ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

SPECTRAL CONTINUITY IN SOME BANACH ALGEBRAS

LAURA BURLANDO

ABSTRACT. In a previous paper by the author, a sufficient condition for continuity of the spectral radius function and two equivalent sufficient conditions for continuity of the spectrum function at a point of a Banach algebra were given. In this paper a new sufficient condition (which improves the one above) for continuity of spectral radius at a point of a Banach algebra is provided. Moreover, these four conditions are proved here to be less restrictive than those already known, not only in the algebra of all linear bounded operators on a generic Banach space, but also in its quotient algebra modulo compact operators. The four conditions above are also characterized in the algebras of all linear and continuous operators on some particular Banach spaces and in their quotient algebras modulo compact operators.

Introduction. Let L be a complex Banach algebra. We set $\underline{L} = L$ if L has an identity. If L has no identity, let $\underline{L}$ denote the Banach algebra obtained by canonical adjunction of an identity to L. We consider the problem of continuity of the following two functions on L: the spectrum function $\sigma : L \to \mathbf{K_C}$ (where $\mathbf{K_C}$ denotes the set of all compact nonempty subsets of the complex plane $\mathbf{C}$, endowed with the Hausdorff metric, and $\sigma(\mathbf{a})$ denotes the spectrum of $\mathbf{a}$ in $\underline{L}$ for any $\mathbf{a} \in L$) and the spectral radius function $r : L \to \mathbf{R}$ (where $r(\mathbf{a})$ denotes the spectral radius of $\mathbf{a}$ in $\underline{L}$ for any $\mathbf{a} \in L$).

We recall that σ is upper semi-continuous on L, which means that, for any $\mathbf{a} \in L$ and for any neighborhood V of $\sigma(\mathbf{a})$, there exists a neighborhood U of $\mathbf{a}$ such that $\sigma(\mathbf{b}) \subset V$ for any $\mathbf{b} \in U$ (see [**19**, (1.6.16)]). Hence r too is upper semi-continuous on L. In general r and σ are not continuous on the whole of L. We remark that if σ is continuous at a point $\mathbf{a} \in L$, then also r is continuous at $\mathbf{a}$.

For any Banach space X, let $L(X)$ denote the Banach algebra of all linear and continuous operators on X, and let I_X denote the identity of $L(X)$. Moreover, let $K(X)$ denote the ideal of compact operators on X.

In the cases of the algebras $L(H)$ and $L(H)/K(H)$, where H is a separable Hilbert space, the continuity points of the functions r and σ were characterized by J.B. Conway and B.B. Morrel (see [**8**–**10**]). Further characterizations of the continuity points of σ in the two algebras above (for a separable Hilbert space H) are provided in [**1**]. It is known that the conditions given by Conway and Morrel and by the authors of [**1**] are sufficient for continuity of r and σ also in the more general cases of the algebras $L(X)$ and $L(X)/K(X)$, where X is a Banach space. Nevertheless, as far as we know, the problem of characterizing the continuity points of spectrum and spectral radius is still open also in the cases of the algebras $L(X)$ and $L(X)/K(X)$ for a generic Banach space X, as well as in the more general one of an abstract Banach algebra.

In our previous papers [**3, 4**] we prove that the conditions given by Conway and Morrel and by the authors of [**1**] are not necessary for continuity of r and σ in $L(X)$ if X is a generic Banach space.

In [**3**] we give the following sufficient condition for continuity of r at a point $\mathbf{a}$ of a complex Banach algebra L.

Let $\mathbf{e}$ denote the identity of $\underline{L}$. We define $S(\mathbf{a}) = \{\lambda \in \mathbf{C} : \lambda\mathbf{e} - \mathbf{a} \in \mathring{S}_{\underline{L}}\}$ (where S_M denotes the set of noninvertible elements of M for any Banach algebra M with identity) and $\gamma(\mathbf{a}) = \sup\{|\lambda| : \lambda \in S(\mathbf{a})\}$ (we also set $\sup\{|\lambda| : \lambda \in \varnothing\} = 0$). Notice that $S(\mathbf{a}) \subset \sigma(\mathbf{a})$. For any $J \in \mathbf{J}_{\underline{L}}$ (where, for any Banach algebra M, $\mathbf{J}_M$ denotes the set of all proper closed two-sided ideals of M), let π_J denote the canonical quotient map from $\underline{L}$ onto $\underline{L}/J$. Then we set $\sigma_J(\mathbf{a}) = \sigma(\pi_J(\mathbf{a}))$ and $\delta_J(\mathbf{a}) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: ω is a component of $\sigma_J(\mathbf{a})\}$ for any $J \in \mathbf{J}_{\underline{L}}$. Notice that $J_1, J_2 \in \mathbf{J}_{\underline{L}}$ and $J_1 \subset J_2$ implies $\sigma_{J_2}(\mathbf{a}) \subset \sigma_{J_1}(\mathbf{a})$. It follows that $\sigma_J(\mathbf{a}) \subset \sigma(\mathbf{a})$ for any $J \in \mathbf{J}_{\underline{L}}$. Since $S(\mathbf{a})$ and $\sigma_J(\mathbf{a})$, $J \in \mathbf{J}_{\underline{L}}$, are subsets of $\sigma(\mathbf{a})$, we have that $\gamma(\mathbf{a}) \leq r(\mathbf{a})$ and $\delta_J(\mathbf{a}) \leq r(\mathbf{a})$ for any $J \in \mathbf{J}_{\underline{L}}$.

Then (see [**3**, 1.5]):

(1) *if* $\mathbf{a} \in L$ *is such that* $r(\mathbf{a}) = \max\{\gamma(\mathbf{a}), \sup\{\delta_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}\}$*, the spectral radius function is continuous at* $\mathbf{a}$.

In [**4**] we give the following sufficient condition for continuity of σ at a point $\mathbf{a}$ of a complex Banach algebra L.

Let $\zeta(\mathbf{a})$ denote the set of all points $\lambda \in \sigma(\mathbf{a})$ such that any neighborhood of λ contains a component of $\sigma_J(\mathbf{a})$ for some $J \in \mathbf{J}_{\underline{L}}$ (notice that $\zeta(\mathbf{a})$ is a closed subset of $\sigma(\mathbf{a})$). Then (see [**4**, 1.5]):

(2) *if* $\mathbf{a} \in L$ *is such that* $\sigma(\mathbf{a}) = \overline{S(\mathbf{a})} \cup \zeta(\mathbf{a})$*, the spectrum function is continuous at* $\mathbf{a}$*.*

We remark that condition (2) implies condition (1). In addition, if $L = L(X)$, where X is a Banach space, it can be proved that Conway and Morrel's conditions in $L(X)$ for r imply condition (1) (see [**3**, 1.7]) and the conditions given by Conway and Morrel and by the authors of [**1**] in $L(X)$ for σ imply condition (2) (see [**4**, 2.1]). Therefore, if X is a separable Hilbert space, conditions (1) and (2) characterize continuity of r and σ, respectively, in $L(X)$. Several examples are constructed in [**3, 4**] to illustrate less restrictiveness of (1) and (2) in $L(X)$, for a generic Banach space X, with respect to the corresponding conditions given in [**8, 9, 1**].

Unfortunately, Conway and Morrel's characterization of continuity points of σ in $L(X)/K(X)$ (where X is a separable Hilbert space) can be used to construct an example which shows how conditions (1) and (2) are not necessary, in general, for continuity of r and σ, respectively (see [**4**, 3.5]). In order to overcome this counterexample, in another paper to appear [**6**], we have given a sufficient condition for continuity of the spectral radius function (implied by condition (1)) and two equivalent sufficient conditions for continuity of the spectrum function (implied by condition (2)) at a point $\mathbf{a}$ of a Banach algebra L. These three conditions will be recalled in Section 1 of this paper.

One of our concerns here is to show how the conditions given in [**6**] are less restrictive than (1) and (2) and are satisfied by the example above. Moreover, in this paper we give a new sufficient condition for continuity of the function r at $\mathbf{a} \in L$, which improves the one given in [**6**]. We are also interested in studying our conditions in the algebras of all linear bounded operators on some special Banach spaces and in their quotient algebras modulo compact operators.

In Section 1 we recall the conditions of [**6**], and state and prove the new sufficient condition above for continuity of the spectral radius function (Theorem 1.1). This condition is implied by the sufficient condition for continuity of r given in [**6**]. Besides, the condition of Theorem 1.1 is (respectively, the two equivalent sufficient conditions for continuity of σ given in [**6**] are) characterized in the algebra $L(X)$, where X is either a complex nonzero Hilbert space, or c_0, or l_p, $p \in [1,+\infty)$, and is (respectively, are) proved to be equivalent to both condition (1) and the sufficient condition for continuity of r given in [**6**] (respectively, to condition (2)) in these particular algebras (Propositions 1.3, 1.4, 1.5 and 1.6). Finally, two examples are given in the algebra $L(X \times Y)$ (where X and Y are any two different spaces of the set $\{c_0\} \cup \{l_p\}_{p\in[1,+\infty)}$) to show how the two equivalent sufficient conditions for continuity of σ and the sufficient condition for continuity of r given in [**6**] are less restrictive than conditions (2) and (1), respectively (Example 1.7), and the condition of Theorem 1.1 is less restrictive than the one given in [**6**] for the spectral radius function (Example 1.8).

In Section 2 we prove that two equivalent sufficient conditions for continuity of r in $L(X)/K(X)$ (where X is a Banach space), due to Conway and Morrel, imply the condition for continuity of r given in [**6**] (and thus imply also the condition of Theorem 1.1), so that all these conditions are equivalent and characterize continuity of the spectral radius function in $L(X)/K(X)$ if X is a separable Hilbert space (Theorem 2.1). Still in the case of the algebra $L(X)/K(X)$ for a Banach space X, in Theorem 2.2 we prove that a condition given by Conway and Morrel and a condition given by the authors of [**1**] for continuity of σ are equivalent; in addition, we give a third sufficient condition for continuity of σ in this algebra, that is equivalent to the two above, and we show how these three equivalent conditions imply the two equivalent sufficient conditions of [**6**] for continuity of spectrum, and are therefore equivalent to them if X is a separable Hilbert space (so that also the two conditions of [**6**] characterize continuity of the spectrum function in $L(X)/K(X)$ if X is Hilbert and separable). In the general case $L = L(X)/K(X)$, where X is a generic Banach space, our conditions for continuity of r and σ given in Theorem 1.1 and in [**6**] are less restrictive than the ones given by Conway and Morrel and by the authors of [**1**].

Finally, the two equivalent sufficient conditions for continuity of σ and the sufficient condition for continuity of r given in [**6**], together with the condition of Theorem 1.1, are characterized in the algebras $L(X)/K(X)$, where X is either a complex infinite-dimensional Hilbert space, or c_0, or l_p, $p \in [1,+\infty)$, and $L(X)/J$, where X is a complex nonseparable Hilbert space and $K(X) \subsetneq J \in \mathbf{J}_{L(X)}$ (Propositions 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9).

Section 1. For any Banach algebra M with identity, let G_M denote the group of invertible elements of M and let H_M denote the union of all components of G_M which do not contain the identity of M.

Now let L be a complex Banach algebra. If $\mathbf{e}$ denotes the identity of $\underline{L}$, for any $\mathbf{a} \in L$ we set $H(\mathbf{a}) = \{\lambda \in \mathbf{C} : \lambda\mathbf{e} - \mathbf{a} \in H_{\underline{L}}\}$ and $\alpha(\mathbf{a}) = \sup\{|\lambda| : \lambda \in H(\mathbf{a})\}$. Notice that $H(\mathbf{a}) \subset \rho(\mathbf{a})$ (where $\rho(\mathbf{a})$ denotes the resolvent set of $\mathbf{a}$, namely $\rho(\mathbf{a}) = \mathbf{C}\backslash\sigma(\mathbf{a})$). Although $H(\mathbf{a})$ is not a subset of $\sigma(\mathbf{a})$, the inequality $\alpha(\mathbf{a}) \leq r(\mathbf{a})$ holds: in fact, $\lambda\mathbf{e} - \mathbf{a}$ can be connected with $\mathbf{e}$ by means of invertible elements of $\underline{L}$ for any $\lambda \in \mathbf{C}$ such that $|\lambda| > r(\mathbf{a})$ (as the range of the continuous map $\varphi : [0,1] \ni t \mapsto \lambda\mathbf{e} - (1-t)\mathbf{a} \in \underline{L}$, connecting $\lambda\mathbf{e} - \mathbf{a}$ with $\lambda\mathbf{e}$, is contained in $G_{\underline{L}}$).

In [**6**, 1.4] we proved that the condition

$$r(\mathbf{a}) = \max\{\alpha(\mathbf{a}), \gamma(\mathbf{a}), \sup\{\delta_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}\} \tag{3}$$

implies continuity of r at $\mathbf{a}$. Now we are going to give a less restrictive sufficient condition for continuity of r at $\mathbf{a}$ than (3).

For any Banach algebra M with identity and for any $J \in \mathbf{J}_M$, let H_M^J denote the union of all components of $\pi_J^{-1}(G_{M/J})$ which do not contain the identity of M and have nonempty intersection with G_M. We remark that $H_M^{\{0\}} = H_M$ and $G \subset \overset{\circ}{S}_M$ for any component G of $\pi_J^{-1}(G_{M/J})$ such that $G \cap G_M = \varnothing$. Moreover, if $\tilde{G}$ is the component of G_M containing the identity, we have $\tilde{G} \cap H_M^J = \varnothing$ for any $J \in \mathbf{J}_M$.

Now let L be a complex Banach algebra and let $\mathbf{e}$ denote the identity of $\underline{L}$. For any $\mathbf{a} \in \mathbf{L}$ and for any $J \in \mathbf{J}_{\underline{L}}$, we set $H_J(\mathbf{a}) = \{\lambda \in \mathbf{C} : \lambda\mathbf{e} - \mathbf{a} \in H_{\underline{L}}^J\}$ and $\alpha_J(\mathbf{a}) = \sup\{|\lambda| : \lambda \in H_J(\mathbf{a})\}$. Notice that $H_{\{0\}}(\mathbf{a}) = H(\mathbf{a})$ and $\alpha_{\{0\}}(\mathbf{a}) = \alpha(\mathbf{a})$.

Since $\lambda\mathbf{e} - \mathbf{a}$ and $\mathbf{e}$ belong to the same component of $G_{\underline{L}}$ for any $\lambda \in \mathbf{C}$ such that $|\lambda| > r(\mathbf{a})$, we have $\alpha_J(\mathbf{a}) \leq r(\mathbf{a})$ for any $J \in \mathbf{J}_{\underline{L}}$, even if $H_J(\mathbf{a})$ may intersect $\rho(\mathbf{a})$.

Theorem 1.1. *Let L be a complex Banach algebra. Then*:

i) *the function $\alpha_J : L \ni \mathbf{a} \mapsto \alpha_J(\mathbf{a}) \in \mathbf{R}$ is lower semi-continuous for any $J \in \mathbf{J}_{\underline{L}}$*;

ii) *for any* $\mathbf{a} \in L$ *which satisfies*

$$r(\mathbf{a}) = \max\{\gamma(\mathbf{a}), \sup\{\delta_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}, \sup\{\alpha_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}\} \tag{4}$$

the spectral radius function is continuous at $\mathbf{a}$;

iii) *the set* $\{\mathbf{a} \in L : \mathbf{a}$ *satisfies condition* (4)$\}$ *is a* G_δ*-set.*

Proof. Assertion i) is a consequence of $H^J_{\underline{L}}$ being an open subset of $\underline{L}$ for any $J \in \mathbf{J}_{\underline{L}}$.

Now we prove ii).

We recall that the functions $\gamma : L \ni \mathbf{a} \mapsto \gamma(\mathbf{a}) \in \mathbf{R}$ and $\delta_J : L \ni \mathbf{a} \mapsto \delta_J(\mathbf{a}) \in \mathbf{R}$ ($J \in \mathbf{J}_{\underline{L}}$) are lower semi-continuous (see [**3**, 1.4]). Then from i) it follows that also the function $L \ni \mathbf{a} \mapsto \max\{\gamma(\mathbf{a}), \sup\{\delta_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}, \sup\{\alpha_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}\} \in \mathbf{R}$ is lower semi-continuous. Consequently, since $\max\{\gamma(\mathbf{a}), \sup\{\delta_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}, \sup\{\alpha_J(\mathbf{a}) : J \in \mathbf{J}_{\underline{L}}\}\} \leq r(\mathbf{a})$ for any $\mathbf{a} \in L$ and r is upper semi-continuous on L, the spectral radius function is continuous at $\mathbf{a}$ for any $\mathbf{a} \in L$ which satisfies condition (4).

Assertion iii) is a consequence of $r(.) - \max\{\gamma(.), \sup\{\delta_J(.) : J \in \mathbf{J}_{\underline{L}}\}, \sup\{\alpha_J(.) : J \in \mathbf{J}_{\underline{L}}\}\}$ being an upper semi-continuous function. □

We recall that the set of all continuity points of every map from a topological space into a metric one is a G_δ-set. We also remark that condition (1) implies condition (3) and condition (3) implies condition (4). Moreover, conditions (1), (3) and (4) are equivalent if $G_{\underline{L}}$ is connected, and conditions (3) and (4) are equivalent if $\mathbf{J}_{\underline{L}} = \{0\}$.

Let L be a complex Banach algebra, let $\mathbf{e}$ denote the identity of $\underline{L}$ and let $\mathbf{a} \in L$. For any $G \in \mathbf{c}(G_{\underline{L}})$ (where $\mathbf{c}(\Xi)$ denotes the set of all components of Ξ for any topological space Ξ) we set $\rho_G(\mathbf{a}) = \{\lambda \in \mathbf{C} : \lambda\mathbf{e} - \mathbf{a} \in G\}$. We recall that $\rho_G(\mathbf{a})$ is an open subset of $\mathbf{C}$ for any $G \in \mathbf{c}(G_{\underline{L}})$, $\rho(\mathbf{a}) = \cup_{G\in\mathbf{c}(G_{\underline{L}})}\rho_G(\mathbf{a})$, $H(\mathbf{a}) = \cup_{G\in\mathbf{c}(H_{\underline{L}})}\rho_G(\mathbf{a})$ and besides $\cap_{G\in\mathbf{c}(G_{\underline{L}})}\overline{(\rho(\mathbf{a})\backslash\rho_G(\mathbf{a}))} \subset \overline{(\cup_{G\in\mathbf{c}(G_{\underline{L}})}\partial\overline{\rho_G(\mathbf{a})})} \subset \sigma(\mathbf{a})$ (see [**6**, 2.6 and remark after 2.1]).

In [**6**] the following result is proved.

Theorem 1.2 [**6**, 2.7]. *Let* L *be a complex Banach algebra, and let*

$\mathbf{a} \in L$. *Then the following two conditions are equivalent*:

$$\sigma(\mathbf{a}) = \overline{S(\mathbf{a})} \cup \zeta(\mathbf{a}) \cup (\cap_{G\in \mathbf{c}(G_{\underline{L}})}\overline{(\rho(\mathbf{a})\backslash\rho_G(\mathbf{a}))}); \tag{5}$$

$$\sigma(\mathbf{a}) = \overline{S(\mathbf{a})} \cup \zeta(\mathbf{a}) \cup \overline{(\cup_{G\in \mathbf{c}(G_{\underline{L}})}\partial\overline{\rho_G(\mathbf{a})})}. \tag{6}$$

Moreover, if the equivalent conditions (5) *and* (6) *are satisfied, the spectrum function is continuous at* $\mathbf{a}$.

We recall that condition (2) implies the two equivalent conditions (5) and (6) and is equivalent to them if $G_{\underline{L}}$ is connected. Moreover, the equivalent conditions (5) and (6) imply condition (3) (see [**6**, remarks after 2.7]), and therefore imply also condition (4).

For any Banach space X and for any $T \in L(X)$, let $N(T)$ and $R(T)$ denote the kernel and the range of T, respectively.

Now let X be a complex nonzero Hilbert space of Hilbert dimension h. Let Δ_h denote the set of all cardinal numbers α such that either $\alpha = 1$ or $\aleph_0 \le \alpha \le h$. We recall that $G_{L(X)}$ is connected (see [**14**, Problem 110]). We also recall that $\mathbf{J}_{L(X)} = \{K_\alpha(X) : \alpha \in \Delta_h\}$, where $K_\alpha(X) = \overline{\{T \in L(X) : \overline{R(T)} \text{ has Hilbert dimension less than } \alpha\}}$ for any $\alpha \in \Delta_h$ (see [**15**, Theorem 6.1, Theorem 6.4 and Corollary 6.1]), so that $\mathbf{J}_{L(X)}$ is well ordered by inclusion. Notice that $K_1(X) = \{0\}$, $K_\alpha(X)$ is self-adjoint for any $\alpha \in \Delta_h$ and $K(X) = K_{\aleph_0}(X)$ if $h \ge \aleph_0$. For any $T \in L(X)$, let $d(T)$ denote the approximate nullity of T (see [**12**, 1.3]). Finally (see [**3**, 1.10]), we recall that $\overset{\circ}{S}_{L(X)} = \{T \in L(X) : d(T) \ne d(T^*)\}$, where T^* denotes the Hilbert adjoint of T for any $T \in L(X)$. Then the result below is a consequence of [**3**, 2.2].

Proposition 1.3. *Let X be a complex Hilbert space of Hilbert dimension $h > 0$. Then, for any $A \in L(X)$, each of conditions* (1), (3) *and* (4) *is equivalent to the following condition*:

$r(A) = \max\{\sup\{|\lambda| : \lambda \in \mathbf{C}$ *and* $d(\lambda I_X - A) \ne d(\bar{\lambda} I_X - A^*)\}$, $\sup\{\delta_{K_\alpha(X)}(A) : \alpha \in \Delta_h$ *and α is not a limit cardinal number if* $\alpha > \aleph_0\}\}$.

For any complex Banach algebra L and for any $\mathbf{a} \in L$, we set $\underline{\psi(\mathbf{a})} = \{\lambda \in \sigma(\mathbf{a}) : \{\lambda\} \text{ is a component of } \sigma(\mathbf{a})\}$. We remark that $\underline{\psi(\mathbf{a})} \subset \zeta(\mathbf{a})$.

Since $\mathbf{J}_{L(X)}$ is well ordered by inclusion for any Hilbert space X, the result below is a consequence of [**5**, 2.8].

Proposition 1.4. *Let X be a complex nonzero Hilbert space. Then for any $A \in L(X)$ each of conditions* (2), (5) *and* (6) *is equivalent to the following condition*:

$$\sigma(A) = \overline{\{\lambda \in \mathbf{C} : d(\lambda I_X - A) \neq d(\bar{\lambda} I_X - A^*)\}} \cup \overline{\psi(A)}.$$

We recall that, if the Hilbert space X is not separable, in $L(X)$ conditions (1) and (2) are less restrictive than the corresponding Conway and Morrel's ones, which therefore are not necessary for continuity of r and σ, respectively (see [**3**, 2.1 and **4**, remarks before 3.4]).

Now let $X \in \{c_0\} \cup \{l_p\}_{p \in [1,+\infty)}$. We recall that $G_{L(X)}$ is connected (see [**16**, corollary of Proposition 2 and corollary of Lemma 11b]) and $\mathbf{J}_{L(X)} = \{\{0\}, K(X)\}$ (see [**11**, (5.4.23)]). If, for any $A \in L(X)$, $\sigma_p^0(A)$ denotes the set of all isolated eigenvalues of A whose spectral projection has finite-dimensional range, the following result is a consequence of the remarks above and of [**3**, 1.6].

Propostion 1.5. *Let $X \in \{c_0\} \cup \{l_p\}_{p \in [1,+\infty)}$. Then for any $A \in L(X)$ each of conditions* (1), (3) *and* (4) *is equivalent to each of the following conditions*:

i) $r(A) = \max\{\gamma(A), \delta_{\{0\}}(A), \delta_{K(X)}(A)\}$;

ii) $r(A) = \max\{\gamma(A), \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: ω is a component of $\sigma_{K(X)}(A) \cup \sigma_p^0(A)\}\}$.

The following result is a consequence of the remarks above and of [**5**, 2.8].

Proposition 1.6. *Let $X \in \{c_0\} \cup \{l_p\}_{p \in [1,+\infty)}$. Then for any $A \in L(X)$ each of conditions* $(2), (5)$ *and* (6) *is equivalent to the following condition*:

$$\sigma(A) = \overline{S(A)} \cup \overline{\psi(A)}.$$

We recall that the groups of invertible operators on several classical Banach spaces are connected, and indeed, contractible, but yet there are examples of Banach spaces X such that $G_{L(X)}$ is disconnected (see [**16**] for a survey on this subject). In the following example we use Douady's construction of a Banach space with disconnected linear group (recorded in [**16**, Section 1]) to show how the equivalent conditions (5) and (6) are less restrictive than condition (2), and condition (3) is less restrictive than condition (1), even in the algebra $L(X)$.

We recall that a linear bounded operator on a Banach space is invertible modulo compact operators if and only if it is a Fredholm operator (see [**11**, (3.2.8)]).

For any $\lambda \in \mathbf{C}$ and for any $\varepsilon > 0$, let $B_{\mathbf{C}}(\lambda, \varepsilon)$ denote the set of all points $\mu \in \mathbf{C}$ such that $|\mu - \lambda| < \varepsilon$.

Example 1.7. Let us consider the complex separable Banach space $l_{p_1} \times l_{p_2}$, with $p_1, p_2 \in \{0\} \cup [1, +\infty)$ (where l_0 denotes c_0) and $p_1 \neq p_2$. We recall that $l_{p_1} \times l_{p_2}$ is reflexive if $p_1, p_2 \in (1, +\infty)$, and is also isomorphic to its dual if, in addition, $1/p_1 + 1/p_2 = 1$.

Now let us consider the operator $A \in L(l_{p_1} \times l_{p_2})$ defined by $A((x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}) = ((x_{n+1})_{n\in\mathbf{N}}, x_0 e_0 + \sum_{n\in\mathbf{N}} y_n e_{n+1})$ (where $\{e_n\}_{n\in\mathbf{N}}$ is the canonical basis of l_{p_2}) for any $((x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}) \in l_{p_1} \times l_{p_2}$.

We recall that $A \in G_{L(l_{p_1}\times l_{p_2})}$ and A cannot be connected with the identity by means of invertible operators (see [**16**, Section 1]. Hence there exists $G \in \mathbf{c}(H_{L(l_{p_1}\times l_{p_2})})$ such that $A \in G$.

Let S_1 and S_2 denote, respectively, the backward shift on l_{p_1} and the unilateral shift on l_{p_2} (which means that $S_1(x_n)_{n\in\mathbf{N}} = (x_{n+1})_{n\in\mathbf{N}}$ for any $(x_n)_{n\in\mathbf{N}} \in l_{p_1}$ and $S_2(y_n)_{n\in\mathbf{N}} = \sum_{n\in\mathbf{N}} y_n e_{n+1}$ for any $(y_n)_{n\in\mathbf{N}} \in l_{p_2}$). Let $T \in L(l_{p_1} \times l_{p_2})$ be defined by $T(x, y) = (S_1 x, S_2 y)$ for any $(x, y) \in l_{p_1} \times l_{p_2}$. It is not difficult to verify that $\lambda I_{l_{p_1}\times l_{p_2}} - T$ is a Fredholm operator of index zero for any $\lambda \in \mathbf{C}\backslash\partial B_{\mathbf{C}}(0, 1)$ and $\sigma_{K(l_{p_1}\times l_{p_2})}(T) = \partial B_{\mathbf{C}}(0, 1) = \sigma_{K(l_{p_j})}(S_j)$ for any $j = 1, 2$.

Since $A - T \in K(l_{p_1} \times l_{p_2})$, it follows that $\sigma_{K(l_{p_1}\times l_{p_2})}(A) = \partial B_{\mathbf{C}}(0, 1)$ and $\lambda I_{l_{p_1}\times l_{p_2}} - A$ is a Fredholm operator of index zero for any $\lambda \in \mathbf{C}\backslash\partial B_{\mathbf{C}}(0, 1)$ (see [**11**, (4.4.2)]). It is not difficult to verify that

$N(\lambda I_{l_{p_1}\times l_{p_2}} - A) = \{0\}$ for any $\lambda \in \mathbf{C}\backslash\partial B_{\mathbf{C}}(0,1)$. Therefore, $\sigma(A) = \partial B_{\mathbf{C}}(0,1)$ and $r(A) = 1$.

Hence, $B_{\mathbf{C}}(0,1) \subset \rho_G(A) \subset H(A) \subset B_{\mathbf{C}}(0, r(A)) = B_{\mathbf{C}}(0,1)$, and consequently $\rho_G(A) = B_{\mathbf{C}}(0,1)$. It follows that $\partial\rho_G(A) = \partial B_{\mathbf{C}}(0,1) = \sigma(A)$.

Therefore A satisfies condition (6) (so that the spectrum function is continuous at A).

Now we prove that A does not satisfy condition (2).

We remark that $S(A) \subset \overset{\circ}{\sigma}(A)$, and consequently $S(A) = \varnothing$.

Now we prove that $\zeta(A) = \varnothing$.

For any $k = 1,2$, let $P_k \in L(l_{p_k}, l_{p_1} \times l_{p_2})$ and $Q_k \in L(l_{p_1} \times l_{p_2}, l_{p_k})$ be defined by $P_k x = (\delta_{1k}x, \delta_{2k}x)$ for any $x \in l_{p_k}$ and $Q_k(x_1, x_2) = x_k$ for any $(x_1, x_2) \in l_{p_1} \times l_{p_2}$. We remark that $Q_k P_k = I_{l_{p_k}}$ for any $k = 1,2$, $Q_k P_h$ is the null operator if $k \neq h$, $P_1 Q_1 + P_2 Q_2 = I_{l_{p_1}\times l_{p_2}}$ and $T = \sum_{j,k=1,2} P_j Q_j T P_k Q_k$ for any $T \in L(l_{p_1} \times l_{p_2})$. Moreover, we have $Q_k A P_k = S_k$ for any $k = 1,2$.

For any $k = 1,2$, we set $J_k = \{T \in L(l_{p_1} \times l_{p_2}) : Q_k T P_k \in K(l_{p_k})\}$.

Since either any linear bounded operator from l_{p_1} into l_{p_2} or any linear bounded operator from l_{p_2} into l_{p_1} is compact (see [**18**, 5.1.2 and following remark]) and $I_{l_{p_1}\times l_{p_2}} \notin J_k$, it follows that $J_k \in \mathbf{J}_{L(l_{p_1}\times l_{p_2})}$.

Let us consider the linear and continuous operator $\Phi : L(l_{p_k}) \to L(l_{p_1} \times l_{p_2})/J_k$ defined by $\Phi(T) = P_k T Q_k + J_k$ for any $T \in L(l_{p_k})$.

We remark that for any $T \in L(l_{p_1} \times l_{p_2})$, we have $Q_k T P_k = Q_k(\sum_{j,h=1,2} P_j Q_j T P_h Q_h) P_k = Q_k P_k Q_k T P_k Q_k P_k$ and consequently, $T - P_k Q_k T P_k Q_k \in J_k$.

Then Φ is surjective, as $T + J_k = \Phi(Q_k T P_k)$ for any $T \in L(l_{p_1} \times l_{p_2})$. Using the equality $Q_k P_k = I_{l_{p_k}}$, it is not difficult to prove that Φ is a homomorphism of Banach algebras and $N(\Phi) = K(l_{p_k})$. Hence, Φ induces an isomorphism of Banach algebras from $L(l_{p_k})/K(l_{p_k})$ onto $L(l_{p_1} \times l_{p_2})/J_k$. Therefore, we have $\sigma_{J_k}(A) = \sigma(\Phi(Q_k A P_k)) = \sigma_{K(l_{p_k})}(Q_k A P_k) = \sigma_{K(l_{p_k})}(S_k) = \partial B_{\mathbf{C}}(0,1) = \sigma(A)$.

We recall that J_1 and J_2 are the maximal ideals of $L(l_{p_1} \times l_{p_2})$ (see [**18**, 5.3.2], where only the case $p_1, p_2 \in [1, +\infty)$ is treated; the proof can be repeated in the case $0 \in \{p_1, p_2\}$). Consequently, we have

$\sigma_J(A) = \partial B_{\mathbf{C}}(0,1)$ for any $J \in \mathbf{J}_{L(l_{p_1} \times l_{p_2})}$. Hence $\zeta(A) = \varnothing$.

We have thus proved that $\overline{S(A)} \cup \zeta(A) = \varnothing \subsetneq \sigma(A)$. Therefore A does not satisfy condition (2).

Finally, we remark that the operator $I_{l_{p_1} \times l_{p_2}} + A$ satisfies condition (3) (indeed, it satisfies the equivalent conditions (5) and (6)) but does not satisfy condition (1). $\square$

The following example shows how condition (4) is less restrictive than condition (3) in the case of the algebra of all linear and continuous operators on a generic Banach space.

Example 1.8. Let us consider the complex separable Banach space $l_{p_1} \times l_{p_2}$, with $p_1, p_2 \in \{0\} \cup [1, +\infty)$ (where l_0 denotes c_0) and $p_1 \neq p_2$. Let $A, T \in L(l_{p_1} \times l_{p_2})$ be defined as in Example 1.7.

We remark that $\sigma(T) = \overline{B_{\mathbf{C}}(0,1)}$. Hence $H(T) = \varnothing$.

We recall that $\lambda I_{l_{p_1} \times l_{p_2}} - T$ is a Fredholm operator of index zero for any $\lambda \in B_{\mathbf{C}}(0,1)$. Then there exists a component F of $\pi^{-1}_{K(l_{p_1} \times l_{p_2})}(G_{L(l_{p_1} \times l_{p_2})/K(l_{p_1} \times l_{p_2})})$ such that $\lambda I_{l_{p_1} \times l_{p_2}} - T \in F$ for any $\lambda \in B_{\mathbf{C}}(0,1)$.

Since $A - T \in K(l_{p_1} \times l_{p_2})$ and $A \in G_{L(l_{p_1} \times l_{p_2})}$ cannot be connected with the identity by means of invertible operators, it follows that $A \in F$ and $I_{L(l_{p_1} \times l_{p_2})} \notin F$ (see [**11**, (6.2.5)]). Hence, $F \subset H^{K(l_{p_1} \times l_{p_2})}_{L(l_{p_1} \times l_{p_2})}$. Consequently, we have $B_{\mathbf{C}}(0,1) = H_{K(l_{p_1} \times l_{p_2})}(T)$.

For any $\lambda \in B_{\mathbf{C}}(0,1)$, since $\lambda I_{l_{p_1} \times l_{p_2}} - T$ is a Fredholm operator of index zero, it follows that $\lambda I_{l_{p_1} \times l_{p_2}} - T \notin \overset{\circ}{S}_{L(l_{p_1} \times l_{p_2})}$ (see, for instance, [**13**], or [**17**, 2.1]. Hence, $S(T) = \varnothing$.

Since $T - A$ is a finite-dimensional operator, it follows that $\sigma_J(T) = \sigma_J(A) = \partial B_{\mathbf{C}}(0,1)$ for any nonzero $J \in \mathbf{J}_{L(l_{p_1} \times l_{p_2})}$ (see [**11**, (5.2.1)]).

It follows that $\sigma(I_{l_{p_1} \times l_{p_2}} + T) = \overline{B_{\mathbf{C}}(1,1)}$ and $\sigma_J(I_{l_{p_1} \times l_{p_2}} + T) = \partial B_{\mathbf{C}}(1,1)$ for any nonzero $J \in \mathbf{J}_{L(l_{p_1} \times l_{p_2})}$.

Therefore, $\delta_J(I_{l_{p_1} \times l_{p_2}} + T) = 0$ for any $J \in \mathbf{J}_{L(l_{p_1} \times l_{p_2})}$.

Since $H(T) = S(T) = \varnothing$, it follows that $H(I_{l_{p_1} \times l_{p_2}} + T) = S(I_{l_{p_1} \times l_{p_2}} + T) = \varnothing$, and consequently, $\alpha(I_{l_{p_1} \times l_{p_2}} + T) = \gamma(I_{l_{p_1} \times l_{p_2}} +$

$T) = 0$.

We have thus proved that $\max\{\alpha(I_{l_{p_1}\times l_{p_2}} + T),\ \gamma(I_{l_{p_1}\times l_{p_2}} + T), \sup\{\delta_J(I_{l_{p_1}\times l_{p_2}} + T) : J \in \mathbf{J}_{L(l_{p_1}\times l_{p_2})}\}\} = 0 < 2 = r(I_{l_{p_1}\times l_{p_2}} + T)$.

Hence, $I_{l_{p_1}\times l_{p_2}} + T$ does not satisfy condition (3).

Since $H_{K(l_{p_1}\times l_{p_2})}(T) = B_{\mathbf{C}}(0,1)$ it follows that $H_{K(l_{p_1}\times l_{p_2})}(I_{l_{p_1}\times l_{p_2}} + T) = B_{\mathbf{C}}(1,1)$. Therefore, $\alpha_{K(l_{p_1}\times l_{p_2})}(I_{l_{p_1}\times l_{p_2}} + T) = 2 = r(I_{l_{p_1}\times l_{p_2}} + T)$, and consequently $I_{l_{p_1}\times l_{p_2}} + T$ satisfies condition (4). □

Section 2. Let X be a complex infinite-dimensional Banach space. We shall denote the semigroup of all linear bounded Fredholm operators on X by $F(X)$. For any linear bounded semi-Fredholm operator T on X, let $\operatorname{ind}(T)$ ($\in \mathbf{Z} \cup \{-\infty, +\infty\}$) denote the index of T. We recall that $\{T \in L(X) : T$ is semi-Fredholm and $\operatorname{ind}(T) = n\}$ is open for any $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$ (see [**11**, (4.2.1), (4.2.2) and (4.4.1)]). If we set $F_n(X) = \{T \in F(X) : \operatorname{ind}(T) = n\}$ for any $n \in \mathbf{Z}$, from [**11**, (6.2.5)] it follows that $\pi_{K(X)}(\cup_{n\in\mathbf{Z}\setminus\{0\}} F_n(X)) \subset H_{L(X)/K(X)}$.

We remark that $\pi_{K(X)}(\overline{F(X)}) \subset \overline{G}_{L(X)/K(X)}$ and the open set $\pi_{K(X)}(L(X)\setminus\overline{F(X)})$ is contained in $S_{L(X)/K(X)}$. Consequently, we have $L(X)\setminus\overline{F(X)} = \pi^{-1}_{K(X)}(\mathring{S}_{L(X)/K(X)})$.

Now we set $SF^{\pm\infty}(X) = \{T \in L(X) : T$ is semi-Fredholm and $\operatorname{ind}(T) \in \{-\infty, +\infty\}\}$. Then by the preceding remarks we have $\pi_{K(X)}(SF^{\pm\infty}(X)) \subset \mathring{S}_{L(X)/K(X)}$.

For any $T \in L(X)$, we set $\rho_{s-F}(T) = \{\lambda \in \mathbf{C} : \lambda I_X - T$ is semi-Fredholm$\}$, $\rho^n_{s-F}(T) = \{\lambda \in \mathbf{C} : \lambda I_X - T$ is semi-Fredholm and $\operatorname{ind}(\lambda I_X - T) = n\}$ for any $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$ and $\rho^{\pm}_{s-F}(T) = \{\lambda \in \mathbf{C} : \lambda I_X - T$ is semi-Fredholm and $\operatorname{ind}(\lambda I_X - T) \neq 0\}$. By what we recalled above, we have that $\rho_{s-F}(T)$, $\rho^n_{s-F}(T)$, $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$, and $\rho^{\pm}_{s-F}(T)$ are open subsets of $\mathbf{C}$. We remark that $\rho(T) \subset \rho^0_{s-F}(T)$, and consequently $\rho^{\pm}_{s-F}(T) \subset \sigma(T)$. Besides, we set $\sigma_{\mathrm{lre}}(T) = \{\lambda \in \mathbf{C} : \pi_{K(X)}(\lambda I_X - T)$ is neither left nor right invertible$\}$. We remark that $\sigma_{\mathrm{lre}}(T) \subset \sigma_{K(X)}(T) = \sigma_{\mathrm{lre}}(T) \cup \rho^{+\infty}_{s-F}(T) \cup \rho^{-\infty}_{s-F}(T)$ (see [**11**, (4.3.4)]).

Finally, we set

$$\Gamma_{0e}(T) = \{\lambda \in \mathbf{C} : \{\lambda\} \text{ is a component of } (\sigma_{K(X)}(T)\backslash\overline{\rho^{\pm}_{s-F}(T)}) \cup (\cup_{n\in\mathbf{Z}}\overline{(\overset{\circ}{\rho^{n}_{s-F}(T)}}\backslash\rho^{n}_{s-F}(T)))\}.$$

Let us consider the following conditions:

(7) $r(\pi_{K(X)}(T)) = \max\{\sup\{|\lambda| : \lambda \in \rho^{\pm}_{s-F}(T)\},\ \delta_{K(X)}(T)\}$;

(8) $r(\pi_{K(X)}(T)) = \max\{\sup\{|\lambda| : \lambda \in \rho^{\pm}_{s-F}(T)\},\ \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: ω is a component of $\sigma_{\text{lre}}(T)\}$;

(9) any neighborhood of any $\lambda \in \sigma_{K(X)}(T)\backslash\overline{\rho^{\pm}_{s-F}(T)}$ contains a component of $\sigma_{\text{lre}}(T)$, and any neighborhood of any $\lambda \in \overline{\overset{\circ}{\rho^{n}_{s-F}(T)}}\backslash\rho^{n}_{s-F}(T)$ contains a component of $\sigma_{\text{lre}}(T)$ for any $n \in \mathbf{Z}\backslash\{0\}$;

(10) $\sigma_{K(X)}(T) = \partial\rho^{\pm}_{s-F}(T) \cup \rho^{+\infty}_{s-F}(T) \cup \rho^{-\infty}_{s-F}(T) \cup \overline{\Gamma_{0e}(T)}$ and $\sigma_{\text{lre}}(T) \cap \overline{\overset{\circ}{\rho^{n}_{s-F}(T)}} \subset \overline{\Gamma_{0e}(T)}$ for any $n \in \mathbf{Z}\backslash\{0\}$.

It is not difficult to verify that condition (9) implies condition (8). Moreover, we recall that conditions (7) and (8) are equivalent (see [**8**, 2.14], where the space X is supposed to be Hilbert and separable; the proof of this result can be repeated in the general case of a Banach space, see [**8**, Section 4]). In Theorem 2.2 we shall prove that conditions (9) and (10) are equivalent too.

Now suppose X is Hilbert and separable. Then each of conditions (7) and (8) is equivalent to continuity of r at $\pi_{K(X)}(T)$ (see [**8**, 2.15]), and each of conditions (9) and (10) is equivalent to continuity of σ at $\pi_{K(X)}(T)$ (see [**10**, 4.1], and [**1**, Theorem 14.23]). The proof of "(c) implies (a)" given in [**10**, 4.1] uses [**10**, 3.2(b)], which actually proves only that $\overline{\overset{\circ}{P_k(A)}}\backslash P_k(A) \subset \liminf \sigma^0_e(A_n)$, as one of the authors confirmed to us. Nevertheless, the assertion "(c) implies (a)" in [**10**, 4.1] is true. In fact, J.B. Conway kindly provided us with the small modifications that make the proof work, so that the result can be proved in the following way. From [**10**, 3.2] it follows only that $\sigma_e(A)\backslash\overline{P_{\pm}(A)} \subset \liminf \sigma^0_e(A_n)$ and $\overline{\overset{\circ}{P_k(A)}}\backslash P_k(A) \subset \liminf \sigma^0_e(A_n)$ for any nonzero integer k. For any $k \in \mathbf{Z}\backslash\{0\}$ and for any $\lambda \in \partial\overline{P_k(A)}$, since $\sigma_e(A)\backslash\overline{P_\pm(A)} \subset \liminf \sigma_e^0(A_n)$ which is closed, it can be assumed that $\lambda \notin \overline{(\sigma_e(A)\backslash\overline{P_\pm(A)})}$. Thus, any neighborhood of λ intersects $\cup_{j\in(\mathbf{Z}\cup\{-\infty,+\infty\})\backslash\{k\}} P_j(A)$, which implies $\lambda \in \liminf \sigma_e^0(A_n)$ by [**10**, 2.1]. Since $\overline{P_{\pm\infty}(A)} \subset \liminf \sigma_e(A_n)$ by continuity of semi-Fredholm index and closedness of $\liminf \sigma_e(A_n)$, at this point the thesis follows from [**10**, 2.2].

We remark that the proof above works also in the case of a generic Banach space, as the assumption $\sigma(A) = \sigma_e^0(A) \cup P_\pm(A)$ is not needed.

Notice that [**8**, 2.15] deals with continuity of the function $r(\pi_{K(X)}(.))$ at T, and [**10**, 4.1] and [**1**, Theorem 14.23] deal with continuity of the function $\sigma_{K(X)}$ at T. Nevertheless, it is not difficult to verify that continuity of σ (respectively, r) at $\pi_J(\mathbf{a})$ is equivalent to continuity of σ_J (respectively, $r(\pi_J(.))$) at $\mathbf{a}$ for any complex Banach algebra L, for any $J \in \mathbf{J}_L$ and for any $\mathbf{a} \in L$ (see also [**9**, remarks at the beginning of Section 4]).

We recall that each of conditions (7) and (8) (respectively, (9) and (10)) is sufficient for continuity of r (respectively, σ) at $\pi_{K(X)}(T)$ also in the general case of a Banach space X (see [**8**, Section 4, **1**, p. 312] and remarks above about [**10**, 4.1]).

Theorem 2.1. *Let X be a complex infinite-dimensional Banach space and let $A \in L(X)$. Then the following two equivalent conditions*

i) *A satisfies condition* (7)

ii) *A satisfies condition* (8)

imply

iii) *$\pi_{K(X)}(A)$ satisfies condition* (3).

Moreover, iii) *implies*

iv) *$\pi_{K(X)}(A)$ satisfies condition* (4)

and iv) *implies*

v) *the spectral radius function is continuous at $\pi_{K(X)}(A)$.*

Finally, if X is a separable Hilbert space, conditions i), ii), iii), iv) *and* v) *are equivalent.*

Proof. In Theorem 1.1 we proved that (4) is a sufficient condition for continuity of r. We have also already remarked that condition (3) implies condition (4). Finally, we have recalled above that the two equivalent conditions (7) and (8) are necessary for continuity of the spectral radius function at $\pi_{K(X)}(A)$ if X is a separable Hilbert space.

Then it is sufficient to prove that i) implies iii).

Let X be a complex infinite-dimensional Banach space and let $A \in L(X)$. Suppose that A satisfies condition (7).

Then $r(\pi_{K(X)}(A)) = \max\{\sup\{|\lambda| : \lambda \in \rho^{\pm}_{s-F}(A)\}, \delta_{K(X)}(A)\} = \max\{\sup\{|\lambda| : \lambda \in \cup_{n\in\mathbf{Z}\setminus\{0\}}\rho^{n}_{s-F}(A)\}, \sup\{|\lambda| : \lambda \in \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A)\}, \delta_{K(X)}(A)\}$.

By the remarks above, we have $\cup_{n\in\mathbf{Z}\setminus\{0\}}\rho^{n}_{s-F}(A) \subset H(\pi_{K(X)}(A))$ and $\rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \subset S(\pi_{K(X)}(A))$. Moreover, $\delta_{K(X)}(A) = \delta_{\{0\}}(\pi_{K(X)}(A))$.

It follows that $r(\pi_{K(X)}(A)) \le \max\{\alpha(\pi_{K(X)}(A)), \gamma(\pi_{K(X)}(A)), \sup\{\delta_J(\pi_{K(X)}(A)) : J \in \mathbf{J}_{L(X)/K(X)}\}$ and consequently $\pi_{K(X)}(A)$ satisfies condition (3). □

For any complex Banach algebra L and for any $J \in \mathbf{J}_{\underline{L}}$, we set $\psi_J(\mathbf{a}) = \psi(\pi_J(\mathbf{a}))$ for any $\mathbf{a} \in L$. We remark that $\overline{\psi_J(\mathbf{a})} \subset \zeta(\mathbf{a})$.

Theorem 2.2. *Let X be a complex infinite-dimensional Banach space and let $A \in L(X)$. Then the following conditions are equivalent*:

i) $\sigma_{K(X)}(A) = \overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{\psi_{K(X)}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^{n}_{s-F}(A)})}$;

ii) *A satisfies condition* (9);

iii) *A satisfies condition* (10).

Moreover, the equivalent conditions i), ii) *and* iii) *imply*

iv) *$\pi_{K(X)}(A)$ satisfies the equivalent conditions* (5) *and* (6)

and condition iv) *implies*

v) *the spectrum function is continuous at $\pi_{K(X)}(A)$.*

Finally, if X is a separable Hilbert space, conditions i), ii), iii), iv) *and* v) *are equivalent.*

Proof. We have already recalled that the equivalent conditions (5) and (6) are sufficient for continuity of σ (Theorem 1.2) and that conditions (9) and (10) are necessary for continuity of spectrum at $\pi_{K(X)}(A)$ if X is a separable Hilbert space (above). Hence, it is sufficient to prove that conditions i), ii) and iii) are equivalent and imply condition iv).

Let X be a complex infinite-dimensional Banach space and let $A \in L(X)$. First of all, we prove that i) implies ii).

For any $n \in \mathbf{Z}\backslash\{0\}$, we have that $\overline{\overset{\circ}{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)$ does not intersect $\overline{\rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \cup (\cup_{k\in\mathbf{Z}}\partial\overline{\rho^k_{s-F}(A)})}$. Since A satisfies condition i), it follows that $\overline{\overset{\circ}{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A) \subset \overline{\psi_{K(X)}(A)}$.

We prove that also $\partial\overline{\rho^0_{s-F}(A)}\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \overline{\psi_{K(X)}(A)}$.

For any $\mu \in \partial\overline{\rho^0_{s-F}(A)}\backslash\overline{\rho^{\pm}_{s-F}(A)}$, any neighborhood of μ intersects $\mathbf{C}\backslash\overline{\rho^0_{s-F}(A)}$. Since $\mu \notin \overline{\rho^{\pm}_{s-F}(A)}$, it follows that any neighborhood of μ intersects $\mathbf{C}\backslash\overline{\rho_{s-F}(A)} = \sigma_{K(X)}(A)\backslash\overline{\rho_{s-F}(A)}$. Since A satisfies i), we have $\sigma_{K(X)}(A)\backslash\overline{\rho_{s-F}(A)} \subset \overline{\psi_{K(X)}(A)}$ and consequently $\mu \in \overline{\psi_{K(X)}(A)}$.

Hence $\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \overline{\psi_{K(X)}(A)}$, as condition i) is satisfied.

In view of [**2**, 1.6] we have that $\psi_{K(X)}(A) \subset \{\lambda \in \sigma_{\mathrm{lre}}(A) : \{\lambda\}$ is a component of $\sigma_{\mathrm{lre}}(A)\}$. Hence, every neighborhood of any point of $\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ contains a component of $\sigma_{\mathrm{lre}}(A)$ and every neighborhood of any point of $\overline{\overset{\circ}{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)$ contains a component of $\sigma_{\mathrm{lre}}(A)$ for any $n \in \mathbf{Z}\backslash\{0\}$.

We have thus proved that A satisfies condition (9).

Now we prove that ii) implies iii).

We remark that $\sigma_{K(X)}(A)\backslash(\partial\rho^{\pm}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A)) = \sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$. Since condition ii) is satisfied, it follows that any neighborhood of any point of $\sigma_{K(X)}(A)\backslash(\partial\rho^{\pm}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A))$ contains a component of $\sigma_{\mathrm{lre}}(A)$. Since $\sigma_{\mathrm{lre}}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} = \sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ and $\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ is an open set in the relative topology of $(\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}}(\overline{\overset{\circ}{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)))$ (as

$\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A) \subset \sigma_{K(X)}(A)$ for any $n \in \mathbf{Z}$), it follows from [**2**, 1.5] that any neighborhood of any point of $\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ contains a component of $(\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)))$. Hence, $\Gamma_{0e}(A)$ is dense in $\sigma_{K(X)}(A)\backslash(\partial\rho^{\pm}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A))$ by [**5**, 2.1].

We have thus proved that $\sigma_{K(X)}(A) = \partial\rho^{\pm}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \cup \overline{\Gamma_{0e}(A)}$.

Now let $n \in \mathbf{Z}\backslash\{0\}$ and let $\lambda \in \sigma_{\mathrm{lre}}(A) \cap \overset{\circ}{\overline{\rho^n_{s-F}(A)}} = \sigma_{K(X)}(A) \cap \overset{\circ}{\overline{\rho^n_{s-F}(A)}} = \overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)$. Since A satisfies condition (9), any neighborhood of λ contains a component of $\sigma_{\mathrm{lre}}(A)$. Since $\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash \rho^n_{s-F}(A)$ is an open set in the relative topology of $(\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{k\in\mathbf{Z}}(\overset{\circ}{\overline{\rho^k_{s-F}(A)}}\backslash\rho^k_{s-F}(A)))$, it follows from [**2**, 1.5] that any neighborhood of λ contains a component of $(\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{k\in\mathbf{Z}} (\overset{\circ}{\overline{\rho^k_{s-F}(A)}}\backslash\rho^k_{s-F}(A)))$. From [**5**, 2.1] it follows that $\Gamma_{0e}(A)$ is dense in $\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A) = \sigma_{\mathrm{lre}}(A) \cap \overset{\circ}{\overline{\rho^n_{s-F}(A)}}$.

Hence, A satisfies condition (10).

We prove that iii) implies i).

First of all, we prove that $\sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})}) \subset (\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}\backslash\{0\}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)))$.

Let $\lambda \in \sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})})$. Then there exists $\varepsilon > 0$ such that $B_{\mathbf{C}}(\lambda,\varepsilon) \cap (\rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \cup (\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})) = \varnothing$. Consequently, for any $n \in \mathbf{Z}$, either $B_{\mathbf{C}}(\lambda,\varepsilon) \subset \overline{\rho^n_{s-F}(A)}$ or $B_{\mathbf{C}}(\lambda,\varepsilon) \cap \rho^n_{s-F}(A) = \varnothing$.

Hence $\lambda \in (\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}\backslash\{0\}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A)))$.

We have thus proved that $\sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})}) \subset (\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}\backslash\{0\}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash

$\rho^n_{s-F}(A))) = (\sigma_{K(X)}(A)\backslash(\partial\rho^{\pm}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A)))\cup (\cup_{n\in\mathbf{Z}\backslash\{0\}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}} \cap \sigma_{\mathrm{lre}}(A)))$. Since A satisfies condition (10), it follows that $\Gamma_{0e}(A)$ is dense in $\sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})})$.

From the inclusion above it follows also that

$$\begin{aligned}\sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})})&\\ = ((\sigma_{K(X)}(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) \cup (\cup_{n\in\mathbf{Z}}(\overset{\circ}{\overline{\rho^n_{s-F}(A)}}\backslash\rho^n_{s-F}(A))))&\\ \backslash(\overline{\rho^{+\infty}_{s-F}(A)} \cup \overline{\rho^{-\infty}_{s-F}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})}).&\end{aligned}$$

Then by [**2**, 1.5] we have that $\psi_{K(X)}(A)$ is dense in $\sigma_{K(X)}(A)\backslash(\overline{\rho^{+\infty}_{s-F}(A)}\cup\overline{\rho^{-\infty}_{s-F}(A)}\cup\overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})})$. Hence condition i) is satisfied.

We have thus proved that conditions i), ii) and iii) are equivalent.

Now we prove that the equivalent conditions i), ii) and iii) imply iv).

We remark that $\pi_{K(X)}(G)\in\mathbf{c}(G_{L(X)/K(X)})$ and $G=\pi^{-1}_{K(X)}(\pi_{K(X)}(G))$ for any $G \in \mathbf{c}(F(X))$ (see [**11**, (6.2.5)]). Consequently, for any $n \in \mathbf{Z}$, since $F_n(X) = \cup_{G\in\mathbf{c}(F_n(X))}G$ we have that $\rho^n_{s-F}(A) = \cup_{G\in\mathbf{c}(F_n(X))}\rho_{\pi_{K(X)}(G)}(\pi_{K(X)}(A))$. Thus, for any $\lambda \in \partial\overline{\rho^n_{s-F}(A)}$ and for any $\varepsilon > 0$, since $\lambda \in \overline{\cup_{G\in\mathbf{c}(F_n(X))}\rho_{\pi_{K(X)}(G)}(\pi_{K(X)}(A))}$ there exists $G_0 \in \mathbf{c}(F_n(X))$ such that $B_{\mathbf{C}}(\lambda,\varepsilon)\cap\rho_{\pi_{K(X)}(G_0)}(\pi_{K(X)}(A)) \neq \varnothing$. Moreover, since $\lambda \in \partial\overline{\rho^n_{s-F}(A)}$, we have also that $B_{\mathbf{C}}(\lambda,\varepsilon)\cap(\mathbf{C}\backslash\overline{\rho^n_{s-F}(A)}) \neq \varnothing$ and consequently $B_{\mathbf{C}}(\lambda,\varepsilon)\cap(\mathbf{C}\backslash\overline{\rho_{\pi_{K(X)}(G_0)}(\pi_{K(X)}(A))}) \neq \varnothing$. Hence $B_{\mathbf{C}}(\lambda,\varepsilon)\cap\partial\overline{\rho_{\pi_{K(X)}(G_0)}(\pi_{K(X)}(A))} \neq \varnothing$.

We have thus proved that $B_{\mathbf{C}}(\lambda,\varepsilon)$ intersects $\cup_{G\in\mathbf{c}(F_n(X))}\partial\overline{\rho_{\pi_{K(X)}(G)}(\pi_{K(X)}(A))}$ for any $\varepsilon > 0$. Therefore, $\lambda \in \overline{(\cup_{G\in\mathbf{c}(F_n(X))}\partial\overline{\rho_{\pi_{K(X)}(G)}(\pi_{K(X)}(A))})}$.

We have thus proved that $\overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})} \subset \overline{(\cup_{G\in\mathbf{c}(G_{L(X)/K(X)})}\partial\overline{\rho_G(\pi_{K(X)}(A))})}$.

We have already remarked that $\rho^{+\infty}_{s-F}(A)\cup\rho^{-\infty}_{s-F}(A) \subset S(\pi_{K(X)}(A))$.

Moreover, we have $\overline{\psi_{K(X)}(A)} = \overline{\psi(\pi_{K(X)}(A))} \subset \zeta(\pi_{K(X)}(A))$.

Since A satisfies i), it follows that

$\sigma(\pi_{K(X)}(A)) \subset$

$\overline{S(\pi_{K(X)}(A))} \cup \zeta(\pi_{K(X)}(A)) \cup \overline{(\cup_{G\in\mathbf{c}(G_{L(X)/K(X)})}\partial\overline{\rho_G(\pi_{K(X)}(A))})}$

and consequently condition iv) is satisfied.

The proof is thus complete. □

We recall that [**4**, 3.5] provides an example of a linear bounded operator on l_2 which satisfies conditions (9) and (10) and whose coset $\mathbf{x}$ in the Calkin algebra $L(l_2)/K(l_2)$ does not satisfy conditions (1) and (2). We remark that $\mathbf{x}$ satisfies the equivalent conditions (5) and (6) by Theorem 2.2.

Lemma 2.3. *Let X be a Hilbert space of Hilbert dimension $h \geq \aleph_0$. Then $\pi^{-1}_{K_\alpha(X)}(\overset{\circ}{S}_{L(X)/K_\alpha(X)}) = \{T \in L(X) : d(T) \neq d(T^*)$ and $\max\{d(T), d(T^*)\} \geq \alpha\}$ for any $\alpha \in \Delta_h$ with $\alpha \geq \aleph_0$.*

Proof. We have already recalled that $\overset{\circ}{S}_{L(X)} = \{T \in L(X) : d(T) \neq d(T^*)\}$. Consequently, $\pi_{K_\alpha(X)}(\{T \in L(X) : d(T) = d(T^*)\}) \subset \overline{G}_{L(X)/K_\alpha(X)}$. We also recall that any $T \in L(X)$ which satisfies $\max\{d(T), d(T^*)\} < \alpha$ is invertible modulo $K_\alpha(X)$ (see [**12**, 2.6]). Therefore, $\pi^{-1}_{K_\alpha(X)}(\overset{\circ}{S}_{L(X)/K_\alpha(X)}) \subset \{T \in L(X) : d(T) \neq d(T^*)$ and $\max\{d(T), d(T^*)\} \geq \alpha\} = \cup_{\alpha\leq\beta\in\Delta_h}\{T \in L(X) : \min\{d(T), d(T^*)\} < \beta \leq \max\{d(T), d(T^*)\}\}$.

From [**12**, 2.6] it follows also that, for any $\beta \in \Delta_h$ with $\beta \geq \alpha$, $\{T \in L(X) : \min\{d(T), d(T^*)\} < \beta \leq \max\{d(T), d(T^*)\}\} = \{T \in L(X) : T$ is either left or right invertible modulo $K_\beta(X)$ and is not invertible modulo $K_\beta(X)\}$, which is an open subset of $L(X)$ (see [**19**, (1.5.5)]).

Hence, $\pi_{K_\alpha(X)}(\{T \in L(X) : \min\{d(T), d(T^*)\} < \beta \leq \max\{d(T), d(T^*)\}\})$ is an open subset of $L(X)/K_\alpha(X)$ and does not intersect $G_{L(X)/K_\alpha(X)}$ for any $\beta \in \Delta_h$ with $\beta \geq \alpha$. It follows that $\pi_{K_\alpha(X)}(\{T \in L(X) : d(T) \neq d(T^*)$ and $\max\{d(T), d(T^*)\} \geq \alpha\}) \subset \overset{\circ}{S}_{L(X)/K_\alpha(X)}$.

We have thus proved that $\pi^{-1}_{K_\alpha(X)}(\overset{\circ}{S}_{L(X)/K_\alpha(X)}) = \{T \in L(X) : d(T) \neq d(T^*)$ and $\max\{d(T), d(T^*)\} \geq \alpha\}$. □

Let X be a complex infinite-dimensional Banach space such that $G_{L(X)}$ is connected. We recall then that $\mathbf{c}(G_{L(X)/K(X)}) = \{\pi_{K(X)}(F_n(X))\}_{n\in\mathbf{Z}:F_n(X)\neq\varnothing}$ and $F_n(X) = \pi^{-1}_{K(X)}(\pi_{K(X)}(F_n(X)))$ for any $n \in \mathbf{Z}$ (see [**11**, (6.2.5) and corollary of (6.2.6)]).

Now let X be a complex Hilbert space of Hilbert dimension $h \geq \aleph_0$. Since $\mathbf{J}_{L(X)} = \{K_\beta(X) : \beta \in \Delta_h\}$, it follows that $\mathbf{J}_{L(X)/K_\alpha(X)} = \{K_\beta(X)/K_\alpha(X) : \beta \in \Delta_h, \beta \geq \alpha\}$ for any $\alpha \in \Delta_h$ with $\alpha \geq \aleph_0$. Hence, $\mathbf{J}_{L(X)/K_\alpha(X)}$ is well ordered by inclusion for any $\alpha \in \Delta_h$ with $\alpha \geq \aleph_0$. We also recall that $G_{L(X)/K_\alpha(X)}$ is connected for any $\alpha \in \Delta_h$ with $\alpha > \aleph_0$ (see [**7**, Corollary 3]). Hence $H^J_{L(X)/K_\alpha(X)} = \varnothing$ for any $J \in \mathbf{J}_{L(X)/K_\alpha(X)}$ and for any $\alpha \in \Delta_h$ with $\alpha > \aleph_0$. Moreover, for any $\alpha \in \Delta_h$ with $\alpha > \aleph_0$, since $G_{(L(X)/K(X))/(K_\alpha(X)/K(X))}$ is connected we have that $\pi^{-1}_{K_\alpha(X)/K(X)}(G_{L(X)/K(X))/(K_\alpha(X)/K(X))}$ is connected by [**11**, (6.2.5)]. Therefore, $H^J_{L(X)/K(X)} = \varnothing$ for any nonzero $J \in \mathbf{J}_{L(X)/K(X)}$.

We set $\Omega_h = \{\alpha \in \Delta_h : \alpha > \aleph_0$ and α is not a limit cardinal number$\}$. Since $G_{L(X)}$ is connected, the two results below follow from Lemma 2.3 and from [**3**, 2.2].

Proposition 2.4. *Let X be a complex Hilbert space of Hilbert dimension $h \geq \aleph_0$, and let $A \in L(X)$. Then the following conditions are equivalent*:

i) $\pi_{K(X)}(A)$ *satisfies condition* (3);

ii) $\pi_{K(X)}(A)$ *satisfies condition* (4);

iii) $r(\pi_{K(X)}(A)) = \max\{\sup\{|\lambda| : \lambda \in \cup_{n\in\mathbf{Z}\setminus\{0\}}\rho^n_{s-F}(A)\}, \sup\{|\lambda| : \lambda \in \mathbf{C}, d(\lambda I_X - A) \neq d(\bar{\lambda} I_X - A^*)$ *and* $\max\{d(\lambda I_X - A), d(\bar{\lambda} I_X - A^*)\} \geq \aleph_0\}, \delta_{K(X)}(A), \sup\{\delta_{K_\alpha(X)}(A) : \alpha \in \Omega_h\}\}$.

Proposition 2.5. *Let X be a complex Hilbert space of Hilbert dimension $h > \aleph_0$, let $\alpha \in \Delta_h$, $\alpha > \aleph_0$, and let $A \in L(X)$. Then the following conditions are equivalent*:

i) $\pi_{K_\alpha(X)}(A)$ *satisfies condition* (1);

ii) $\pi_{K_\alpha(X)}(A)$ *satisfies condition* (3);

iii) $\pi_{K_\alpha(X)}(A)$ *satisfies condition* (4);

iv) $r(\pi_{K_\alpha(X)}(A)) = \max\{\sup\{|\lambda| : \lambda \in \mathbf{C}, d(\lambda I_X - A) \neq d(\bar{\lambda} I_X - A^*)$

and $\max\{d(\lambda I_X - A), d(\bar{\lambda}_X - A^*)\} \geq \alpha\}$, $\sup\{\delta_{K_\beta(X)}(A) : \beta \in \Omega_h, \beta \geq \alpha\}\}$.

The following two results are consequences of the remarks preceding Proposition 2.4, of Lemma 2.3 and of [**5**, 2.7].

Proposition 2.6. *Let X be a complex Hilbert space of Hilbert dimension $h \geq \aleph_0$, and let $A \in L(X)$. Then $\pi_{K(X)}(A)$ satisfies the equivalent conditions* (5) *and* (6) *if and only if* $\sigma_{K(X)}(A) = \overline{\{\lambda \in \mathbf{C} : d(\lambda I_X - A) \neq d(\bar{\lambda} I_X - A^*) \text{ and } \max\{d(\lambda I_X - A), d(\bar{\lambda} I_X - A^*)\} \geq \aleph_0\}} \cup \overline{\psi_{K(X)}(A)} \cup (\cup_{n \in \mathbf{Z}} \partial \overline{\rho^n_{s-F}(A)})$.

We remark that for any nonseparable Hilbert space X the operator $A \in L(X)$ introduced in [**3**, 2.3] does not satisfy conditions (9) and (10) (indeed, it does not satisfy (7) and (8) either). Nevertheless, A satisfies the condition of Propositon 2.6, so that σ is continuous at $\pi_{K(X)}(A)$. Hence conditions (9) and (10) (respectively, (7) and (8)) are not necessary for continuity of σ (respectively, r) at $\pi_{K(X)}(A)$ in the case of a nonseparable Hilbert space.

Proposition 2.7. *Let X be a complex Hilbert space of Hilbert dimension $h > \aleph_0$, let $\alpha \in \Delta_h$, $\alpha > \aleph_0$, and let $A \in L(X)$. Then the following conditions are equivalent*:

i) $\pi_{K_\alpha(X)}(A)$ *satisfies condition* (2);

ii) $\pi_{K_\alpha(X)}(A)$ *satisfies the equivalent conditions* (5) *and* (6);

iii) $\sigma_{K_\alpha(X)}(A) = \overline{\{\lambda \in \mathbf{C} : d(\lambda I_X - A) \neq d(\bar{\lambda} I_X - A^*) \text{ and } \max\{d(\lambda I_X - A), d(\bar{\lambda} I_X - A^*)\} \geq \alpha\}} \cup \overline{\psi_{K_\alpha(X)}(A)}$.

Now let $X \in \{c_0\} \cup \{l_p\}_{p \in [1, +\infty)}$. We recall that the only proper closed two-sided ideal of $L(X)/K(X)$ is the null one, as $\mathbf{J}_{L(X)} = \{\{0\}, K(X)\}$. Then, since $G_{L(X)}$ is connected and $\pi^{-1}_{K(X)}(\overset{\circ}{S}_{L(X)/K(X)}) = L(X) \backslash \overline{F(X)}$, the following result holds.

Proposition 2.8. *Let* $X \in \{c_0\} \cup \{l_p\}_{p\in[1,+\infty)}$ *and let* $A \in L(X)$. *Then the following conditions are equivalent*:

i) $\pi_{K(X)}(A)$ *satisfies condition* (3);

ii) $\pi_{K(X)}(A)$ *satisfies condition* (4);

iii) $r(\pi_{K(X)}(A)) = \max\{\sup\{|\lambda| : \lambda \in \cup_{n\in\mathbf{Z}\setminus\{0\}}\rho^n_{s-F}(A)\}, \sup\{|\lambda| : \lambda \in \mathbf{C}, \lambda I_X - A \in L(X)\setminus\overline{F(X)}\}, \delta_{K(X)}(A)\}$.

The result below follows from the remarks preceding Propositions 2.4 and 2.8, and from [**5**, 2.7].

Proposition 2.9. *Let* $X \in \{c_0\} \cup \{l_p\}_{p\in[1,+\infty)}$ *and let* $A \in L(X)$. *Then* $\pi_{K(X)}(A)$ *satisfies the equivalent conditions* (5) *and* (6) *if and only if*

$$\sigma_{K(X)}(A) = \overline{\{\lambda \in \mathbf{C} : \lambda I_X - A \in L(X)\setminus\overline{F(X)}\}} \cup \overline{\psi_{K(X)}(A)} \cup \overline{(\cup_{n\in\mathbf{Z}}\partial\overline{\rho^n_{s-F}(A)})}.$$

Finally, we remark that the operator $I_{l_{p_1}\times l_{p_2}} + T$ (where $T \in L(l_{p_1} \times l_{p_2})$ is defined in Example 1.7) does not satisfy conditions (7) and (8) (so that, in particular, it does not satisfy conditions (9) and (10) either), and yet $\pi_{K(l_{p_1}\times l_{p_2})}(I_{l_{p_1}\times l_{p_2}} + T)$ satisfies the equivalent conditions (5) and (6) (and thus also conditions (3) and (4)) in view of what we proved in Example 1.8 and of [**11**, (6.2.5)]. Notice that the operator $I_{l_{p_1}\times l_{p_2}} + A$ (where $A \in L(l_{p_1} \times l_{p_2})$ is defined in Example 1.7) shares the properties above with T, as $A - T \in K(l_{p_1} \times l_{p_2})$.

REFERENCES

1. C. Apostol, L.A. Fialkow, D.A. Herrero and D. Voiculescu, *Approximation of Hilbert space operators*, Volume II, Pitman, 1984.

2. L. Burlando, *Continuity of spectrum and spectral radius in algebras of operators*, Ann. Fac. Sci. Toulouse Math. (5) **9** (1988), 5–54.

3. ———, *On continuity of the spectral radius function in Banach algebras*, Ann. Mat. Pura Appl. (4) **156** (1990), 357–380.

4. ———, *On continuity of the spectrum function in Banach algebras*, Riv. Mat. Pura Appl. **8** (1991), 131–152.

5. ———, *On continuity of the spectrum function in Banach algebras with good ideal structure*, Riv. Mat. Pura Appl. **9** (1991), 7–21.

6. ———, *Spectral continuity* (to appear in Atti Sem. Mat. Fis. Univ. Modena).

7. L.A. Coburn and A. Lebow, *Components of invertible elements in quotient algebras of operators*, Trans. Amer. Math. Soc. **130** (1968), 359–365.

8. J.B. Conway and B.B. Morrel, *Operators that are points of spectral continuity*, Integral Equations Operator Theory **2** (1979), 174–198.

9. ———, *Behaviour of the spectrum under small perturbations*, Proc. Roy. Irish Acad. Sect. A **81** (1981), 55–63.

10. ———, *Operators that are points of spectral continuity*, II, Integral Equations Operator Theory **4** (1981), 459–503.

11. S.R. Caradus, W.E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Dekker, 1974.

12. G. Edgar, J. Ernest and S.G. Lee, *Weighing operator spectra*, Indiana Univ. Math. J. **21** (1971), 61–80.

13. M. Gonzalez, *A perturbation result for generalised Fredholm operators in the boundary of the group of invertible operators*, Proc. Roy. Irish Acad. Sect. A **86** (1986), 123–126.

14. P.R. Halmos, *A Hilbert space problem book*, Van Nostrand, 1967.

15. E. Luft, *The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space*, Czechoslovak Math. J. **18** (1968), 595–605.

16. B.S. Mityagin, *The homotopy structure of the linear group of a Banach space*, Russian Math. Surveys **25** (1970), No. 5, 59–103.

17. M. Ó Searcóid, *A contribution to the solution of the compact correction problem for operators on a Banach space*, Glasgow Math. J. **31** (1989), 219–229.

18. A. Pietsch, *Operator ideals*, North-Holland, 1980.

19. C.E. Rickart, *General theory of Banach algebras*, Van Nostrand, 1960.

Dipartimento di Matematica dell'Università di Genova, Via L. B. Alberti, 4, 16132 Genova, Italy