ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

INVERSE LIMIT MEANS AND SOME FUNCTIONAL EQUATIONS

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ABSTRACT. Nonexistence of inverse limit means on arc-like continua whose coordinate means satisfy some functional equations is demonstrated.

1. Introduction. A *mean* on a topological Hausdorff space X is a (continuous) mapping $\mu : X \times X \to X$ such that $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$ whenever $x, y \in X$. Let an inverse sequence $\{X_n, f_n\}$ be given each coordinate space X_n of which admits a mean $\mu_n : X_n \times X_n \to X_n$ such that for each $n \in \mathbf{N}$ the functional equation,

$$f_n(\mu_{n+1}(x, y)) = \mu_n(f_n(x), f_n(y)) \tag{1}$$

holds for all $x, y \in X_{n+1}$. Then the inverse limit space $X = \lim\{X_n, f_n\}$ admits a mean $\mu : X \times X \to X$ defined by

$$\mu(\{x_n\}, \{y_n\}) = \{\mu_n(x_n, y_n)\} \tag{2}$$

([**1**, Theorem 1], which is called an *inverse limit mean* with respect to the sequence $\{X_n, f_n\}$.

In what follows all mappings are assumed to be continuous. We let I denote the closed unit interval $[0, 1]$, and let $g : I \to I$ stand for the surjection defined by

$$g(x) = \begin{cases} 2x, & \text{if } x \in [0, 1/2], \\ 2 - 2x, & \text{if } x \in [1/2, 1]. \end{cases} \tag{3}$$

The following result is Theorem 4 of [**1**].

Received by the editors on December 14, 1990.
1991 AMS *Mathematics Subject Classification.* Primary 39B12, 39B22, 54B10. Secondary 26B35, 54F15.
Key words and phrases. Arclike continuum, closed interval, functional equation, inverse limit, mapping, mean, open.

Theorem 4. (Baker and Wilder). *There does not exist any sequence* $\{\mu_n\}$ *of means on* I *such that, for all* $n \in \mathbf{N}$ *and for all* $x, y \in I$ *the functional equation*

$$g(\mu_{n+1}(x,y)) = \mu_n(g(x), g(y)) \tag{5}$$

holds.

A much stronger version of Theorem 4 is proved in [**4**, p. 550]. It reads as follows.

Theorem 6. (Wilder). *There exists an integer* $n_0 \geq 3$ *such that there does not exist any* n_0*-term sequence* $\mu_1, \mu_2, \dots, \mu_{n_0}$ *of means on* I *such that, for each* $n \in \{1, 2, \dots, n_0 - 1\}$, *the functional equation*

$$g(\mu_{n+1}(x,y)) = \mu_n(g(x), g(y)) \tag{5}$$

holds for all $x, y \in I$.

The aim of this paper is to prove similar results to that of Theorems 4 and 6 above, with some other mappings in place of g in functional equation (5). In particular, a countable family of open surjective mappings on I is exhibited, each member of which can replace g in (5) provided the means μ_n satisfy some additional conditions.

Two surjective mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between topological spaces are said to be *equivalent* provided there are homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2 \circ h_X = h_Y \circ f_1$.

2. The results. The theorem below generalizes, and is an application of, Theorem 6 of Wilder.

Theorem 7. *Let a mapping* $f : I \to I$ *be such that there are subintervals* $[a,b]$ *and* $[c,d] = f([a,b])$ *of* I *and homeomorphisms*

$$h_1 : I \to [a,b] \quad \text{and} \quad h_2 : [c,d] \to I \tag{8}$$

which satisfy the condition

$$g = h_2 \circ (f|[a,b]) \circ h_1. \tag{9}$$

Then there exists an integer $n_0 \geq 3$ with the property that there does not exist any n_0-term sequence $\mu_1, \mu_2, \dots, \mu_{n_0}$ of means on I such that, for each $n \in \{1, 2, \dots, n_0\}$, the following conditions are satisfied:

$$h_1(\mu_n(x, y)) = \mu_n(h_1(x), h_1(y)) \qquad \text{for all } x, y \in I, \tag{10}$$

$$\mu_n([c, d] \times [c, d]) \subset [c, d], \tag{11}$$

$$h_2(\mu_n(x, y)) = \mu_n(h_2(x), h_2(y)) \qquad \text{for all } x, y \in [c, d], \tag{12}$$

and that, for each $n \in \{1, 2, \dots, n_0 - 1\}$, the functional equation

$$f(\mu_{n+1}(x, y)) = \mu_n(f(x), f(y)) \tag{13}$$

holds for all $x, y \in I$.

(14) *Remarks.* 1) The existence of homeomorphisms h_1 and h_2 satisfying (9) is another way of asserting that the restriction $f|[a, b] : [a, b] \to [c, d]$ and the mapping $g : I \to I$ defined by (3) are equivalent.

2) Condition (11) ensures that the composition $h_2 \circ \mu_n$ is well defined.

(15) *Proof of Theorem* 7. Suppose on the contrary that for every integer $n_0 \geq 3$ there is an n_0-term sequence $\mu_1, \mu_2, \dots, \mu_{n_0}$ of means on I such that, for each $n \in \{1, 2, \dots, n_0\}$, conditions (10)–(13) hold. To get a contradiction to Theorem 6, it is enough to show that the assumed conditions imply that the means μ_n satisfy functional equation (5). For shortness, put $f_0 = f|[a, b]$, take an arbitrary $n_0 \geq 3$ as above, $n \in \{1, 2, \dots, n_0 - 1\}$ and $x, y \in I$, and observe the following sequence of equivalences.

$$\begin{aligned} g(\mu_{n+1}(x, y)) &= h_2(f_0(h_1(\mu_{n+1}(x, y)))) && \text{by (9)} \\ &= h_2(f_0(\mu_{n+1}(h_1(x), h_1(y)))) && \text{by (10)} \\ &= h_2(\mu_n(f_0(h_1(x)), f_0(h_1(y)))) && \text{by (13)} \\ &= \mu_n(h_2(f_0(h_1(x))), h_2(f_0(h_1(y)))) && \text{by (12)} \\ &= \mu_n(g(x), g(y)) && \text{by (9).} \end{aligned}$$

Thus the proof is complete. □

As a consequence of Theorem 7 we obtain the following result which corresponds to Theorem 4 above, and which can obviously be proved independently from Theorems 6 and 7 using the same sequence of equivalences to reach a contradiction with Theorem 4.

Theorem 15. *Let a mapping $f : I \to I$ be such that there are subintervals $[a,b]$ and $[c,d] = f([a,b])$ of I and homeomorphisms* (8) *which satisfy* (9). *Then there does not exist any sequence $\{\mu_n\}$ of means on I satisfying conditions* (10)–(12) *and such that the functional equation* (13) *holds for all $x, y \in I$.*

3. Applications. A mapping $f : I \to I$ is said to be *open* if it maps open subsets of the domain onto open subsets of the range. Now we construct a countable family of open mappings of I onto itself. Fix a positive integer k and define a surjection $g_k : I \to I$ as follows. Let $m \in \{0, 1, \ldots, k\}$. If m is even, then $g_k(m/k) = 0$; if m is odd, then $g_k(m/k) = 1$; for each m, the restriction

$$g_k|[m/k,(m+1)/k] : [m/k,(m+1)/k] \to I$$

is defined to be affine. Thus for each $k \in \mathbf{N}$ the mapping g_k is an open surjection. Note that $g_k(0) = 0$ and that $g_k(1)$ is either 0 or 1 according to whether k is either even or odd. Observe that g_1 is the identity and $g_2 = g$.

Now we apply Theorems 7 and 15 to prove certain versions of Theorems 6 and 4 in which the function g is replaced by g_k for an arbitrary fixed $k \geq 2$, provided that the means μ_n satisfy an extra homogeneity condition. Recall that a mapping $\mu : I \times I \to I$ is said to be *homogeneous* if, for each $t \in I$, the equality

$$\mu(tx, ty) = t\mu(x, y) \tag{16}$$

holds for every $x, y \in I$.

Theorem 17. *Let an integer $k \geq 2$ be fixed. Then there exists an integer $n_0 \geq 3$ with the property that there does not exist any n_0-term sequence $\mu_1, \mu_2, \ldots, \mu_{n_0}$, of means on I such that, for each $n \in \{1, 2, \ldots, n_0\}$, the equality*

$$\mu_n((2/k)x,(2/k)y) = (2/k)\mu_n(x,y) \qquad \textit{for all } x, y \in I \tag{18}$$

holds, and if $n \neq n_0$*, the functional equation*

$$g_k(\mu_{n+1}(x,y)) = \mu_n(g_k(x), g_k(y)) \tag{19}$$

is satisfied for all $x, y \in I$*.*

Proof. In Theorem 7 put $a = 0$, $b = 2/k$, $c = 0$ and $d = 1$. Define $h_1 : I \to [a,b] = [0, 2/k]$ by $h_1(x) = (2/k)x$ for all $x \in I$ and take $h_2 : I \to I$ as the identity, i.e., $h_2(x) = x$ for all $x \in I$. We have to verify that all the assumptions of Theorem 7 are fulfilled. Indeed, (8) follows by the definitions of h_1 and h_2. It can easily be observed that $g_2 = (g_k|[0, 2/k]) \circ h_1$, whence (9) follows. Further, (10) is an immediate consequence of the definition of h_1 and of (18). Since $[c,d] = [0,1]$ and since h_2 is the identity, conditions (11) and (12) trivially hold. Finally, (19) leads to (13). Thus, Theorem 7 can be applied, and thereby the conclusion follows as needed. □

(20) *Remark.* Note that if $k = 2$, then the coefficient $2/k$ equals 1, so that (18) is redundant. Indeed, if $k = 2$, Theorem 17 is identical with Theorem 6 of Wilder. Thus, the following question is natural.

(21) **Question.** Is condition (18) on the means μ_n an essential assumption in Theorem 17 for $k > 2$?

Since condition (18) is a very particular case of the homogeneity condition (16) for the means μ_n, we get the following corollary to Theorem 17.

Corollary 22. *For each integer* $k \geq 2$ *there exists an integer* $n_0 \geq 3$ *such that there does not exist any* n_0*-term sequence* $\mu_1, \mu_2, \ldots, \mu_{n_0}$ *of homogeneous means on* I *satisfying, for each* $n \in \{1, 2, \ldots, n_0 - 1\}$ *and* $x, y \in I$*, the functional equation*

$$g_k(\mu_{n+1}(x,y)) = \mu_n(g_k(x), g_k(y)). \tag{19}$$

In the same way as Theorem 7 leads to Theorem 17 and Corollary 22, Theorem 15 leads to the next two results which can also be deduced from Theorem 17 and Corollary 22.

Theorem 23. *Let an integer $k \geq 2$ be fixed. Then there does not exist any sequence $\{\mu_n\}$ of means on I such that, for all $n \in \mathbf{N}$ and all $x, y \in I$, we have*

$$\mu_n((2/k)x, (2/k)y) = (2/k)\mu_n(x, y) \tag{18}$$

and

$$g_k(\mu_{n+1}(x, y)) = \mu_n(g_k(x), g_k(y)). \tag{19}$$

Corollary 24. *Let an integer $k \geq 2$ be fixed. Then there does not exist any sequence $\{\mu_n\}$ of homogeneous means on I such that, for all $n \in \mathbf{N}$, the functional equation*

$$g_k(\mu_{n+1}(x, y)) = \mu_n(g_k(x), g_k(y)) \tag{19}$$

holds for all $x, y \in I$.

Recall that a surjective mapping $f : I \to I$ is open if and only if f is equivalent to $g_k : I \to I$ for some positive integer k (see [**3**, (1.3), p. 184]). In the light of this characterization, it is tempting to attain a much more general result, namely to replace the mapping g in (5) of Theorems 4 and 6 or the mapping g_k in (19) of Theorems 17 and 23 by an arbitrary open mapping f not being a homeomorphism. In other words we have the following question.

(25) **Question.** Is it true that for each open mapping $f : I \to I$ which is not a homeomorphism there exists an integer $n_0 \geq 3$ having the property that there is no n_0-term sequence $\mu_1, \mu_2, \dots, \mu_{n_0}$ of means on I such that, for each $n \in \{1, 2, \dots, n_0 - 1\}$ and for all $x, y \in I$ the functional equation

$$f(\mu_{n+1}(x, y)) = \mu_n(f(x), f(y))$$

holds?

4. Final conclusions. A continuum (i.e., a compact connected metric space) X is said to be *arclike* provided for each positive number

ε there exists a mapping of X onto an arc such that all the point inverses have diameters less than ε. It is well known that every arclike continuum is homeomorphic to an inverse limit space $X = \lim\{X_n, f_n\}$, where each coordinate space X_n is the unit interval I and each bonding mapping f_n is surjective.

Consider now an arclike continuum $X = \lim\{X_n, f_n\}$ such that, for each $n \in \mathbf{N}$, we have $X_n = I$ and $f_n = g_k$ for some fixed $k \geq 2$, and assume that for each $n \in \mathbf{N}$ the coordinate mean μ_n satisfies condition (18). Then, by Theorem 23, the functional equation (1) cannot be satisfied, and we obtain the following result.

Corollary 26. *If an arclike continuum X is the inverse limit of an inverse sequence of closed unit intervals $X_n = I$ with bond mappings $f_n = g_k$ for some fixed integer $k \geq 2$, then X does not admit an inverse limit mean μ defined by* (2) *such that the coordinate means μ_n satisfy condition* (18). *In particular, X does not admit an inverse limit mean whose coordinate means are homogeneous.*

Let us recall that taking in Theorem 4 a constant sequence of means, i.e., $\mu_n = \mu$ for all $n \in \mathbf{N}$, it follows (see [**1**, Corollary 4.2]) that the functional equation

$$g(\mu(x, y)) = \mu(g(x), g(y))$$

has no solution among the means on I. Similarly, Theorem 23 leads to the following corollary.

Corollary 27. *Let an integer $k \geq 2$ be fixed. Then the functional equation*

$$g_k(\mu(x, y)) = \mu(g_k(x), g_k(y)) \tag{28}$$

for $x, y \in I$ has no solution μ among means on I satisfying the condition

$$\mu((2/k)x, (2/k)y) = (2/k)\mu(x, y) \tag{29}$$

for all $x, y \in I$, thus among homogeneous means on I.

Remark. The methods presented in this paper can also be applied to produce a corresponding version of Baker and Wilder's Theorem 5 of [**1**, p. 92]. For details, see [**2**].

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