSEN NUMBER ON NON-ORIENTABLE MANIFOLDS

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ABSTRACT. The coincidence Nielsen theory is generalized onto the pairs of maps between non-orientable manifolds. The formula for the Nielsen number for a pair of self-maps of the Klein bottle is given.

Introduction. The Nielsen fixed point theory [**3, 10**] was extended [**11, 3**] into the coincidences of maps $(f,g) : M \to N$ between closed oriented manifolds of the same dimension. This generalization used the notion of coincidence index which is defined in orientable case [**12**]. In this paper we drop the orientability assumption of M and N. Now we cannot follow the procedure from [**3**] since there is no suitable index theory. Instead of this, we introduce in Section 1 the semi-index of a Nielsen class. The idea of this semi-index was inspired by the paper of B.J. Jiang [**9**]. At first we define the reducibility relation on the Nielsen class. Then we try to split this class onto pairs of points in this relation. The number of remaining points is taken for our semi-index. It is a nonnegative integer, a homotopy invariant, which allows us to get the Nielsen number as before. Now any pair of continuous maps between two smooth manifolds $(f,g) : M \to N$ has at least $N(f,g)$ coincidence points. On the other hand, we prove the Wecken theorem under the assumption $\dim M = \dim N \geq 3$. Thus, in this case $N(f,g)$ is the best lower bound of the number of coincidence points.

In Section 2 we find a dependence between this Nielsen number of a pair (f,g) and of its lifts to covering spaces. We apply this dependence in Section 3 to compute $N(f,g)$ for all pairs of self maps of the Klein bottle.

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1. The Nielsen number. Let u and v denote paths in a topological

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space X. We will call them homotopic and we will write $u \simeq v$ if and only if $u(0) = v(0)$, $u(1) = v(1)$ and they are fixed end-points homotopic. If $(f, g) : X \to Y$ are continuous maps, then we denote the coincidence set of these maps by $\Phi(f, g) = \{x \in X : fx = gx\}$. We will say that two coincidence points $x, y \in \Phi(f, g)$ are Nielsen equivalent if and only if there exists a path ω joining them such that $f\omega \simeq g\omega$. This is an equivalence relation, and we will denote the quotient set by $\Phi'(f, g)$.

Let E be a real vector space of finite dimension. We will denote by $\alpha = [(a_1, \dots, a_k)]$ the orientation of E determined by the ordered basis $(a_1, \dots, a_k)$. Let $E = E_1 \oplus E_2$, and let $\alpha = [(a_1, \dots, a_k)]$, $\beta = [(b_1, \dots, b_1)]$ be fixed orientations of E_1 and E_2, respectively. Then $\alpha \wedge \beta$ will stand for the orientation of E determined by the basis $(a_1, \dots, a_k, b_1, \dots, b_1)$. This operation is associative: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$. Let $\phi : E \to E'$ be a monomorphism, and let $\alpha = [(a_1, \dots, a_k)]$ be an orientation of E. Then ϕ determines an orientation of $\phi(E)$ by

$$\phi(\alpha) = [(\phi(a_1), \dots, \phi(a_k)].$$

By a (local) orientation of a smooth manifold M at the point x, we will mean an orientation of the tangent space T_xM. Let V and W be two smooth manifolds. The pair of maps $(f, g) : V \to W$ will be called transverse if and only if both f and g are smooth and, for any coincidence point $x \in \phi(f, g)$, the difference of the induced tangent maps $T_xf - T_xg : T_xV \to T_{fx}W$ is an epimorphism. Two smooth maps f and g are transverse if and only if the graphs $\Gamma_f = \{(x, fx) \in V \times W; x \in V\}$, $\Gamma_g = \{(x, gx) \in V \times W; x \in V\}$ are transverse as the submanifolds of $V \times W$ [**7**]. Then $\Phi(f, g)$ is a submanifold of V, $\dim \Phi(f, g) = \dim V - \dim W$ and $\delta\Phi(f, g) \subseteq \delta V$ (here δ denotes the boundary of a manifold).

The following lemma is an easy consequence of the transversality theorems [**7**]:

Lemma (1.1). *Any pair of maps $(f, g) : V \to W$ is homotopic to a transverse pair. Moreover, if the restriction $(f, g)|_{\delta V} : \delta V \to W$ is transverse, then we may assume that this homotopy is constant on δV.*

Let M, N be two closed smooth connected manifolds of the same dimension, $(f, g) : M \to N$ a transverse pair, and let $x, y \in \Phi(f, g)$. We fix local orientations:

$\alpha_0(f)$—of the graph Γ_f at the point (x, fx) and

$\alpha_0(g)$— of the graph Γ_g at the point (x, gx).

Since $x \in \Phi(f, g)$ and (f, g) is regular, so both graphs are transverse one to another at the point $(x, fx) = (x, gx)$, and we get the local orientation $\alpha_0 = \alpha_0(f) \wedge \alpha_0(g)$ of the manifold $M \times N$ at this point.

Definition (1.2). Under the above assumptions, we will say that x and y reduce one another if and only if there exists a path ω from x to y such that

a) $f\omega \simeq g\omega$.

b) if $\alpha_1(f)$ denotes the translation of $\alpha_0(f)$ in Γ_f along the path $(\omega, f\omega)$,

$\alpha_1(g)$ denotes the translation of $\alpha_0(g)$ in Γ_g along the path $(\omega, g\omega)$,

α_1 denotes the translation of α_0 in $M \times N$ along the path $(\omega, f\omega)$,

then $\alpha_1(f) \wedge \alpha_1(g) = -\alpha_1$.

We will say that the above path ω reverses orientation in graphs.

Let $A \subset \Phi(f, g)$ be any subset. It may be represented as $\Phi(f, g) = \{a_1, b_1, \dots, a_k, b_k; c_1, \dots, c_s\}$ where a_i, b_i reduce one another but so does no pair c_i, c_j, $i \neq j$. We call the elements $\{c_1, \dots, c_s\}$ free in this decomposition.

Lemma (1.3). *The number of free elements does not depend on the decomposition of A.*

Proof. Let us consider two decompositions $\mathcal{U}$ and $\mathcal{B}$. For any $a, b \in A$ we will write $a \top b$ if a and b form a pair in $\mathcal{U}$, $a \perp b$ if a and b form a pair in $\mathcal{B}$, $a \leftrightarrow b$ if they reduce one another. Let us notice that the last relation has the following odd-transitivity property: if $a_0 \leftrightarrow a_1, \dots, a_{k-1} \leftrightarrow a_k$ and k is odd, then $a_0 \leftrightarrow a_k$. We will prove our lemma inductively with respect to $\#A$ ($\#$ denotes cardinality).

For $\#A = 0$ the lemma is obvious, so we may assume that it holds for $\#A < n$ and we assume that $\#A = n$. If in both decompositions there is no free element, then the lemma is evident so we suppose that a_1 is a free element in $\mathcal{U}$. If a_1 is also free in $\mathcal{B}$ then we apply the inductive assumption to $A - a_1$. Otherwise, let us suppose that $a_1 \perp a_2$ for some a_2. Then a_2 cannot be free in $\mathcal{U}$ so $a_2 \top a_3$ for some a_3. Then we look for a_4 such that $a_3 \perp a_4$. If such a_4 exists, then the odd transitivity implies $a_1 \leftrightarrow a_4$ so a_4 cannot be free in $\mathcal{U}$ and $a_4 \top a_5$ for some a_5. Following this procedure, we get a sequence $a_1 \perp a_2 \top a_3 \perp \dots \perp a_{2k} \top a_{2k+1}$ where $a_i \neq a_j$ for $i \neq j$, a_1 is free in $\mathcal{U}$ and a_{2k+1} is free in $\mathcal{B}$. By the inductive assumption the numbers of free elements in $\mathcal{U}$ and $\mathcal{B}$ contained in $A - (a_1, \dots, a_{2k+1})$ are also the same. □

Thus, we may define for a transverse pair $(f,g) : M \to N$ the semi-index of a Nielsen class A as the number of free elements in any decomposition and we will denote it by $|\text{ind}|(f,g : A)$.

Lemma (1.4). *Let (f_0,g_0), $(f_1,g_1) : M \to N$ be transverse pairs and let $(F,G) : M \times [0,1] \to N$ be a homotopy between them. Let $A_0 \in \Phi'(f_0,g_0)$ correspond to $A_1 \in \Phi'(f_1,g_1)$. Then $|\text{ind}|(f_0,g_0 : A_0) = |\text{ind}|(f_1,g_1 : A_1)$.*

Proof. For notational convenience, we assume that the considered homotopy is given by $(F,G) : [0,1] \times M \to N$, and we may assume that it is transverse (1.1). We may also assume that

$$F(t,x) = F(0,x), \qquad G(t,x) = G(0,x) \quad \text{for } t \leq 1/3$$

and

$$F(t,x) = F(1,x), \qquad G(t,x) = G(1,x) \quad \text{for } t \geq 2/3.$$

Then $\Phi(F,G)$ is a one-dimensional submanifold of $I \times M$ orthogonal to $\delta I \times M$.

At first we will consider a connected component of $\Phi(f,g)$ with ends $(0,x),(0,x')$. We will show that the points $x, x' \in \Phi(f_0,g_0)$ reduce one another. Then we will show that if $(0,x_0),(1,x_1)$ are ends of a component and $(0,x_0'),(1,x_1')$ are ends of the other one, then the points $x_0, x_0' \in \Phi(f_0, g_0)$ reduce one another if and only if so do $x_1, x_1' \in \Phi(f_1, g_1)$.

The above information gives rise to the decompositions of A_0 and A_1 with the same number of free points.

Let $\Lambda \subset \Phi(F, G)$ be a component parametrized by $\omega(t) = (\omega_1(t), \omega_2(t))$ where $\omega_1(0) = \omega_1(1) = 0$. We are going to show that $x_0 = \omega_2(0)$ and $x_1 = \omega_2(1) \in \Phi(f_0, g_0)$ reduce one another. Here we identify M with $0 \times M \subset I \times M$. We denote by τ_t and ν_t the tangent and the normal space to $\Lambda \subset I \times M$ at the point $\omega(t)$. Let α_t, β_t denote continuous families of orientations of ν_t and τ_t, respectively. Let β_0 be directed inwards, then β_1 is directed outwards. Since Λ is orthogonal to $M = 0 \times M \subset I \times M$, α_0 is an orientation of $T_{x_0}M$. Let $\tilde{\alpha}_t$ denote the translation of α_0 along ω_2. We will show at first that $\tilde{\alpha}_1 = -\alpha_1$. Let η_t denote the orientation of the tangent bundle to $I \times \omega_2(t)$ at the point $(0, \omega_2(t))$ directed inwards. Then $\alpha_t \wedge \beta_t$, $\tilde{\alpha}_t \wedge \eta_t$ are translations of $\alpha_0 \wedge \beta_0$ in $I \times M$ along homotopic paths ω and $(0, \omega_2)$ so $\alpha_1 \wedge \beta_1 = \tilde{\alpha}_1 \wedge \eta_1$. But $\beta_1 = -\eta_1$ since β_1 is directed outwards and η_1 inwards so $\tilde{\alpha}_1 = -\alpha_1$.

Now $(1_M, f_0)_*\alpha_0$ and $(1_M, g_0)_*\alpha_0$ are orientations of Γ_{f_0} and Γ_{g_0} at the point $(x_0, f_0x_0) = (x_0, g_0x_0)$ and $(1_M, f_0)_*\tilde{\alpha}_t$, $(1_M, g_0)_*\tilde{\alpha}_t$ are their translations along the paths $(\omega_2, f_0\omega_2)$ and $(\omega_2, g_0\omega_2)$, respectively. We denote $\alpha = (1_M, f_0)_*(\tilde{\alpha}_1) \wedge (1_M, g_0)_*(\tilde{\alpha}_1)$. Let b denote the translation of $(1_M, f_0)_*(\alpha_0) \wedge (1_M, g_0)_*(\alpha_0)$ in $M \times N$ along $(\omega_2, f_0\omega_2)$. We have to show that $\alpha = -b$. Let us consider the orientation $(1_{I\times M}, F)_*(\eta_0) \wedge (1_M, f_0)_*(\alpha_0) \wedge (1_M, g_0)_*(\alpha_0)$ of the space $I \times M \times N$ at the point $(0, x_0, f_0x_0)$. Since the paths $(0, \omega_2, f_0\omega_2)$, $(\omega, F\omega)$ are homotopic in $I \times M \times N$, so the translations of the considered orientation along these paths are equal. Translating along $(0, \omega_2, f_0\omega_2)$ we get $(1_{I\times M}, F)_*(\eta_1) \wedge b$; on the other hand, the translation along $(\omega, F\omega)$ has the form $(1_{I\times M}, F)_*(\beta_t) \wedge (1_{I\times M}, F)_*(\alpha_t) \wedge (1_{I\times M}, G)_*(\alpha_t)$ so for $t = 1$ we get

$$\begin{aligned}
(1_{I\times M}, F)_*(\beta_1) &\wedge (1_{I\times M}, F)_*(\alpha_1) \wedge (1_{I\times M}, G)_*(\alpha_1) \\
&= (1_{I\times M}, F)_*(\beta_1) \wedge (1_{I\times M}, F)_*(\tilde{\alpha}_1) \wedge (1_{I\times M}, G)_*(\tilde{\alpha}_1) \\
&= (1_{I\times M}, F)_*(\beta_1) \wedge \alpha.
\end{aligned}$$

This way we get the equality $(1_{I\times M}, F)_*(\eta_1) \wedge b = (1_{I\times M}, F)_*(\beta_1) \wedge \alpha$. Now $\eta_1 = -\beta_1$ implies $\alpha = -b$.

Now we consider two components $\Lambda, \Lambda' \subset \Phi(f,g)$ parametrized by $\omega(t) = (\omega_1(t), \omega_2(t))$, $\omega'(t) = (\omega_1'(t), \omega_2'(t))$ such that $\omega_1(i) = \omega_1'(i) = i$, $i = 0, 1$. Let us denote $x_i = \omega_2(i)$ $x_i' = \omega_2'(i)$, $i = 0, 1$ and suppose that the points $x_0, x_0' \in \Phi(f_0, g_0)$ reduce one another by the path σ. We will show that then $x_1, x_1' \in \Phi(f_1, g_1)$ reduce one another by the path $\omega_2^{-1} * \sigma * \omega_2'$.

Let us denote by τ_t, ν_t, (τ_t', ν_t') the tangent and normal spaces to $\Lambda \subset I \times M \times N$, $\Lambda' \subset I \times M \times N$, at the point $\omega(t)$ $(\omega'(t))$. Let α_t, β_t be orientations of ν_t and τ_t. Let $\bar{\alpha}_t, \bar{\beta}_t$ be such orientations of $T_{(\sigma(t))} \times M$ and $T_{(\sigma(t))}(I \times \sigma(t))$ that $\bar{\alpha}_0 = \alpha_0$ and $\bar{\beta}_0 = \beta_0$. Let α_t', β_t' be orientations of ν_t', τ_t' such that $\alpha_0' = \bar{\alpha}_1, \beta_0' = \bar{\beta}_1$. Since β_1 and β_1' have the same direction so after translating α_1 along $\tilde{\sigma} = \omega_2^{-1} * \sigma * \omega_2'$ we obtain α_1'. Thus the translation of $Tf_1(\alpha_1)$, $(Tg_1(\alpha_1))$ in the graph Γ_{f_1} (Γ_{g_1}) along $(\tilde{\sigma}, f\tilde{\sigma})$ $((\tilde{\sigma}, g\tilde{\sigma}))$ gives $Tf_1(\alpha_1')$ $(Tg_1(\alpha_1'))$. Let us denote $\alpha = Tf_1(\alpha_1') \wedge Tg_1(\alpha_1')$. On the other hand, let us denote by b the translation of $Tf_1(\alpha_1) \wedge Tg_1(\alpha_1)$ in $M \times N$ along $(\tilde{\sigma}, f\tilde{\sigma})$. It remains to show that $\alpha = -b$. To do this, we notice that the translation of $TF(\beta_1) \wedge Tf_1(\alpha_1) \wedge Tg_1(\alpha_1)$ along $(0, \tilde{\sigma}, f\tilde{\sigma})$ is $TF(\beta_1') \wedge b$. But the path $(0, \tilde{\sigma})$ is homotopic to $\omega^{-1} * (0, \sigma) * \omega'$, so we translate the considered orientation in $I \times M \times N$ along $(\omega, F\omega)^{-1} * (0, \sigma, f_0\sigma) * (\omega', F\omega')$. After the path $(\omega, F\omega)^{-1}$, we get $TF(\beta_0) \wedge Tf_0(\alpha_0) \wedge Tg_0(\alpha_0)$. The assumption that x_0 and x_0' reduce one another implies that after $(0, \sigma, f_0\sigma)$ we get $-TF(\bar{\beta}_1) \wedge Tf_0(\bar{\alpha}_1) \wedge Tg_0(\bar{\alpha}_1)$. At last, after $(\omega_2', F\omega_2')$, we get $-TF(\beta_1') \wedge Tf(\alpha_1') \wedge Tg(\alpha_1')$. Thus, $b = -Tf_1(\alpha_1') \wedge Tg_1(\alpha_1') = -\alpha$. □

The above lemma allows us to extend the definition of semi-index onto arbitrary continuous pairs $(f,g) : M \to N$. We put

$$|\mathrm{ind}\,|(f, g : A) = |\mathrm{ind}\,|(f', g' : A')$$

where (f', g') is any transverse pair homotopic to (f,g) and the class A' corresponds to A by this homotopy.

Definition (1.5). Let $(f,g) : M \to N$ be continuous. A Nielsen class $A \in \Phi'(f,g)$ will be called *essential* if and only if $|\mathrm{ind}\,|(f, g : A) \neq 0$. Otherwise, it is called *unessential*. We define the *Nielsen number* of (f,g) as the number of its essential classes, and we denote it by $N(f,g)$.

The above definition generalizes the definitions given in [**12**]. It follows from

Lemma (1.6). *Let* $(f, g) : M \to N$ *be a pair of maps between orientable manifolds, and let* $A \subset \Phi(f, g)$ *be a Nielsen class. Then* $|\mathrm{ind}|(f, g : A) = |\mathrm{ind}\,(f, g : A)|$ *where the right side denotes the absolute value of the ordinary coincidence index* [**12**].

Proof. Let us fix orientations in M and N. We may assume that (f, g) is transverse. Then A is finite and $\mathrm{ind}\,(f, g : a) = +1$ or -1 for any $a \in A$. It follows from [**3**] that two points $a, b \in A$ reduce one another if and only if $\mathrm{ind}\,(f, g : a) = -\mathrm{ind}\,(f, g : b)$. Thus, for a decomposition $A = \{a_1, b_1, \dots, a_k, b_k : c_1, \dots, c_s\}$

$$\mathrm{ind}\,(f, g : A) = \sum_{i=1}^{s} \mathrm{ind}\,(f, g : c_i) = +s \quad (\text{or } -s)$$

since $\mathrm{ind}\,(f, g : a_i) = -\mathrm{ind}\,(f, g : b_i)$ and either all $\mathrm{ind}\,(f, g : c_i) = +1$ or all $\mathrm{ind}\,(f, g : c_i) = -1$. □

Now it is evident that any pair of maps (f, g) has at least $N(f, g)$ coincidence points. On the other hand, we will prove:

Wecken Theorem (1.7). *Let* $(f, g) : M \to N$ *be two continuous maps between two smooth closed manifolds of the same dimension* $k \geq 3$*. Then there exists a pair* (f_1, g_1) *homotopic to* (f, g) *such that* $\#\Phi(f_1, g_1) = N(f, g)$*.*

The proof needs some preparations. Let us recall at first "the Whitney trick" [**9**]:

Whitney Lemma (1.8). *Let* R *denote a smooth manifold without boundary. Let* P *and* Q *be its smooth submanifolds such that their dimensions satisfy* $p + q = r$*. Let* $x, y \in P \cap Q$ *be transverse intersection points. Let* u *and* v *be smooth arcs joining these points in* P *and* Q*, respectively, which are homotopic in* R*, and let* $u(I) \cap v(I) = u\{0, 1\} = v\{0, 1\} = \{x, y\}$*. Let* $\alpha_0 (\beta_0)$ *be a local orientation of* P (Q) *at the point*

x, and let α_1 (β_1) be its translation to the point y along the path u, (v). Moreover, let us assume that $\alpha_1 \wedge \beta_1$ is opposite to the translation of $\alpha_0 \wedge \beta_0$ in R along the path α. Let $p, q \geq 3$. Then, for any open neighborhood U of $\alpha(I)$ there exists a smooth isotopy constant outside U carrying P onto a submanifold P' such that $P' \cap Q = P \cap Q - \{x, y\}$.

Now we modify *cancelling and creating* procedures into the nonorientable case [**10, 3**].

Cancelling procedure (1.9). *Let M, N be smooth manifolds without boundary of dimension $k \geq 3$. Let $f, g : M \to N$ be such continuous maps that $\Phi(f, g)$ is finite and $x_0, x_1 \in \Phi(f, g)$ are transverse coincidence points reducing one another. Then f is homotopic to a map f_1 such that $\Phi(f_1, g_1) = \Phi(f, g) - \{x_0, x_1\}$ and any two points in $\Phi(f_1, g_1)$ reduce one another if and only if they do so in $\Phi(f, g)$.*

Proof. Since x_0 and x_1 reduce one another, so there exists a path ω joining them such that $f\omega \simeq g\omega$ in N, reversing orientation in graphs. We may assume that ω is a smooth arc avoiding other coincidence points. Then the arcs $u = (\omega, f\omega)$ and $v = (\omega, g\omega)$ are homotopic in $M \times N$. Let us fix an open neighborhood $U \subset M$ of the set $\omega(I)$ such that $U \cap \Phi(f, g) = \{x_0, x_1\}$. Let $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ denote projections. It is easy to check that: the manifolds $P = \pi_1^{-1} U \cap \Gamma_f$, $Q = \pi_1^{-1} U \cap \Gamma_g$, $R = U \times N$ the points $x = (x_0, fx_0)$, $y = (x_1, fx_1)$ and the arcs u, v satisfy the conditions of Whitney lemma. Thus, there exists a smooth isotopy $H : P \times I \to R \subset M \times N$ with a compact support constant outside U such that $H(x, fx, 0) = (x, fx)$ for $x \in M$ and $H(x, fx, 1) \in Q$ for $x \in U$. Now we define $F, G : M \times I \to N$

$$F(x,t) = \begin{cases} \pi_2 H(x, fx, t) & \text{for } x \in U \\ fx & \text{for } x \in M - U \end{cases}$$
$$G(x,t) = \begin{cases} g\pi_1 H(x, fx, t) & \text{for } x \in U \\ gx & \text{for } x \in M - U. \end{cases}$$

It is easy to check that $f_1(x) = F(x, 1)$ and $g_1(x) = G(x, 1)$ are desired maps. □

Creating procedure (1.10). *Let M, N be smooth manifolds without boundary of dimension $k \geq 2$. Let $(f, g) : M \to N$ be such smooth maps that $\Phi(f, g)$ is finite. Let, moreover, $x_0 \in \Phi(f, g)$ be a transverse coincidence point. For any integer m there is a smooth homotopy $F : M \times I \to N$ joining f with $f_1 : M \to N$ such that $\Phi(f_1, g) = \Phi(f, g) \cup \{x', x_1, \dots, x_m\}$ where each point x_i reduces x_0. Moreover, x_0 and x' are in Nielsen relation and any points $y, z \in \Phi(f, g)$ reduce one another if and only if they do so in $\Phi(f_1, g)$.*

Proof. Let us choose Euclidean neighborhoods $x_0 \in U \subset M$ and $fx \in V \subset N$ such that $f(U) \cup g(U) \subset V$. Let us fix their orientations so that the degree of the map $f - g : (U, U - x_0) \to (V, V - 0)$ equals $+1$ (the linear structure is carried out from R^n to U and V). Let B_r denote the closed ball in U of radius equal to r centered at 0. We may assume that $x_0 \in B_4 - B_3$ and $U \cap \Phi(f, g) = \{x_0\}$. We define smooth map $\Theta : B_4 \to V$ by $\Theta(x) = fx - gx$. Then we may follow the proof of *creating procedure* in [**9**] to get the map $\bar{\Theta}$ defined there. At last, we put

$$f_1(x) = \begin{cases} \bar{\Theta}(x) + g(x) & \text{for } x \in B_4 \\ f(x) & \text{for } x \in M - B_4. \end{cases}$$

Now the zeros of $\bar{\Theta}$ give new coincidence points all in Nielsen relation with x_0. □

Proof of the Wecken Theorem. We may assume that the pair (f, g) is transverse. Let $\Phi(f, g) = \{a_1, b_1, \dots, a_k, b_k; c_1, \dots, c_s\}$ be a decomposition. We may apply the cancelling procedure to all pairs $\{a_i, b_i\}$ and we get a pair of maps such that no two coincidence points reduce one another. Let us notice that all unessential Nielsen classes disappear. Let $A = \{c_1, \dots, c_1\}$ be a Nielsen class. We apply creating procedure to the point c_1 and the number 1. Then A becomes $A' = \{c_1, \dots, c_1, c_1', \dots, c_1'; c_0\}$ where c_i and c_i' reduce one another. Applying cancelling procedure to these pairs, we reduce A to the only point c_0. □

Remark (1.11). The above construction of semi-index could be easily modified to get a homotopy invariant with the properties similar to classical index (more exactly to its absolute value): let $(f, g) : M \to N$

be a transverse pair and let $U \subseteq M$ be an open subset. We decompose as before $\Phi(f,g) = \{a_1, b_1, \dots, a_k, b_k; c_1, \dots, c_s\}$ and we may define the semi-index of a transverse pair on the subset $U \subseteq M$ as $|\mathrm{ind}|(f,g;U) = s$.

This definition may be extended onto any continuous pair of maps such that $\Phi(f,g) \cap U$ is compact, since the restriction $(f,g) : U \to N$ is homotopic to a transverse pair by a homotopy with compact coincidence set.

This semi-index has the following properties:

Coincidence points, (1.11a). If $|\mathrm{ind}|(f,g;U) \neq 0$, then $\Phi(f,g) \cap U \neq 0$.

Homotopy invariance, (1.11b). Let $(F,G) : M \times I \to N$ be a homotopy between pairs (f_0, g_0) and (f_1, g_1). Let $U \subseteq M \times I$ be an open subset such that $U \cap \Phi(F,G)$ is compact. Let $U_t = \{x \in M; (x,t) \in U\}$. Then $|\mathrm{ind}|(f_0, g_0; U_0) = |\mathrm{ind}|(f_1, g_1; U_1)$.

Subadditivity, (1.11c). If U_1, U_2 are open subsets of M such that $\Phi(f,g) \cap U_1$ and $\Phi(f,g) \cap U_2$ are compact, then $|\mathrm{ind}|(f,g : U_1 \cup U_2) \leq |\mathrm{ind}|(f,g;U_1) + |\mathrm{ind}|(f,g;U_2)$.

The above inequality may be sharp; for example, if $\Phi(f,g) \cap U_1 = \{x_1\}$, $\Phi(f,g) \cap U_2 = \{x_2\}$ where x_1 and x_2 reduce one another.

Unfortunately, despite these formal similarities with the classical index, the definition of $|\mathrm{ind}|(f,g;U)$ is not local (the reducibility relation depends on the behavior of f and g on the whole M). We will focus on the Nielsen classes only. In this case this formal analogy is even stronger:

Additivity, (1.11d). If $U_1, U_2 \subset M_2$ are open subsets and $A_1 = \Phi(f,g) \cap U_1$, $A_2 = \Phi(f,g) \cap U_2$ are two different Nielsen classes, then $|\mathrm{ind}|(f,g;U_1 \cap U_2) = |\mathrm{ind}|(f,g;U_1) + |\mathrm{ind}|(f,g;U_2)$.

2. Covering spaces. Now we will discuss some relations between the Nielsen number of (f,g) and of its lifts to covering spaces. The main result of this section is Theorem (2.5).

Let $p : \tilde{M} \to M$, $p' : \tilde{N} \to N$ denote connected covering spaces of n-manifolds M and N.

Lemma (2.1). *Let us consider a commutative diagram*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{(\tilde{f},\tilde{g})} & \tilde{N} \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle p'} \\ M & \xrightarrow{(f,g)} & N \end{array}$$

where (f,g) is transverse (then so is $(\tilde{f},\tilde{g})$). Let $\tilde{x},\tilde{y} \in \Phi(\tilde{f},\tilde{g})$ and let $\tilde{\omega}$ be a path from $\tilde{x}$ to $\tilde{y}$ such that $\tilde{f}\tilde{\omega} \simeq \tilde{g}\tilde{\omega}$. Then for $x = p\tilde{x}$, $y = p\tilde{y}$, $\omega = p\tilde{\omega}$, we have $x, y \in \Phi(f,g)$ and ω is a path from x to y satisfying $f\omega \simeq g\omega$. Under these assumptions $\tilde{\omega}$ reverses orientations in graphs if and only if so does ω.

Proof. We consider coverings

$$\begin{array}{ccccc} \Gamma_{\tilde{f}}, & & \Gamma_{\tilde{g}} & \hookrightarrow & \tilde{M} \times \tilde{N} \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle p\times p'} \\ \Gamma_{f}, & & \Gamma_{g} & \hookrightarrow & M \times N \end{array}$$

We choose a local orientation $\alpha(f)$ of the manifold Γ_f at (x, fx) and a local orientation $\alpha(g)$ of the manifold Γ_g at (x, gx). Then induce the local orientation $\alpha(\tilde{f})$ of the manifold $\Gamma_{\tilde{f}}$ at $(\tilde{x}, f\tilde{x})$ and the local orientation $\alpha(\tilde{g})$ of the manifold $\Gamma_{\tilde{g}}$ at $(\tilde{x}, g\tilde{x})$. Then we translate $\alpha(f), \alpha(g)$ along $(\omega, f\omega)$ and $(\omega, g\omega)$, respectively. It induces the translations of $\alpha(\tilde{f}), \alpha(\tilde{g})$ along $(\tilde{\omega}, \tilde{f}\tilde{\omega})$ and $(\tilde{\omega}, \tilde{g}\tilde{\omega})$, respectively. The same way we translate the local orientation $\alpha = \alpha(f) \wedge \alpha(g)$ of the manifold $M \times N$ along $(\omega, f\omega)$ which induces the translation of the local orientation $\alpha = \alpha(\tilde{f}) \wedge \alpha(\tilde{g})$ of the manifold $M \times N$ along $(\tilde{\omega}, \tilde{f}\tilde{\omega})$. To get our thesis we compare final orientations at the points $y = \omega(1)$ and $\tilde{y} = \tilde{\omega}(1)$. □

Corollary (2.2). *If the points $x, y \in \Phi(f,g)$ reduce one another, then $p^{-1}\{x,y\} \cap \Phi(\tilde{f},\tilde{g})$ splits into pairs of points reducing one another.*

Proof. Let ω be a path from x to y reversing orientations in graphs. Let $\tilde{x} \in \Phi(\tilde{f}, \tilde{g}) \cap p^{-1}x$ and let $\tilde{\omega}$ denote the lift of the path ω starting from this point. Since $f\omega \simeq g\omega$, $\tilde{\omega}(1) \in \Phi(\tilde{f}, \tilde{g})$ and $\tilde{f}\tilde{\omega} \simeq \tilde{g}\tilde{\omega}$. This way we get a bijective correspondence between sets $\Phi(\tilde{f}, \tilde{g}) \cap p^{-1}x$ and $\Phi(\tilde{f}, \tilde{g}) \cap p^{-1}y$. On the other hand, (2.1) implies that the points $\tilde{x} = \tilde{\omega}(0)$ and $\tilde{\omega}(1)$ reduce one another. □

Lemma (2.3). *Let $p : \tilde{M} \to M$, $p' : \tilde{N} \to N$ be connected regular coverings and let the diagram*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}, \tilde{g}} & \tilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{f, g} & N \end{array}$$

commute. Then

(a) $\Phi(f, g) = \cup_{(\tilde{f}, \tilde{g})} p\Phi(\tilde{f}, \tilde{g})$ *where the summation runs all pairs of lifts.*

(b) *The sets $p\Phi(\tilde{f}, \tilde{g})$, $p\Phi(\tilde{f}', \tilde{g}')$ are either equal or disjoint.*

(c) $p\phi(\tilde{f}, \tilde{g}) = p\Phi(\tilde{f}', \tilde{g}')$ *if and only if these pairs are conjugated, i.e., $(f', g') = \beta(f, g)\alpha$ for some covering transformations α, β of p and p', respectively.*

Thus, $\Phi(f, g) = \cup_{(\tilde{f}, \tilde{g})} p\Phi(\tilde{f}, \tilde{g})$ is a disjoint sum where the summation runs one $(\tilde{f}, \tilde{g})$ from each conjugacy (Reidemeister) class.

Proof. We follow the proof of [**10**, Theorem 1.5]. □

Let M, N be nonorientable manifolds, and let $p : \tilde{M} \to M$, $p' : \tilde{N} \to N$ denote the two-folded covering spaces corresponding to the subgroups of loops preserving orientation. Let $\alpha : \tilde{M} \to \tilde{M}$ and $\beta : \tilde{N} \to \tilde{N}$ be involutions of these coverings. Let us notice that both α and β reverse orientation of orientable manifolds $\tilde{M}$ and $\tilde{N}$. Let us suppose that a continuous map $f : M \to N$ admits a lift $\tilde{f}$

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{f} & N \end{array}$$

Then for all $\tilde{x} \in \tilde{M}$ either $\tilde{f}(\alpha\tilde{x}) = \beta\tilde{f}\tilde{x}$ or $\tilde{f}(\alpha\tilde{x}) = \tilde{f}\tilde{x}$. In the first case, the lift $\tilde{f}$ will be called odd and in the second case, it will be called even. (For the explanation of these notions, consider $\tilde{M} = \tilde{N} = S^n$, $\alpha = \beta =$ antipodism.) In the rest of this section, only such two-folded coverings will be considered. Then (2.3) implies $\Phi(f,g) = p\Phi(\tilde{f},\tilde{g}) \cup p\Phi(\tilde{f},\beta\tilde{g})$ if and only if both $\tilde{f}$ and $\tilde{g}$ are simultaneously even or odd and $\Phi(f,g) = p\Phi(\tilde{f},\tilde{g})$ otherwise.

Under the above notations one can easily check:

Lemma (2.4). *Let $f : M \to N$ be a continuous map. It admits a lift*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{f} & N \end{array}$$

if and only if $f_\# p_\# \pi_1 \tilde{M} \subset p'_\# \pi_1 \tilde{N}$.

If the lift $\tilde{f}$ exists, then

$\tilde{f}$ is even if and only if $f_\#(\pi_1 M) \subset p'_\#(\pi_1 \tilde{N})$ and

$\tilde{f}$ is odd if and only if $f_\#(\pi_1 M) \not\subset p'_\#(\pi_1 \tilde{N})$.

Now we are going to compare the Nielsen numbers of (f,g) and of its lifts. Let us notice that if (f,g) admits a lift $(\tilde{f},\tilde{g})$, then it admits exactly four $(\tilde{f},\tilde{g})$, $(\tilde{f},\beta\tilde{g})$, $(\beta\tilde{f},\tilde{g})$, $(\beta\tilde{f},\beta\tilde{g})$. Moreover, if $\tilde{f}$ and $\tilde{g}$ are simultaneously both even or odd, then the two first pairs of lifts are conjugated and so are the two last. Then by (2.3), $\Phi(f,g) = p\Phi(\tilde{f},\tilde{g}) \cup p\Phi(\tilde{f},\beta\tilde{g})$ is a disjoint sum. If one of $\tilde{f},\tilde{g}$ is even and the other is odd, then all four pairs of lifts form on Reidemeister class and $\Phi(f,g) = p\Phi(\tilde{f},\tilde{g})$.

Let us denote by

$$\text{average } N(\tilde{f},\tilde{g}) = (1/4)[N(\tilde{f},\tilde{g}) + N(\beta\tilde{f},\tilde{g}) + N(\tilde{f},\beta\tilde{g}) + N(\beta\tilde{f},\beta\tilde{g})].$$

Since the Nielsen numbers of conjugated pairs are equal,

$$\text{average } N(f,g) = (1/2)[N(\tilde{f},\tilde{g}) + N(\tilde{f},\beta\tilde{g})].$$

Let us denote $C(f_\#, g_\#)_x = \{\alpha \in \pi_1(M, x) : f_\#\alpha = g_\#\alpha\}$ for $x \in \Phi(f, g)$.

Theorem (2.5). *Suppose $C(f_\#, g_\#)_{p\tilde{x}} \subset p_\#\pi_1(\tilde{M}, \tilde{x})$ for any $\tilde{x} \in \Phi(\tilde{f}, \tilde{g})$. Then $N(f, g) = \text{average}\, N(\tilde{f}, \tilde{g})$.*

Proof. Let us assume at first that both $\tilde{f}, \tilde{g}$ are simultaneously either even or odd. Then $\Phi(f, g) = p\Phi(\tilde{f}, \tilde{g}) \cup p\Phi(\tilde{f}, \beta\tilde{g})$ is the disjoint sum. We will show that if a Nielsen class $A \subset \Phi(f, g)$ satisfies $A \subset p\Phi(\tilde{f}, \tilde{g})$, then $p^{-1}A$ is the sum of two Nielsen classes of $(\tilde{f}, \tilde{g})$ both of the same semi-index as A. The same is true for any class in $p\phi(\tilde{f}, \beta\tilde{g})$, and it will imply our theorem in this case. We may assume that the pair (f, g) is transverse; hence, $A = \{x_0, \dots, x_k\}$ is finite. Let us fix $\tilde{x}_0 \in p^{-1}x_0 \cap \phi(\tilde{f}, \tilde{g})$. We choose a path ω_i from x_0 to x_i establishing the Nielsen relation ($i = 1, \dots, k$). Let $\tilde{\omega}_i$ be its lift starting from $\tilde{x}_0$. Then the homotopy between $f\omega_i$ and $g\omega_i$ lifts to a homotopy from $\tilde{f}\tilde{\omega}_i$ to $\tilde{g}\tilde{\omega}_i$. It implies that $\tilde{\omega}_1(1), \dots, \tilde{\omega}_k(1)$ are coincidence points all in Nielsen relation with $\tilde{x}_0$. On the other hand, since $\tilde{f}$ and $\tilde{g}$ are of the same parity, $\alpha\tilde{x}_0 \in \Phi(\tilde{f}, \tilde{g})$ and by a similar argument, $\alpha\tilde{x}_0, \alpha\tilde{\omega}_1(1), \dots, \alpha\tilde{\omega}_k(1)$ are coincidence points in Nielsen relation. We will show that $p^{-1}A$ splits into two Nielsen classes:

$$p^{-1}A = \tilde{A} \cup \alpha\tilde{A} = \{x_0, \omega_1(1), \dots, \omega_k(1)\} \cup \{\alpha x_0, \alpha\omega_1(1), \dots, \alpha\omega_k(1)\}.$$

Suppose otherwise; then there exists a path $\tilde{\omega}$ from $\tilde{x}_0$ to $\alpha\tilde{x}_0$ such that $\tilde{f}\tilde{\omega} \simeq \tilde{g}\tilde{\omega}$. It implies $\langle p\tilde{\omega}\rangle \in C(f_\#, g_\#)_{x_0}$. But $\langle p\tilde{\omega}\rangle \notin p_\#\pi_1(\tilde{M}, \tilde{x})$ since it lifts to the open path $\tilde{\omega}$. This contradicts the assumption of the lemma. Moreover, Lemma (2.1) implies $|\text{ind}\,|(f, g : A) = |\text{ind}\,|(\tilde{f}, \tilde{g} : \tilde{A}) = |\text{ind}\,|(\tilde{f}, \tilde{g} : \alpha\tilde{A})$.

Now let us assume that $\tilde{f}$ is even and $\tilde{g}$ is odd. Then it is easy to see that $p : \Phi(\tilde{f}, \tilde{g}) \to \Phi(f, g)$ is a homeomorphism preserving Nielsen relation. Lemma (2.1) implies that the semi-indices of the corresponding Nielsen classes are equal. Thus, $N(f, g) = N(\tilde{f}, \tilde{g})$ for any pair of lifts. $\square$

3. Klein bottle. Now we will apply the results of previous sections to find the Nielsen number formula for self maps of the Klein bottle.

Let $C, D, E : R^2 \to R^2$ be given by

$$C(x,y) = (x+1,y), D(x,y) = (x,y+1), E(x,y) = (x+(1/2),-y)$$

and let G (G_0) be the subgroup of self-homomorphisms generated by $\{C,D,E\}$ ($\{D,E\}$). Then we may represent the Klein bottle B as the quotient space R^2/G and the torus T as R^2/G_0. We will denote the points of these spaces by $[x,y]_T \in T$, $[x,y]_B \in B$. There are natural covering maps $p_2 : R^2 \to T$, $p_1 : T \to B$ given by $p_2(x,y) = [x,y]_T$, $p_1[x,y]_T = [x,y]_B$. Let us denote $p = p_1 p_2 : R^2 \to B$. Then p and p_2 are universal coverings and p_1 is a two-folded covering corresponding to the subgroup of elements preserving orientation of B.

Let us notice that any point of B may be represented as $[x,y]_B$ for some $(x,y) \in [0,1/2] \times [0,1]$; hence, any map on B is given by a map defined on $[0,1/2]\times[0,1]$ satisfying $f(0,y) = f(1/2, 1-y)$ and $f(x,0) = f(x,1)$. Let us fix a point $b_0 = [0,0]_B \in B$ and the loops $X(t) = [t/2,0]_B$, $Y(t) = [0,t]_B$. Then $\pi_1(B,b_0) = F(X,Y)/\{YX = XY^{-1}\}$ where $F(X,Y)$ denotes the free group generated by X and Y. Now each element of $\pi_1(B,b_0)$ may be uniquely expressed as X^aY^b $(a,b \in Z)$. Lemma (2.2) in [**6**] asserts that any map $f : (B,b_0) \to (B,b_0)$ induces

$$f_\# X = X^a Y^b$$
$$f_\# Y = Y^d$$

where $a,b,d \in Z$, a is odd or $d = 0$. Moreover, two maps are homotopic if their numbers a,b,d are the same. We will need explicit formulae for the representatives of all homotopy classes: when a is odd, then the formula

$$(3.1) \qquad f[x,y]_B = [ax, -2bx+dy]_B, \qquad 0 \le x \le 1/2, \quad 0 \le y \le 1,$$

and when a is even, the formula

$$(3.2) \qquad f[x,y]_B = [ax, 2bx]_B, \qquad 0 \le x \le 1/2, \quad 0 \le y \le 1$$

gives maps which induce all possible homomorphims of homotopy groups. Any self map of the Klein bottle can be lifted to a self map of torus. We will need explicit formulae for these lifts. To do this, we define a real function $\chi(t) = |2(t-k)-1| - 1$ for $t \in [k, k+1]$. This is a continuous function satisfying

$$\chi(0) = 0, \qquad \chi(t+1/2) + \chi(t) = -1, \qquad \chi(t+1) = \chi(t).$$

Let a be odd, and let $f : B \to B$ be given by $f[x,y]_B = [ax, -2bx + dy]_B$. We put

$$\begin{aligned} \hat{f}_1(x,y) &= (ax, \chi(x)b + dy) \\ \hat{f}_2(x,y) &= (ax + 1/2, -\chi(x)b - dy). \end{aligned}$$

It is easy to check that the above formulae define lifts:

$$\begin{array}{ccc} R^2 & \xrightarrow{\hat{f}_i} & R^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p}, \\ B & \xrightarrow{f} & B \end{array} \qquad i = 1, 2.$$

The same formulae define also two different lifts:

$$\begin{array}{ccc} T & \xrightarrow{\hat{f}_i} & T \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{f} & B \end{array}$$

Let us notice that $\tilde{f}_1, \tilde{f}_2 : T \to T$ are homotopic to

$$\begin{aligned} \tilde{f}_1[x,y]_T &= [ax, dy]_T \\ \tilde{f}_2[x,y]_T &= [ax, -dy]_T \end{aligned} \tag{3.3}$$

since the homology group homomorphisms $\tilde{f}_{1*}, \tilde{f}_{2*} : H_1T \to H_1T$ induced by these lifts are represented by the matrices

$$\begin{pmatrix} a, & 0 \\ 0, & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a, & 0 \\ 0, & -d \end{pmatrix} \tag{3.4}$$

and the torus is a $K(\pi, 1)$ space.

Now let a be even. Then the lifts of $f[x,y]_B = [ax, 2bx]_B$ are given by

$$\tilde{f}_1[x,y]_T = [ax, 2bx]_T, \qquad \tilde{f}_2[x,y]_T = [ax + 1/2, -2bx]_T$$

so the corresponding homology homomorphisms $\tilde{f}_{1*}, \tilde{f}_{2*} : H_1T \to H_1T$ are represented by

$$(3.5) \qquad \begin{pmatrix} a, & 0 \\ 2b, & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a, & 0 \\ -2b, & 0 \end{pmatrix}.$$

The Nielsen number for self-maps of the torus is given in the following.

Theorem (3.6) [**8**]. *Let $(f, g) : T \to T$ be continuous. If the induced homology group homomorphisms $f_*, g_* : H_1T \to H_1T$ are represented by the matrices A and B, then*

$$N(f, g) = |\det(A - B)|. \qquad \square$$

Lemma (3.7). *For any pair of self-maps (f, g) of the Klein bottle*

$$N(f, g) = \text{average}\, N(\tilde{f}, \tilde{g}).$$

Proof. We may assume that f and g are of the form (3.1) or (3.2). Then they are fiber maps

$$\begin{array}{ccc} B & \xrightarrow{(f,g)} & B \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_3} \\ S^1 & \xrightarrow{(\bar{f},\bar{g})} & S^1 \end{array}$$

where $\bar{f}[x]_{S^1} = [ax]_{S^1}$, $\bar{g}[x]_{S^1} = [a'x]_{S^1}$, $p_3[x, y]_B = [x]_{S^1}$ and $S^1 = [0, 1/2]/\{0, 1/2\}$ is a circle.

Let us suppose at first that $\bar{f}$ and $\bar{g}$ are homotopic. Then $(\bar{f}, \bar{g})$ is homotopic to a pair of coincidence free maps; hence, so is (f, g) and its lifts to torus. So in this case $N(f, g) = N(\bar{f}, \bar{g}) = 0$.

Now we assume that $\bar{f}$ and $\bar{g}$ are not homotopic. We will show that the assumptions of (2.5) are satisfied. Suppose otherwise; then we get $\omega \in \pi_1 B$ satisfying $f_\# \omega = g_\# \omega$ and reversing orientation of B. The last means that $\omega = X^k Y^1$ where k is an odd number and implies that

$\bar{\omega} = p_{3\#}\omega \in \pi_1 S^1$ is a nontrivial element satisfying $\bar{f}_\# \bar{\omega} = \bar{g}_\# \bar{\omega}$. Thus, the induced homomorphisms $\bar{f}_\# = \bar{g}_\#$ which implies $\bar{f}$ is homotopic to $\bar{g}$, contradicting our assumption. Our lemma now follows from (2.5). □

Now we will give more explicit formulae:

Let us denote

$$f_\# X = X^a Y^b \qquad g_\# X = X^{a'} Y^{b'}$$
$$f_\# Y = Y^d \qquad g_\# Y = Y^{d'}$$

Let a and a' be odd. Then (3.4), (3.6) and (3.7) imply

$$\begin{aligned} N(f,g) &= \text{average}\, N(\tilde{f},\tilde{g}) = 1/2(N(\tilde{f},\tilde{g}) + N(\tilde{f},\beta\tilde{g})) \\ &= 1/2\left(|\det \begin{bmatrix} a-a', & 0 \\ 0, & d-d' \end{bmatrix}| + |\det \begin{bmatrix} a-a', & 0 \\ 0, & d+d' \end{bmatrix}| \right) \\ &= 1/2(|a-a'||d-d'| + |a-a'||d+d'|) \\ &= |a-a'| \max(|d|,|d'|). \end{aligned}$$

Now let a be odd and let a' be even. Then $d' = 0$ so (3.5), (3.6) and (3.7) imply

$$N(f,g) = N(\tilde{f},\tilde{g}) = |\det \begin{bmatrix} a-a', & 0 \\ -2b', & d \end{bmatrix}| = |a-a'||d|.$$

Let now both a and a' be even. Then $d = d' = 0$ so (3.4), (3.5), (3.6) and (3.7) imply

$$\begin{aligned} N(f,g) &= \text{average}\, N(\tilde{f},\tilde{g}) = 1/2(N(\tilde{f},\tilde{g}) + N(\tilde{f},\beta\tilde{g}) \\ &= 1/2\left(|\det \begin{bmatrix} a-a', & 0 \\ 2b-2b', & 0 \end{bmatrix}| + |\det \begin{bmatrix} a-a', & 0 \\ 2b+2b', & 0 \end{bmatrix}| \right) = 0. \end{aligned}$$

This way we get the final

Theorem (3.8). *Let (f,g) be a pair of self-maps of the Klein bottle preserving the base point b_0. Let the induced homomorphisms on the fundamental group be given by*

$$f_\# X = X^a Y^b \quad \textit{and} \quad g_\# X = X^{a'} Y^{b'}$$
$$f_\# Y = Y^d \qquad\qquad g_\# Y = Y^{d'}$$

Then $N(f,g) = |a - a'| \max(|d|, |d'|)$.

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