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ZEROS OF THE WRONSKIAN OF CHEBYSHEV AND ULTRASPHERICAL POLYNOMIALS

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ABSTRACT. For a polynomial $f(z)$, the Wronskian $Wf(z)$ is defined by $f(z)f''(z) - (f'(z))^2$. The zero distribution of $Wf(z)$ is studied in the cases where $f(z)$ is a Chebyshev polynomial of the first kind or an ultraspherical polynomial of order $0 < \lambda \leq 1$.

1. Introduction. In the theory of special functions, two related classes of inequalities have attracted some attention. These are the Laguerre inequality [**4**, p. 171f.]

$$[P'(x)]^2 - P(x)P''(x) \geq 0, \qquad -\infty < x < \infty, \tag{1.1}$$

which holds for all polynomials $P(x)$ with only real zeros, and the Turan inequality

$$[P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad x \in I, \tag{1.2}$$

which has been proven for various sequences $\{P_n(x)\}$ of classical orthogonal polynomials, for appropriate intervals I. For a discussion of these inequalities and the relationships between them, see [**3, 6**].

We have considered the inequality (1.1) and studied the zero distribution of the polynomial to the left, in particular the distance of the zeros from the real axis. General results in this direction were obtained in [**2**]. There it was also shown that the zeros of the left-hand side of (1.1) lie in the strip $|\text{Im}\,(z)| < (4 + \log n)/2\pi$ if $P(z)$ is a polynomial of degree $2n + 1$ having only real zeros that, in addition, are evenly

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distributed in the interval $[-1,1]$. Similar results hold when the zeros are slightly perturbed.

The main objective of this paper is to prove corresponding results for the case where P is a Chebyshev polynomial of the first kind or an ultraspherical polynomial of order $0<\lambda\le 1$.

2. Some basics. We define the nonlinear differential operator W by

$$Wf(z) := [f(z)]^2 \frac{d^2}{dz^2} \log f(z), \tag{2.1}$$

in analogy to the ordinary differential operator $Df(z) = f(z)(d/dz) \log f(z)$. Throughout this paper, we assume that f is a polynomial. Obviously, we have

$$Wf(z) = f(z)f''(z) - [f'(z)]^2 = \begin{vmatrix} f(z) & f'(z) \\ f'(z) & f''(z) \end{vmatrix}. \tag{2.2}$$

Because of this last determinant form we call the polynomial $Wf(z)$ the "Wronskian of the polynomial $f(z)$."

The following properties are easy to verify. Let f, g and h be polynomials. Then

$$\begin{aligned} W(gh) &= g^2 Wh + h^2 Wg; \\ W(g^n) &= n g^{2n-2} Wg, \quad n \in \mathbf{R}; \\ W(z-a) &= -1. \end{aligned}$$

It is also easy to see that for a polynomial $f(z) = (z-\alpha_1)^{m_1} \cdots (z-\alpha_k)^{m_k}$, $m_j \in \mathbf{N}$, $j = 1, \dots, k$, we have

$$Wf(z) = -[f(z)]^2 \left\{ \frac{m_1}{(z-\alpha_1)^2} + \cdots + \frac{m_k}{(z-\alpha_k)^2} \right\}. \tag{2.3}$$

This immediately gives the inequality (1.1) for polynomials having only real zeros. An upper bound on the zeros of $Wf(z)$ is given by the following result.

Lemma 2.1. *If all the zeros of the polynomial $f(z)$ are real and lie on the interval $[-1,1]$, then the zeros of $Wf(z)$ lie inside or on the unit circle.*

This was proved in [**2**]; it is also a special case of [**5**, Theorem 8.1].

3. Chebyshev polynomials. Let $T_n(z)$ be the n-th degree Chebyshev polynomial, defined by

$$T_n(\cos\theta) = \cos n\theta, \qquad z = \cos\theta. \tag{3.1}$$

As an immediate consequence of (3.1), we get the explicit expression for the zeros of $T_n(x)$, namely

$$\alpha_k := \cos\frac{2k-1}{2n}\pi, \qquad k = 1, 2, \dots, n.$$

This means that the α_k are not evenly distributed in the sense of [**2**]; they *crowd* toward the endpoints -1 and 1. More precisely, the distance between two consecutive zeros is of the order $1/n^2$ near the endpoints, and of the order $1/n$ near the middle. Using this fact, it is easy to see that the results in [**2**] give bounds on the zeros of $WT_n(z)$ that are weaker than Lemma 2.1. In spite of this, we shall see that the zeros of $WT_n(z)$ lie very close to the real axis.

Theorem 3.1. *For $n \geq 2$, the zeros of $WT_n(z)$ lie inside the ellipse*

$$\frac{y^2}{A_n^2} + \frac{x^2}{B_n^2} = 1, \qquad z = x + iy,$$

where $A_n := (1/n)(\log n - (1/2)\log\log n + 1)$, $B_n := \sqrt{1 + A_n^2}$.

As an immediate consequence, we get the following.

Corollary 3.2. *For $n \geq 2$, the zeros of $WT_n(z)$ lie in the strip*

$$|y| < \frac{1}{n}\left(\log n - \frac{1}{2}\log\log n + 1\right).$$

FIGURE 1.

Theorem 3.1 and Corollary 3.2, for $n = 20$, are illustrated by Figure 1. The four *corner* zeros are approximately $\pm 0.9849 \pm 0.01293i$, and the two zeros on the imaginary axis are $\pm 0.1568i$.

Proof of Theorem 3.1. Differentiating (3.1) twice with respect to θ, we get

$$T_n'(\cos\theta) = n\frac{\sin(n\theta)}{\sin\theta},$$
$$T_n''(\cos\theta) = -n^2\frac{\cos(n\theta)}{\sin^2\theta} + n\frac{\cos\theta\sin(n\theta)}{\sin^3\theta};$$

hence

$$WT_n(\cos\theta) = \frac{n\cos\theta}{2\sin^3\theta}\left[\sin(2n\theta) - 2n\frac{\sin\theta}{\cos\theta}\right]. \tag{3.2}$$

We shall show that the $\sin(2n\theta)$ term dominates the $2n\tan\theta$ term outside the ellipse. If we set $\theta = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$, then

$$\sin\theta = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta, \tag{3.3}$$
$$\cos\theta = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta. \tag{3.4}$$

Since $\bar{z} = \cos\bar{\theta}$, it suffices to consider the case $\beta > 0$. Now (3.3) and (3.4) imply

$$|\tan\theta|^2 = \frac{\sin^2\alpha\cos^2\alpha + \sinh^2\beta\cosh^2\beta}{(\cos^2\alpha + \sinh^2\beta)^2} \leq \frac{\sinh^2\beta\cosh^2\beta}{\sinh^4\beta},$$

hence

$$|\tan\theta| \le \coth\beta. \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}|\sin(2n\theta)|^2 &= \sin^2(2n\alpha)\cosh^2(2n\beta) + \cos^2(2n\alpha)\sinh^2(2n\beta)\\ &\ge (\sin^2(2n\alpha) + \cos^2(2n\alpha))\sinh^2(2n\beta),\end{aligned}$$

and therefore

$$|\sin(2n\theta)| \ge \sinh(2n\beta). \tag{3.6}$$

Now we see with (3.5), (3.6) and (3.2) that $WT_n(z) \neq 0$ if we can show that

$$\sinh(2n\beta) > 2n\coth\beta, \tag{3.7}$$

or equivalently,

$$(e^{2n\beta} - e^{-2n\beta})(e^{\beta} - e^{-\beta}) > 4n(e^{\beta} + e^{-\beta}). \tag{3.8}$$

If we set $\beta_0 = (\log n - (1/2)\log\log n + 9/10)/n$ and use the fact that $e^{\beta} - e^{-\beta} \ge 2\beta$, we see that (3.8) with $\beta = \beta_0$ follows from

$$(1 - a^{-2})\left(1 - \frac{1}{2}\frac{\log\log n}{\log n} + \frac{9/10}{\log n}\right) > \frac{2}{e^{9/5}}(a^{1/2n} + a^{-1/2n}),$$

where $a := e^{2n\beta} = e^{9/5}n^2/\log n$. This holds for $n \ge 3$. Now $\sinh(2n\beta)\sinh\beta/\cosh\beta$ is an increasing function for $\beta \ge 0$, so (3.7) holds whenever

$$\beta \ge \frac{1}{n}\left(\log n - \frac{1}{2}\log\log n + \frac{9}{10}\right). \tag{3.9}$$

Now we use the Maclaurin expansions for $\sinh\beta$ and $\cosh\beta$ to see that for $\beta \ge 0$, we have

$$\begin{aligned}\sinh\beta &= \beta\left[1 + \frac{1}{3!}\beta^2\left(1 + \frac{3!}{5!}\beta^2 + \frac{3!}{7!}\beta^4 + \cdots\right)\right]\\ &\le \beta\left(1 + \frac{1}{6}\beta^2\cosh\beta\right).\end{aligned}$$

Since $z = \cos\theta$, (3.4) implies $|z|^2 = \cosh^2\beta - \sin^2\alpha$. Furthermore, by Lemma 2.1 we may restrict our attention to $|z| \le 1$. Hence, we have $\cosh\beta \le \sqrt{2}$, and consequently,

$$\sinh\beta \le \beta\left(1 + \frac{1}{6}\beta^2\sqrt{2}\right) \le \beta\left(1 + \frac{1}{4}\beta^2\right).$$

This shows that (3.9) follows from

$$\sinh\beta \ge \frac{1}{n}\left(\log n - \frac{1}{2}\log\log n + 1\right) \tag{3.10}$$

if we can verify that

$$\frac{1}{n}b\left(1 + \frac{1}{4n^2}b^2\right) \le \frac{1}{n}\left(b + \frac{1}{10}\right),$$

where $b := \log n - (\log\log n)/2 + 9/10$. But this is equivalent to $b^3/n^2 \le 2/5$ which holds for $n \ge 6$.

Finally, we note that for each $\beta > 0$, the equation

$$\frac{y^2}{\sinh^2\beta} + \frac{x^2}{\cosh^2\beta} = 1 \tag{3.11}$$

defines an ellipse with foci ± 1 and principal half axes $\sinh\beta$, $\cosh\beta$. Each point $z = x + iy \in \mathbf{C}\backslash[-1, 1]$ lies on an ellipse (3.11) for exactly one $\beta > 0$. It is now clear that (3.10) holds whenever $z = x + iy$ lies outside the ellipse given in the statement of the theorem. But (3.10) implies (3.9) and hence $WT_n(z) \ne 0$; this completes the proof for $n \ge 6$. The cases $n = 2, \dots, 5$ were verified by numerical computation.

Remarks. (1) It is easy to see from formula (3.2) that $WT_n(z) = 0$ is equivalent to

$$zT'_{2n}(z) = 4n^2$$

and to

$$zU_{2n-1}(z) = 2n,$$

where $U_n(z) := \sin[(n+1)\theta]/\sin\theta$, $z = \cos\theta$, is the n-th Chebyshev polynomial of the second kind.

(2) Theorem 3.1 could be somewhat sharpened. For instance, in the case $\pi/2 \leq 2n\alpha \leq 3\pi/2$ (modulo 2π) it is easy to see that $\mathrm{Im}\,(\sin(2n\theta)) \leq 0$, and it is an easy consequence of (3.3) and (3.4) that $\mathrm{Im}\,(\tan\theta) > 0$ for $\beta > 0$. Hence, $WT_n(\cos\theta) \neq 0$ for such α and for all $\beta > 0$. Also, it can be shown that Theorem 3.1 and Corollary 3.2 are asymptotically sharp for the zeros of $WT_n(z)$ that lie on the imaginary axis. More specifically, only the constant term 1 in $\log n - (\log\log n)/2 + 1$ can be slightly improved.

4. Ultraspherical and Legendre polynomials. If $f(z)$ is a polynomial of degree n whose zeros lie close to those of $T_n(z)$, it is reasonable to expect the zeros of $Wf(z)$ to lie close to those of $WT_n(z)$. In this section we will see that this is, to some extent, the case when $f(z)$ is an ultraspherical (or Gegenbauer) polynomial; this class of polynomials includes the Legendre and the Chebyshev polynomials of the second kind as special cases. Although the results in this section are probably far from best possible, they show that the zeros of the Wronskians of these polynomials approach the real axis (uniformly) with increasing degrees of the polynomials.

Lemma 4.1. *Let* $-1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq 1$ *and* $d := \max_{1\leq k\leq n} |\beta_k - \cos((2k-1)/2n)\pi|$. *Then*

$$F_n(z) := \sum_{k=1}^{n} \frac{1}{(z-\beta_k)^2} \neq 0, \qquad z = x + iy \tag{4.1}$$

provided that

$$\sqrt{2}\sinh\beta\sinh(2n\beta) > 4n + 8\frac{2 + d/|y|}{(1-d/|y|)^2}\frac{d}{(\sinh\beta)^3}\cosh^2(n\beta), \tag{4.2}$$

where $z = \cos(\alpha + i\beta) = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = x + iy$.

Proof. We compare $F_n(z)$ with

$$f_n(z) := \sum_{k=1}^{n} \frac{1}{(z-\alpha_k)^2}, \qquad \alpha_k := \cos\frac{2k-1}{2n}\pi.$$

By (2.3), (3.2) and (3.1) we have

$$f_n(z) = \frac{n\cos\theta}{2\sin^3\theta\cos^2 n\theta}\left[2n\frac{\sin\theta}{\cos\theta} - \sin(2n\theta)\right] \tag{4.3}$$

(recall that $z = \cos\theta$). Now let $R_n(z) := F_n(z) - f_n(z)$; then with (4.3) we have

(4.4)
$$F_n(z) = \frac{n\cos\theta}{2\sin^3\theta\cos^2(n\theta)} \cdot \left[\frac{2}{\cos\theta}\left(n\sin\theta + \frac{1}{n}\sin^3\theta\cos^2(n\theta)R_n(z)\right) - \sin(2n\theta)\right].$$

We will prove (4.1) by showing that the term in brackets in (4.4) is nonzero. This is achieved if we can verify

$$|\sin(2n\theta)| > \left|\frac{2}{\cos\theta}\left(n\sin\theta + \frac{1}{n}\sin^3\theta\cos^2(n\theta)R_n(z)\right)\right|. \tag{4.5}$$

By (3.6), we have

$$|\sin(2n\theta)| \geq \sinh(2n\beta), \tag{4.6}$$

and (3.4) yields

$$\begin{aligned} |\cos\theta|^2 &= \cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta \\ &= \sinh^2\beta + \cos^2\alpha \geq \sinh^2\beta. \end{aligned}$$

Hence

$$\left|\frac{1}{\cos\theta}\right| \leq \frac{1}{\sinh\beta}. \tag{4.7}$$

Now (4.6) and (4.7) imply that (4.5) holds when

$$\sinh(2n\beta) > \frac{2}{\beta}\left[n|\sin\theta| + \frac{1}{n}|\sin\theta|^3|\cos(n\theta)|^2|R_n(z)|\right]. \tag{4.8}$$

It follows from the definition of $R_n(z)$ that

$$R_n(z) = \sum_{k=1}^{n}(\beta_k - \alpha_k)\frac{2z - \alpha_k - \beta_k}{(z-\beta_k)^2(z-\alpha_k)^2}.$$

We may assume $y > 0$ since the zeros are symmetric about the real axis. Since

$$|\beta_k - \alpha_k| \le d, \qquad |z - \alpha_k| = |x - \alpha_k + iy| \ge y,$$

we have

$$|2z - \alpha_k - \beta_k| \le 2|z - \alpha_k| + |\beta_k - \alpha_k| \le |z - \alpha_k| \left(2 + \frac{d}{y}\right),$$

$$|z - \beta_k| = |(z - \alpha_k) - (\beta_k - \alpha_k)| \ge |z - \alpha_k| - d \ge |z - \alpha_k| \left(1 - \frac{d}{y}\right).$$

Hence

$$|R_n(z)| \le dD \sum_{k=1}^{n} |z - \alpha_k|^{-3} \le ndD|y|^{-3} \tag{4.9}$$

where

$$D := \frac{2 + d/|y|}{(1 - d/|y|)^2}, \qquad d := \max_{1 \le k \le n} \{\beta_k - \alpha_k\}.$$

Next we note that by Lemma 2.1, we may restrict our attention to $|z| = |\cos\theta| \le 1$, or, by (3.4),

$$\cos^2\alpha \cosh^2\beta + \sin^2\alpha \sinh^2\beta \le 1$$

which is equivalent to $|\sin\alpha| \ge |\sinh\beta|$. By symmetry we may restrict our attention to $0 \le \alpha \le \pi$, $\beta \ge 0$ (i.e., $y = -\sin\alpha \sinh\beta < 0$ by (3.4)), so $\sin\alpha \ge 0$, $\sinh\beta \ge 0$ and

$$\sin\alpha \ge \sinh\beta. \tag{4.10}$$

Also, by (3.3),

$$|\sin\theta|^2 = \sin^2\alpha \cosh^2\beta + \cos^2\alpha \sinh^2\beta = \sin^2\alpha + \sinh^2\beta.$$

We note that $(\sin^2\alpha + \sinh^2\beta)^{3/2} / \sin^3\alpha \sinh^3\beta$ (for $\beta \ne 0$) is decreasing as $\sin\alpha$ increases. Hence, by (4.10),

$$\frac{|\sin\theta|^3}{|y|^3} = \frac{(\sin^2\alpha + \sinh^2\beta)^{3/2}}{\sin^3\alpha \sinh^3\beta} \le \frac{2\sqrt{2}}{\sinh^3\beta}. \tag{4.11}$$

We also have

$$|\cos(n\theta)|^2 = \cosh^2 n\beta - \sin^2 n\alpha \le \cosh^2 n\beta \tag{4.12}$$

and, with (3.3) and (4.10),

$$|\sin\theta| = (\sin^2\alpha + \sinh^2\beta)^{1/2} \le (2\sin^2\alpha)^{1/2} \le \sqrt{2}. \tag{4.13}$$

Finally, we see with (4.9), (4.11), (4.12) and (4.13) that (4.8) holds when

$$\sinh(2n\beta) > \frac{2}{\beta}\left[\sqrt{2}n + 2\sqrt{2}dD\cosh^2(n\beta)(\sinh\beta)^{-3}\right].$$

Now (4.2) follows, and the result is proved. □

In applying Lemma 4.1, we restrict our attention to the special case that will be applicable to the polynomials mentioned in the introductory paragraph.

Theorem 4.2. *Let $c \le 1.6$, and let $f(z)$ be a polynomial of degree n with zeros $\beta_k = \cos[(2k-1)\pi/2n] + \varepsilon_k$, where $|\beta_k| \le 1$, $|\varepsilon_k| \le c/n$, $k = 1, 2, \dots, n$. Then for $n \ge 20$, the zeros of $Wf(z)$ lie in the ellipse*

$$\frac{y^2}{A_n^2} + \frac{x^2}{B_n^2} = 1, \qquad z = x + iy,$$

where $A_n := \max\{(15c/n)^{1/4}, \sqrt{1/n}\}$, $B_n := \sqrt{1 + A_n^2}$.

Proof. We may restrict our attention to the lower half plane, $y < 0$; then $\sinh\beta \ge 0$. We will show that there are no zeros of $Wf(z)$, i.e., that (4.1) holds, whenever

$$\sinh\beta \ge \max\{(15c/n)^{1/4}, \sqrt{1/n}\}. \tag{4.14}$$

The theorem now follows from (4.14) just as Theorem 3.1 does from (3.13).

We suppose that (4.14) holds and estimate the various terms in (4.2). By (3.4) and (4.10), we have

$$|y| = \sin\alpha \sinh\beta \ge \sinh^2\beta \tag{4.15}$$

since we may restrict our attention to $|z| \le 1$, by Lemma 2.1. Since $d \le c/n$, we get with $c \le 1.6$ and $n \ge 20$ that

$$\frac{d}{|y|} \le \frac{c}{n}\frac{\sqrt{n}}{\sqrt{15c}} \le \frac{2}{5\sqrt{30}}.$$

Therefore,

$$D = \frac{29}{12}(1-\gamma), \qquad \gamma > 1/250. \tag{4.16}$$

Next we note that $\beta/\sinh\beta$ is decreasing for $\beta \ge 0$; therefore,

$$\beta \ge \frac{1}{\sinh(1)}\sinh\beta \ge \frac{1}{\sinh(1)}\sqrt{1/n} \tag{4.17}$$

by (4.14). Hence,

$$e^{-2n\beta} \le e^{-2\sqrt{n}/\sinh(1)} < 1/2000$$

and therefore

$$\sinh(2n\beta) = \frac{1}{2}(1-e^{-4n\beta})e^{2n\beta} \ge \frac{1}{2}(1-\delta)e^{2n\beta}, \tag{4.18}$$

where $\delta := 2.5 \cdot 10^{-7}$. Also

$$\cosh^2(n\beta) = \frac{1}{4}(1+e^{-2n\beta})^2 e^{2n\beta} \le \frac{1}{4}(1+\varepsilon)e^{2n\beta}, \tag{4.19}$$

where $\varepsilon < 1.1 \cdot 10^{-3}$. Furthermore, again by (4.17),

$$e^{2n\beta}\sinh\beta \ge ne^{2\sqrt{n}/\sinh(1)}n^{-3/2} \ge ne^{2\sqrt{20}/\sinh(1)}20^{-3/2} > \frac{45}{2}n.$$

Hence, with (4.16), (4.18) and (4.19), hypothesis (4.2) holds when

$$\sinh\beta\frac{1}{2}(1-\delta)e^{2n\beta} \ge \frac{8}{45}e^{2n\beta}\sinh\beta + \frac{58}{3}d(\sinh\beta)^{-3}\frac{1}{4}(1+\varepsilon)(1-\gamma)e^{2n\beta}$$

which is equivalent to

$$\sinh^4\beta \ge 15\frac{(1+\varepsilon)(1-\gamma)}{1-45\delta/29}d;$$

this holds for $\sinh\beta \geq (15c/n)^{1/4}$ since $(1+\varepsilon)(1-\gamma)/(1-45\delta/29) < 1$. This verifies (4.2), and the proof is complete. □

Remark . The proofs of Lemma 4.1 and Theorem 4.2 indicate that the results are not best possible. In particular, the constant "15" in Theorem 4.2 can be improved if we take n sufficiently large. Also, the restriction $c \leq 1.6$ can easily be modified. The term $\sqrt{1/n}$ in the definition of A_n is of a technical nature; it can be avoided by imposing a lower bound on c.

To conclude this section, we apply Theorem 4.2 to the ultraspherical polynomials $C_n^\lambda(z)$ which can be defined by the generating function

$$(1-2zt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(z)t^n, \qquad |t|<1, \lambda \neq 0.$$

For $\lambda > -1/2$, $\lambda \neq 0$, the $C_n^\lambda(z)$ have the explicit expression

$$C_n^\lambda(z) = \frac{1}{\Gamma(\gamma)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(\gamma+n-m)}{m!(n-2m)!}(2z)^{n-2m}$$

(see, e.g., [**1**, Chapter 22]).

Corollary 4.3. *For $0 < \lambda \leq 1$ and $n \geq 20$, the zeros of WC_n^λ lie inside the ellipse*

$$\frac{y^2}{A_n^2} + \frac{x^2}{B_n^2} = 1, \qquad z = x+iy$$

where $A_n := (15\pi/2n)^{1/4}, B_n := \sqrt{1+A_n^2}$.

Proof. We determine the constant c in Theorem 4.2. The k-th zero β_k of $C_n^\lambda(z)$ is located in the interval

$$\cos\left(\frac{k+\lambda-1}{n+\lambda}\pi\right) \leq \beta_k \leq \cos\left(\frac{k\pi}{n+\lambda}\right), \qquad k = 1, \dots, n$$

(see, e.g., [**1**, p. 787]). Now

$$\begin{aligned} d'_k : &= \left| \cos\left(\frac{2k-1}{2n}\pi\right) - \cos\left(\frac{k+\lambda-1}{n+\lambda}\pi\right) \right| \\ &= 2\left| \sin\left[\frac{\pi}{2}\left(\frac{2k-1}{2n} + \frac{k+\lambda-1}{n+\lambda}\right)\right] \sin\left[\frac{\pi}{2}\left(\frac{k+\lambda-1}{n+\lambda} - \frac{2k-1}{2n}\right)\right]\right| \\ &\le 2\left| \sin\left(\frac{\pi}{2}\frac{2n\lambda - 2k\lambda - n + \lambda}{2n(n+\lambda)}\right)\right|. \end{aligned}$$

Since $-n \le -n+\lambda \le 2n\lambda - 2k\lambda - n + \lambda \le n(2\lambda-1) - \lambda \le n$, we have

$$d'_k \le 2\sin\left(\frac{\pi}{2}\frac{n}{2n(n+\lambda)}\right) \le 2\sin\frac{\pi}{4n} < \frac{\pi}{2n}.$$

Similarly, we find

$$d''_k := \left| \cos\left(\frac{2k-1}{2n}\pi\right) - \cos\left(\frac{k\pi}{n+\lambda}\right)\right| < \frac{\pi}{2n}.$$

Hence we may choose $c = \pi/2$ in Theorem 4.2. Finally, we observe that $(15\pi/2n)^{1/4} > \sqrt{1/n}$ for all $n \ge 1$. The proof is now complete. □

Remarks. (1) Corollary 4.3 covers, in particular, the Chebyshev polynomials of the second kind $U_n(z)$ and the Legendre polynomials $P_n(z)$. Note that $P_n(z) = C_n^{1/2}(z)$, $U_n(z) = C_n^1(z)$. For $U_n(z)$, see also Corollary 5.2 below.

(2) We conjecture that Corollary 4.3 and Theorem 4.2 can be improved to give bounds of the same order as those in Theorem 3.1. The following section is related to this question.

5. Wronskians of asymptotics. There are various asymptotic expressions for the ultraspherical polynomials; see, e.g., [**7**, Chapter 8]. One such expression is

$$C_n^\lambda(\cos\theta) = 2^{1-\lambda}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}\frac{\cos[(n+\lambda)\theta - \pi\lambda/2]}{\sin^\lambda\theta} + 0(n^{\lambda-2}), \tag{5.1}$$

valid for all $\lambda \ne 0, -1, -2, \dots$ and $0 < \theta < \pi$ (see, e.g., [**7**, p. 197]). The aim of this section is to determine the zero distribution of the Wronskian of the main term in (5.1). We denote

$$T_n^\lambda(\cos\theta) := \frac{\cos[(n+\lambda)\theta - \pi\lambda/2]}{\sin^\lambda\theta}. \tag{5.2}$$

Theorem 5.1. *Let λ be any fixed real number. Then for all $n \geq \max\{8, e^{4\lambda}\} - \lambda$ the zeros of $WT_n^\lambda(z)$ lie inside the ellipse*

$$\frac{y^2}{A_n^2} + \frac{x^2}{B_n^2} = 1, \qquad z = x + iy, \tag{5.3}$$

where $A_n = (1/(n+\lambda))(\log(n+\lambda) - (1/2)\log\log(n+\lambda) + (3/2))$, $B_n = (1 + A_n^2)^{1/2}$.

The Chebyshev polynomials of the second kind can be defined by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\cos\theta}.$$

Then by (5.2) we have $U_n(z) = T_n^1(z)$; thus we get the following consequence of Theorem 5.1.

Corollary 5.2. *For $n \geq 54$, the zeros of $WU_n(z)$ lie inside the ellipse* (5.3) *with* $A_n = (\log(n+1) - (\log\log(n+1))/2 + 3/2)/(n+1)$.

Proof of Theorem 5.1. To simplify notation we set $\psi := (n+\lambda)\theta - \pi\lambda/2$. With $z = \cos\theta$ we differentiate (5.2) to obtain

$$\frac{d}{dz}T_n^\lambda(z) = (n+\lambda)\frac{\sin\psi}{\sin^{\lambda+1}\theta} + \lambda\frac{\cos\theta\cos\psi}{\sin^{\lambda+2}\theta} \tag{5.4}$$

and

(5.5)
$$\frac{d^2}{dz^2}T_n^\lambda(z) = \frac{1}{\sin^{\lambda+2}\theta}\Big\{[\lambda - (n+\lambda)^2]\cos\psi + \lambda(\lambda+2)\frac{\cos^2\theta}{\sin^2\theta}\cos\psi + (n+\lambda)(2\lambda+1)\frac{\cos\theta}{\sin\theta}\sin\psi\Big\}.$$

With (5.3)–(5.5) we get

(5.6)
$$WT_n^\lambda(z) = \frac{(n+\lambda)\cos\theta}{2\sin^{2\lambda+3}\theta}\Big\{\sin(2(n+\lambda)\theta - \lambda\pi) - 2(n+\lambda)\tan\theta + \frac{2\lambda}{n+\lambda}\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\cos^2\psi\Big\}.$$

We estimate now the last term in (5.6). With (3.6) we get

$$|\sin\theta\cos\theta| = \frac{1}{2}|\sin(2\theta)| \geq \frac{1}{2}\sinh(2\beta) \geq \beta$$

and with (4.12), $|\cos\psi| \leq \cosh[(n+\lambda)\beta]$ so that with $|\cos\theta| = |z| \leq 1$ (recall that we may restrict our attention to $|z| \leq 1$) we have

$$\left|\frac{2\lambda}{n+\lambda}\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\cos^2\psi\right| \leq \frac{4\lambda\cosh[(n+\lambda)\beta]}{(n+\lambda)\beta}. \tag{5.7}$$

Now we proceed as in the proof of Theorem 3.1. We note that, similar to (3.7), we have

$$|\sin(2(n+\lambda)\theta - \lambda\pi| \geq \sinh(2(n+\lambda)\beta). \tag{5.8}$$

Also, with (3.5),

$$|\tan\theta| \leq \coth\beta = (e^\beta + e^{-\beta})/(e^\beta - e^{-\beta}) \leq 2e^\beta/2\beta = e^\beta/\beta. \tag{5.9}$$

We see now with (5.6)–(5.9) that $WT_n^\lambda(z) \neq 0$ if we can show that

$$\sinh(2(n+\lambda)\beta) > 2(n+\lambda)\frac{e^\beta}{\beta} + \frac{4\lambda}{n+\lambda}\frac{\cosh^2[(n+\lambda)\beta]}{\beta} \tag{5.10}$$

or, equivalently,

$$\left(\beta - \frac{2\lambda}{n+\lambda}\right)e^{2(n+\lambda)\beta} - \left(\beta + \frac{2\lambda}{n+\lambda}\right)e^{-2(n+\lambda)\beta} > 4(n+\lambda)e^\beta + \frac{4\lambda}{n+\lambda}. \tag{5.11}$$

It is easy to verify that the left-hand side of (5.11) grows faster than the right-hand side as functions of β. Hence, (5.11) holds for all $\beta \geq \beta_0$ if it holds for

$$\beta = \beta_0 = \frac{1}{n+\lambda}\left(\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{5}{4}\right). \tag{5.12}$$

Then

$$e^{2(n+\lambda)\beta} = e^{5/2}\frac{(n+\lambda)^2}{\log(n+\lambda)}, \tag{5.13}$$

and with the assumption

$$n \geq e^{4\lambda} - \lambda \tag{5.14}$$

we get

$$\left(\beta - \frac{2\lambda}{n+\lambda}\right) \geq \frac{1}{n+\lambda}\left(\frac{1}{2}\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{5}{4}\right) \tag{5.15}$$

and, for $n+\lambda \geq 8$,

$$\left(\beta + \frac{2\lambda}{n+\lambda}\right) \leq \frac{1}{n+\lambda}\left(\frac{3}{2}\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{5}{4}\right) \leq 1. \tag{5.16}$$

We also use the fact that for $n+\lambda \geq 8$ we have with (5.12),

$$4e^{\beta} < 5.8. \tag{5.17}$$

Now with (5.13)–(5.17), we see that (5.11) holds when

$$\frac{1}{n+\lambda}\left(\frac{1}{2}\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{5}{4}\right)e^{5/2}\frac{(n+\lambda)^2}{\log(n+\lambda)} \\ - \frac{\log(n+\lambda)}{e^{5/2}(n+\lambda)^2} > 5.8(n+\lambda) + \frac{\log(n+\lambda)}{n+\lambda}$$

or

$$\left(\frac{1}{2} - \frac{1}{2}\frac{\log\log(n+\lambda)}{\log(n+\lambda)} + \frac{5/4}{\log(n+\lambda)}\right)e^{5/2} > \\ 5.8 + \frac{\log(n+\lambda)}{(n+\lambda)^2}\left(1 + \frac{1}{e^{5/2}(n+\lambda)}\right). \tag{5.18}$$

It is now easy to verify that the term in parentheses on the left-hand side of (5.18) is minimal when $\log\log(n+\lambda) = 7/2$, with minimum $(1-e^{-7/2})/2$. With this we see that (5.18) holds whenever $n+\lambda \geq 8$. Thus, we have shown that (5.10) is true when $n+\lambda \geq 8$, (5.14) and

$$\beta \geq \frac{1}{n+\lambda}\left(\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{5}{4}\right) \tag{5.19}$$

are all satisfied. As in the discussion around (3.10), we see that (5.19) follows from

$$\sinh\beta \geq \frac{1}{n+\lambda}\left(\log(n+\lambda) - \frac{1}{2}\log\log(n+\lambda) + \frac{3}{2}\right) \tag{5.20}$$

if we can verify that

$$\frac{b}{n+\lambda}\left(1 + \frac{1}{4}\left(\frac{b}{n+\lambda}\right)^2\right) \leq \frac{1}{n+\lambda}\left(b + \frac{1}{4}\right),$$

where $b := \log(n+\lambda) - (1/2)\log\log(n+\lambda) + 5/4$. But this is equivalent to $b^3/(n+\lambda)^2 \leq 1$ which holds for $n + \lambda \geq 8$. The statement of the theorem now follows from (5.20), as in the conclusion of the proof of Theorem 3.1. □

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