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NOTES ON ANALYTIC FEYNMAN INTEGRABLE FUNCTIONALS

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ABSTRACT. In this paper we establish the analytic Feynman integrability (and the Fresnel integrability) for a very large class of functionals on multi-parameter Wiener space. Many previous results in the literature, including results by Chang, Johnson, Park and Skoug, then follow from our results as corollaries.

1. Introduction. In a recent expository essay [**9**], Nelson calls attention to some functionals on Wiener space which were discussed in the book of Feynman and Hibbs [**6**] and in Feynman's original paper [**5**]. These functionals have the form

$$F(x) = \exp\left\{ \int_0^T \int_0^T W(s_1, s_2; x(s_1), x(s_2))\, ds_1\, ds_2 \right\}. \tag{1.1}$$

In [**8**], Johnson and Skoug examine the Feynman integrability of functionals on Wiener space of the form

$$F(\vec{x}) = \exp\left\{ -\int_a^b \langle A(s)\vec{x}(s), \vec{x}(s)\rangle\, ds \right\}. \tag{1.2}$$

Since then, Chang, Johnson and Skoug [**3**], and Park and Skoug [**10**] extended the theory to include functionals of the form

$$F(x) = \exp\left\{ -\int_0^T \cdots \int_0^T \langle A(s_1, \dots, s_n)(x(s_1), \dots, x(s_n)), (x(s_1), \dots, x(s_n))\rangle\, ds_1 \cdots ds_n \right\}. \tag{1.3}$$

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Throughout this paper, we consider the analytic Feynman (and Fresnel) integrability of certain generalized functionals on multi-parameter Wiener space and formulate the counterparts of the results in [**3, 8, 10**] for multi-parameter Wiener space.

Remark 1.1. It is interesting to note that, while the functionals considered in [**3, 8, 10**] only involve the one-parameter Wiener process, the functionals we consider involve multi-parameter Wiener process. However, the proofs in [**3, 10**], as well as our proofs, involve various multi-parameter Wiener processes in a most natural way.

2. Preliminaries. Let $C_N \equiv C_N(P)$ denote N-parameter Wiener space, that is, the space of real valued continuous functions $x(s_1,\dots,s_N)$ on $P = [0,T]^N$ such that $x(0,s_2,\dots,s_N) = x(s_1,0,s_3,\dots,s_N) = \cdots = x(s_1,\dots,s_{N-1},0) = 0$ for all $(s_1,\dots,s_N)$ in P, and let m_N be Wiener measure on C_N. Let ν be a positive integer, let $C_N^\nu \equiv \times_1^\nu C_N$, and let $m_N^\nu \equiv \times_1^\nu m_N$. A subset E of C_N^ν is said to be scale-invariant measurable provided ρE is Wiener measurable for every $\rho > 0$. For a rather detailed discussion of scale-invariant measurability, see [**2, 3, 8, 12**].

Definition 2.1. Let F be a complex valued functional on C_N^ν which is s-almost everywhere defined and scale-invariant measurable, and such that the Wiener integral

$$J(\lambda) = \int_{C_N^\nu} F(\lambda^{-1/2}\vec{x})\, dm_N^\nu(\vec{x})$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}^+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the *analytic Wiener integral of F over C_N^ν with parameter* λ, and for $\lambda \in \mathbf{C}^+$ we write

$$\int_{C_N^\nu}^{\text{anw}\ \lambda} F(\vec{x}) dm_N^\nu(\vec{x}) = J^*(\lambda). \tag{2.1}$$

Definition 2.2. Let q be a nonzero real parameter, and let F be a functional whose analytic Wiener integral (2.1) exists for $\lambda \in \mathbf{C}^+$. If the limit (2.2) exists, we call it the *analytic Feynman integral of F over C_N^ν with parameter q*, and we write

$$\int_{C_N^\nu}^{\mathrm{anf}\ q} F(\vec{x})dm_N^\nu(\vec{x}) = \lim_{\lambda\to -iq}\int_{C_N^\nu}^{\mathrm{anw}\ \lambda} F(\vec{x})dm_N^\nu(\vec{x}) \tag{2.2}$$

where λ approaches $-iq$ through $\mathbf{C}^+$.

Notation. We introduce the notation $f([x_k]_n)$ for the function $f(x_1,\dots,x_n)$ of n variables, $f([x_k]_n;[y_k]_m)$ for the function $f(x_1,\dots,x_n;y_1,\dots,y_m)$ of $n+m$ variables.

Let $M_N(\nu)\equiv M_N(L_2^\nu(P))$ be the collection of complex valued countably additive measures on $\mathcal{B}(L_2^\nu)$, the Borel class of $L_2^\nu(P)$. Then $M_N(\nu)$ is a Banach algebra under the total variation norm where the convolution is taken as the multiplication. Let $S_N(\nu)$ be the space of functionals on C_N^ν expressible in the form

$$F(\vec{x}) = \int_{L_2^\nu} \exp\left\{ i\sum_{j=1}^{\nu}\int_p v_j([s_k]_N)\widetilde{dx}_j([s_k]_N)\right\} d\mu(\vec{v}) \tag{2.3}$$

for s-a.e. $\vec{x}\in C_N^\nu$ and $\mu\in M_N(\nu)$ where $\int_p v([s_k]_N)\widetilde{dx}([s_k]_N)$ means the Paley-Wiener-Zygmund integral [**2, 3, 10, 11, 12**]. The following theorem is a well-known result whose proof is similar to that of Theorems 2.3 and 5.1 in [**2**].

Theorem 2.3. *$S_N(\nu)$ is a Banach algebra, and every element F in $S_N(\nu)$ is analytic Feynman integrable, and for nonzero real q,*

$$\int_{C_N^\nu}^{\mathrm{anf}\ q} F(\vec{x})dm_N^\nu(\vec{x}) = \int_{L_2^\nu} \exp\left\{(1/2qi)\sum_{j=1}^{\nu}||v_j||_2^2\right\} d\mu(\vec{v}). \tag{2.4}$$

Next we give the necessary information for our discussion of the Banach algebra $\mathcal{F}(H)$ of Fresnel integrable functions. The fundamental work on the space $\mathcal{F}(H)$ was done by Albeverio and Hoegh-Krohn [**1**].

Let H_N be the set of all functions $r : P \to \mathbf{R}$ for which there exists v in $L_2(P)$ such that

$$r([s_k]_N) = \int_{s_1}^{T} \cdots \int_{s_N}^{T} v([t_k]_N)\, dt_1 \cdots dt_N$$

for all $(s_1, \dots, s_N)$ in P. The inner product on H_N is defined by

$$(r_1, r_2) = \int_p [D^* r_1([s_k]_N)][D^* r_2([s_k]_N)]\, ds_1 \cdots ds_N \tag{2.5}$$

where $D^*(\cdot) = \partial^N(\cdot)/\partial s_1 \dots \partial s_N$. Then H_N, equipped with this inner product, is a real separable Hilbert space. Let $H_N^\nu \equiv \times_1^\nu H_N$ denote the space of functions $\vec{r}$ on P to $\mathbf{R}^\nu$, each of whose components belongs to H_N, and let $M(H_N^\nu)$ be the collection of complex valued countably additive measures on $\mathcal{B}(H_N^\nu)$, the Borel class of H_N^ν. Given μ in $M(H_N^\nu)$, $\hat{\mu}$ is defined on H_N^ν by

$$\hat{\mu}(\vec{r}) = \int_{H_N^\nu} \exp\{i(\vec{r}, \vec{h})\}\, d\mu(\vec{h}).$$

Let $\mathcal{F}(H_N^\nu) = \{\hat{\mu} : \mu \in M(H_N^\nu)\}$. Then, letting $||\hat{\mu}|| = ||\mu||$, we know, as in [**1**], that $\mathcal{F}(H_N^\nu)$ is a Banach algebra, and the Fresnel integral $F(\hat{\mu})$ is defined for $\hat{\mu}$ in $\mathcal{F}(H_N^\nu)$ by

$$\mathcal{F}(\hat{\mu}) = \int_{H_N^\nu} \exp\left\{(-1/2)\sum_{j=1}^{\nu} ||h_j||^2\right\} d\mu(\vec{h}).$$

Remark 2.4. Albeverio and Hoegh-Krohn's space $\mathcal{F}(H)$ of Fresnel integrable functions consists of Fourier transforms of finite Borel measures on H [**1**]. Also the spaces $\mathcal{F}(H)$ and S are isometrically as Banach algebras which was shown by Johnson [**7**]. Similarly, we know that the Banach algebra $\mathcal{F}(H_N^\nu)$ is isometrically isomorphic to the Banach algebra $S_N(\nu)$.

3. Feynman integrabilities of certain functionals. In this section we discuss the analytic Feynman and Fresnel integrability of certain generalized functionals on N-parameter Wiener space and formulate the counterparts for this Wiener space containing the important results in [**3, 8, 10**].

Theorem 3.1. *Let m be a positive integer, let $n = mN$, and let $P = [0,T]^N$, $Q = [0,T]^n$, and η a finite Borel measure on Q. Let $\varphi_j : P \to L_2(P)$ be Borel measurable for $j = 1, \dots, m$, and let $\theta : Q \times \mathbf{R}^{m\nu} \to \mathbf{C}$ be such that, for all $\vec{s} = (s_1, \dots, s_n) \in Q$,*

$$\theta([s_k]_n; [\vec{U}_k]_m) = \int_{\mathbf{R}^{m\nu}} \exp\left\{ i \sum_{j=1}^{m} \langle \vec{U}_j, \vec{V}_j \rangle \right\} d\sigma_{\vec{s}}([\vec{V}_k]_m) \tag{3.1}$$

where $\sigma_{\vec{s}} \in M(\mathbf{R}^{m\nu})$, the measure algebra of $\mathbf{R}^{m\nu}$, $\vec{U}_j = (u_{j1}, \dots, u_{j\nu}) \in \mathbf{R}^\nu$,

(3.2)
for every $E \in \mathcal{B}(\mathbf{R}^{m\nu}), \sigma_{\vec{s}}(E)$ is a Borel measurable function of $\vec{s}$,

and

$$||\sigma_{\vec{s}}|| \in L_1(Q, \mathcal{B}(Q), \eta). \tag{3.3}$$

Then the function $F : C_N^\nu(P) \to \mathbf{C}$ defined by

$$\begin{aligned} F(\vec{x}) = \int_Q \theta\Bigg([s_k]_n; \bigg(\int_P \varphi_1([s_k]_N)([t_k]_N)\, \widetilde{dx}_j([t_k]_N) \bigg)_{j=1}^{\nu}, \\ \dots, \bigg(\int_P \varphi_m([s_{(m-1)N+k}]_N)([t_k]_N)\, \widetilde{dx}_j([t_k]_N) \bigg)_{j=1}^{\nu} \Bigg) d\eta(\vec{s}) \end{aligned} \tag{3.4}$$

belongs to the Banach algebra $S_N(\nu)$ and hence is analytic Feynman integrable.

Proof. We first define a Borel measure μ on $Q \times \mathbf{R}^{m\nu}$ by $\mu(E) = \int_Q \sigma_{\vec{s}}(E^{(\vec{s})})\, d\eta(\vec{s})$ for $E \in \mathcal{B}(Q \times \mathbf{R}^{m\nu})$. Then μ is an element of $M(Q \times \mathbf{R}^{m\nu})$. Now let $\Phi \equiv (\Phi_1, \dots, \Phi_\nu) : Q \times \mathbf{R}^{m\nu} \to L_2^\nu(P)$ be defined by

$$\begin{aligned} \Phi_j([t_k]_N) &\equiv \Phi_j([s_k]_n; [\vec{V}_k]_m)([t_k]_N) \\ &= \sum_{i=1}^{m} v_{ij} \varphi_i([s_{(i-1)N+k}]_N)([t_k]_N) \end{aligned}$$

for $j = 1, \dots, \nu$, and let $\sigma = \mu \circ \Phi^{-1}$. Then σ belongs to $M_N(\nu)$ and, for $\rho > 0$, it follows from the change of variable theorem and the unsymmetric Fubini theorem that, for a.e. $\vec{x}$ in $C_N^\nu(P)$,

$$\begin{aligned} F(\rho\vec{x}) &= \int_Q \theta\bigg([s_k]_n : \bigg(\rho\int_P \varphi_1([s_k]_N)([t_k]_N)\widetilde{dx}_j([t_k]_N)\bigg)_{j=1}^{\nu}, \dots, \\ &\qquad \bigg(\rho\int_P \varphi_m([s_{(m-1)N+k}]_N)([t_k]_N)\widetilde{dx}_j([t_k]_N)\bigg)_{j=1}^{\nu}\bigg)\, d\eta(\vec{s}) \\ &= \int_Q \bigg[\int_{\mathbf{R}^{m\nu}} \exp\bigg\{i\rho\sum_{i=1}^{m}\sum_{j=1}^{\nu} v_{ij}\int_P \varphi_i([s_{(i-1)N+k}]_N)([t_k]_N) \\ &\qquad \widetilde{dx}_j([t_k]_N)\bigg\}\, d\sigma_{\vec{s}}([\vec{V}_k]_N)\bigg]\, d\eta(\vec{s}) \\ &= \int_{Q\times\mathbf{R}^{m\nu}} \exp\bigg\{i\rho\sum_{j=1}^{\nu}\int_P \varphi_j([t_k]_N)\widetilde{dx}_j([t_k]_N)\bigg\}\, d\mu([s_k]_n; [\vec{V}_k]_m) \\ &= \int_{L_2^\nu(P)} \exp\bigg\{i\rho\sum_{j=1}^{\nu}\int_P u_j([t_k]_N)\widetilde{dx}_j([t_k]_N)\bigg\}\, d\sigma(\vec{u}). \end{aligned}$$

Thus the function F is in $S_N(\nu)$, which completes the proof of Theorem 3.1. □

The above theorem is a generalization of Theorem 1 in [**4**]. Moreover, this theorem insures that various functional on $C_N^\nu(P)$ are in the Banach algebra $S_N(\nu)$ which is an extension of the Banach algebra S introduced by Cameron and Storvick [**2**].

Next we state a stochastic integration formula established by Park and Skoug (see [**11**, Corollary 2.2] or [**12**, Corollary 2.2]). This formula, which follows from a very general Fubini theorem by Park and Skoug [**11**, Theorem 2], plays a major role in the proof of our main results.

Theorem 3.2. *Let $N \in \{1, 2, \dots, n\}$, $P = [0,T]^N$, $Q = [0,T]^n$, and $v \in L_2(Q)$. Then for a.e. $(x, y) \in C_N(P) \times C_n(Q)$ we have that*

$$\begin{aligned} &\int_Q v([s_k]_n)x([s_{i_k}]_N)\widetilde{dy}([s_k]_N) \\ &\qquad = \int_P \bigg(\int_{E_N^*(s)} v([t_k]_n)\widetilde{dy}([t_k]_n)\bigg)\widetilde{dx}([s_{i_k}]_N) \end{aligned}$$

where $E_N^*(s) = E_N([s_{i_k}]_N)$ *is obtained from* $Q = [0,T]^n$ *by replacing all* i_k*-th factors by* $[s_{i_k}, T]$ *for* $k = 1, 2, \dots, N$.

Theorem 3.3. *Let* m *be a positive integer, let* $n = mN$, *and let* $P = [0,T]^N$ *and* $Q = [0,T]^n$. *Assume that for s-almost everywhere* $\vec{x}$ *in* $C_N^\nu(P)$

(3.5)

$$F(\vec{x}) = \exp\bigg\{ -\int_0^T \cdots \int_0^T \langle A([s_k]_n)(\vec{x}([s_k]_N), \dots, \vec{x}([s_{(m-1)N+k}]_N)), (\vec{x}([s_k]_N), \dots, \vec{x}([s_{(m-1)N+k}]_N))\rangle \, ds_1 \cdots ds_n \bigg\}$$

where $\{A([s_k]_n) = (a_{ij}([s_k]_n)) : (s_1, \dots, s_n) \in Q\}$ *is a commutative family of* $\nu m \times \nu m$ *real, symmetric, nonnegative definite matrices such that the eigenvalues* $p_1([s_k]_n), \dots, p_{\nu m}([s_k]_n)$ *are each elements of* $L_1(Q)$. *Then the functional* F *is in the Banach algebra* $S_N(\nu)$ *and hence is analytic Feynman integrable.*

Proof. Let $B = (b_{ij})$ be a $\nu m \times \nu m$ orthogonal matrix such that $BA([s_k]_n)B^{-1} = P([s_k]_n)$ throughout Q where $P([s_k]_n)$ is a $\nu m \times \nu m$ diagonal matrix with nonnegative entries $p_1([s_k]_n), \dots, p_{\nu m}([s_k]_n)$, the eigenvalues of $A([s_k]_n)$.

Let $\rho > 0$ be given. Then for a.e. $\vec{x} \in C_N^\nu(P)$, we obtain that

$$\begin{aligned} F(\rho\vec{x}) &= \exp\bigg\{ -\rho^2 \int_0^T \cdots \int_0^T \langle PB(\vec{x}([s_k]_N), \dots, \vec{x}([s_{(m-1)N+k}]_N)), B(\vec{x}([s_k]_N), \dots, \vec{x}([s_{(m-1)N+k}]_N))\rangle \, ds_1 \cdots ds_n \bigg\} \\ &= \exp\bigg\{ -\rho^2 \int_0^T \cdots \int_0^T \sum_{j=1}^{\nu m} p_j([s_k]_n) \bigg[\sum_{i=1}^{\nu m} b_{ji} x_{1+(i-1)\,\mathrm{mod}(\nu)}([s_{N[\frac{i-1}{\nu}]+k}]_N) \bigg]^2 ds_1 \cdots ds_n \bigg\} \end{aligned}$$

$$= \int_{C_n(Q)} \cdots \int_{C_n(Q)} \exp\Big\{ i\rho\sqrt{2} \sum_{j=1}^{\nu m} \sum_{i=1}^{\nu m} \cdot \int_P \Big[\int_{E_N([s_{N[\frac{i-1}{\nu}]+k}]_N)} b_{ji} \sqrt{p_j([t_k]_n)} \widetilde{dy}_j([t_k]_n) \Big] \cdot \widetilde{dx}_{1+(i-1)\bmod(\nu)}([s_{N[\frac{i-1}{\nu}]+k}]_N) \Big\} dm_n(y_1)\cdots dm_n(y_{\nu m})$$

where the last equality above follows from the Fourier transformation formula, Paley-Wiener-Zygmund theorem, and Theorem 3.2.

Next we define $T \equiv (T_1, \dots, T_\nu) : C_n(Q) \times \cdots \times C_n(Q) \to L_2^\nu(P)$ by

$$T_\alpha([y_k]_{\nu m})([s_k]_N) = \sqrt{2} \sum_{l=0}^{m-1} \sum_{j=1}^{\nu m} \int_{E_l^*([s_k]_N)} b_{j(\nu l+\alpha)} \sqrt{p_j([t_k]_n)} \widetilde{dy}_j([t_k]_n)$$

for $\alpha = 1, \dots, \nu$, where $E_l^*([s_k]_N) = E_N([s_{Nl+k}]_N)$. Then each T_α is in $L_2(P)$ and $\mu = [m_n]^{\nu m} o T^{-1}$ is an element of $M_N(\nu)$, and, for almost everywhere $\vec{x} \in C_N^\nu(P)$, we have, using the change of variable theorem, that

$$F(\rho\vec{x}) = \int_{L_2^\nu(P)} \exp\Big\{ i\rho \sum_{j=1}^{\nu} \int_P v_j([s_k]_N) \widetilde{dx}_j([s_k]_N) \Big\} d\mu(\vec{v}).$$

Thus the functional F is an element of $S_N(\nu)$ which completes the proof of Theorem 3.3. □

Corollary 3.4. *Under the hypotheses of Theorem* 3.1 *and Theorem* 3.3*, the product of functionals* (3.4) *and* (3.5) *also belongs to the Banach algebra* $S_N(\nu)$ *and hence is analytic Feynman integrable.*

Remark 3.5. The Theorem and Corollaries in [**8**, Section 3] now follow from Theorem 3.3 and Corollary 3.4 by letting $n = 1$. Also, Theorem 4.1 and Corollary 4.5 in [**3**] follow from Theorem 3.3 and Corollary 3.4 above by letting $n = 2$ and $N = 1$. Moreover, Theorem 3.1, Theorem 4.1, and Corollary 3.4 in [**10**] now follow by letting $N = 1$ in Theorem 3.3 and Corollary 3.4 above.

Next we consider the Fresnel integrability of certain functionals on H_N^ν. Recall that we briefly described the space $F(H_N^\nu)$ of Fresnel integrable functions in Section 2. Using Theorem 3.1 and Theorem 3.3 and the isometrically isomorphic property of $S_N(\nu)$ and $F(H_N^\nu)$, we obtain the following theorems.

Theorem 3.6. *Let η and θ be as in Theorem 3.1. For $\vec{r}$ in H_N^ν, let*

$$F(\vec{r}) = \exp\bigg\{\int_0^T \cdots \int_0^T \theta([s_k]_n; \Delta_1 \cdots \Delta_N \vec{r}([s_k]_N), \dots, \Delta_1 \cdots \Delta_N \vec{r}([s_{(m-1)N+k}]_N))\bigg\}\, d\eta(\vec{s}) \tag{3.6}$$

where $\Delta_i r([s_k]_N) = r(s_1, \dots, s_N) - r(s_1, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_N)$ for $i = 1, \dots, N$. Then the function F belongs to the Banach algebra $F(H_N^\nu)$.

Theorem 3.7. *For each $\vec{r}$ in H_N^ν, let*

$$F(\vec{r}) = \exp\bigg\{-\int_0^T \cdots \int_0^T \langle A([s_k]_n)(\Delta_1 \cdots \Delta_N \vec{r}([s_k]_N), \dots, \Delta_1 \cdots \Delta_N \vec{r}([s_{(m-1)N+k}]_N)), (\Delta_1 \cdots \Delta_N \vec{r}([s_k]_N), \dots, \Delta_1 \cdots \Delta_N \vec{r}([s_{(m-1)N+k}]))\rangle\, ds_1 \cdots ds_n\bigg\} \tag{3.7}$$

where $\Delta_i r([s_k]_N)$ is as in Theorem 3.6 *and $\{A([s_k]_n)\}$ is as in Theorem* 3.3*. Then the function F is in the Banach algebra $F(H_N^\nu)$, that is, F is Fresnel integrable on H_N^ν.*

Remark 3.8. Corollaries 4.6 and 4.7 in [**3**] now follow from Theorems 3.6 and 3.7 above by letting $n = 2$ and $N = 1$. Moreover, Theorem 5.1 and Corollary 5.1 in [**10**] follow by letting $N = 1$ in Theorems 3.6 and 3.7 above.

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