ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 3, Summer 1993

UNIQUENESS OF BEST APPROXIMATION WITH COEFFICIENT CONSTRAINTS

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ABSTRACT. For given $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, with $-\infty \le \alpha_i < \beta_i \le \infty$, $i = 1, \dots, n$, and continuous functions $u_1, \dots, u_n$, set

$$U(\alpha, \beta) = \left\{ u = \sum_{i=1}^{n} a_i u_i \mid \alpha_i \le a_i \le \beta_i,\ i = 1, \dots, n \right\}.$$

This paper is concerned with the uniqueness and strong uniqueness of best approximation of continuous functions from $U(\alpha, \beta)$. We improve some results of [**5**] and construct an example to answer a question raised in [**5**].

1. Introduction. Let B denote a compact Hausdorff space containing at least $n+1$ points, and let $C(B)$ be the normed linear space of real-valued continuous functions on B with the uniform norm:

$$||f|| = \max_{x \in B} |f(*)|.$$

Let $U_n = \operatorname{span}\{u_1, \dots, u_n\}$, a subspace of $C(B)$. For given $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ with $-\infty \le \alpha_i < \beta_i \le +\infty$, $i = 1, \dots, n$, set

$$U(\alpha, \beta) = \left\{ u = \sum_{i=1}^{n} a_i u_i \mid \alpha_i \le a_i \le \beta_i, \quad i = 1, \dots, n \right\}.$$

In [**5**] the problem of best approximation of functions in $C(B)$ from $U(\alpha, \beta)$ was studied and the following theorems were proved.

Theorem 1.1. *Let $f \in C(B) \backslash U(\alpha, \beta)$. Then $\tilde{u} = \sum_{1 \le i \le n} \tilde{a}_i u_i$ is a best approximant to f from $U(\alpha, \beta)$ if and only if there exist distinct*

Received by the editors on March 28, 1991, and in revised form on November 13, 1991.
AMS *Subject Classifications.* 41A52, 41A29, 41A50.
Key words and phrases. Unique best approximation, strong uniqueness, coefficient constraints, Haar system.

points $\{x_i, i = 1, \dots, r\}$ *and nonzero numbers* $\{c_i, i = 1, \dots, r\}$ *with* $1 \le r \le n+1$, *satisfying*

$$(f - \tilde{u})(x_i) = \operatorname{sgn}(c_i)||f - \tilde{u}||, \ \ i = 1, \dots, r$$

and

$$\sum_{j=1}^{r} c_j u_i(x_j) \begin{cases} \ge 0 & \text{if } \tilde{a}_i = \beta_i, \\ = 0 & \text{if } \alpha_i < \tilde{a}_i < \beta_i \\ \le 0 & \text{if } \tilde{a}_i = \alpha_i, \end{cases} \qquad i = 1, \dots, n.$$

Theorem 1.2. *Let* $N = \{i \mid \alpha_i = -\infty, \beta_i = \infty\}$. *If* $\{u_{i_1}, \dots, u_{i_k}\}$ *is a Haar system for every choice of distinct* $i_1, \dots, i_k \in \{1, \dots, n\}$ *satisfying* $N \subset \{i_1, \dots, i_k\}$, *then* $U(\alpha, \beta)$ *is a unicity set for* $C(B)$, *i.e., for each* $f \in C(B)$ *there exists a unique best approximant to* f *from* $U(\alpha, \beta)$.

Theorem 1.3. *Let* $N = \{i \mid \alpha_i = -\infty, \beta_i = \infty\}$, *and assume that, for all* $i \notin N$, $-\infty < \alpha_i < \beta_i < \infty$. *If* $U(\alpha, \beta)$ *is a unicity set for* $C(B)$, *then* $\{u_{i_1}, \dots, u_{i_k}\}$ *is a Haar system for every choice of distinct* $i_1, \dots, i_k \in \{1, \dots, n\}$ *satisfying* $N \subset \{i_1, \dots, i_k\}$.

Theorem 1.4. *Let* $N = \{i \mid \alpha_i = -\infty, \beta_i = \infty\}$, *and assume that, for all* $i \notin N$, $-\infty < \alpha_i < \beta_i < \infty$. *Assume that* $\{u_{i_1}, \dots, u_{i_k}\}$ *is a Haar system for every choice of distinct* $i_1, \dots, i_k$ *satisfying* $N \subset \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. *Let* $f \in C(B)$, *and let* $\tilde{u}$ *denote the unique best approximant to* f *from* $U(\alpha, \beta)$. *Then there exists* $\gamma = \gamma(f) > 0$ *such that, for all* $u \in U(\alpha, \beta)$,

$$||f - u|| \ge ||f - \tilde{u}|| + \gamma||\tilde{u} - u||.$$

As mentioned in [**5**] there is a gap between Theorem 1.2 and Theorem 1.3. Is the converse of Theorem 1.2 valid? In this paper an example is constructed to show the converse of Theorem 1.2 is not valid. However, we can further improve Theorem 1.3 and Theorem 1.4. The strong unicity constant in this setting is also discussed.

2. Main results.

Theorem 2.1. *Let* $N = \{i \mid \alpha_i = -\infty, \beta_i = \infty\}$ *and* $J = \{i \mid \alpha_i = -\infty, \beta_i < \infty\} \cup \{i \mid \alpha_i > -\infty, \beta_i = \infty\}$. *If* $U(\alpha, \beta)$ *is a unicity set for* $C(B)$, *then for every choice of* k *distinct points* $x_1, \dots, x_k \in B$ *and every choice of* $S = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ *with* $N \subset S$, *the matrix*

$$[u_{i_j}(x_s)] \equiv \begin{bmatrix} u_{i_1}(x_1) & \cdots & u_{i_1}(x_k) \\ \cdots & \cdots & \cdots \\ u_{i_k}(x_1) & \cdots & u_{i_k}(x_k) \end{bmatrix}$$

has its rank at least $k - \max\{|J \backslash S| - 1, 0\}$, *where* $|A|$ *is the number of indices in* A.

Proof. Suppose the rank of $[u_{i_j}(x_s)]$ is $r < k - \max\{|J\backslash S| - 1, 0\}$. Let W be the null space of $[u_{i_j}(x_s)]$. Then $\dim(W) = k - r \geq \max\{|J\backslash S|, 1\}$. Suppose $J\backslash S = \{m_1, \dots, m_{p+1}\}$ with $p = |J\backslash S| - 1$. Since $k - r > p$, we can find $c = (c_1, \dots, c_k) \neq 0$ in W such that

$$\sum_{s=1}^{k} c_s u_{m_j}(x_s) = 0, \qquad j = 1, \dots, p. \tag{2}$$

Replacing c by $-c$, if necessary, we can also have

$$\sum_{s=1}^{k} c_s u_{m_{p+1}}(x_s) \begin{cases} \geq 0 & \text{if } \beta_{m_{p+1}} < \infty \\ \leq 0 & \text{if } \alpha_{m_{p+1}} > -\infty. \end{cases} \tag{3}$$

Choose $(a_1, \dots, a_k) \neq 0$ such that

$$\sum_{j=1}^{k} a_j u_{i_j}(x_s) = 0, \qquad s = 1, \dots, k$$

and let $Q(x) = \sum_{1 \leq j \leq k} a_j u_{i_j}(x) \in U_k = \text{span}\,\{u_{i_1}, \dots, u_{i_k}\}$. We may assume $||Q|| < 1$. Choose $g \in C(B)$ with $||g|| = 1$ and $g(x_s) = \text{sgn}\,(c_s)$, $s = 1, \dots, k$. Set

$$f(x) = g(x)(1 - |Q(x)|).$$

Then $\lambda Q(x)$ are all best approximants to f from U_k for $0 \le \lambda \le 1$ (see [**2**]) and

$$(4) \quad (f - \lambda Q)(x_s) = \operatorname{sgn}(c_s)||f - \lambda Q||, \qquad s = 1, \dots, k, \ 0 \le \lambda \le 1.$$

For $i \in I \equiv \{1, \dots, n\} \backslash (J \cup S)$, let

$$(5) \qquad \gamma_i = \begin{cases} \alpha_i & \text{if } \sum c_s u_i(x_s) < 0 \\ \beta_i & \text{if } \sum c_s u_i(x_s) \ge 0. \end{cases}$$

Also let

$$(6) \qquad \gamma_{m_i} = \begin{cases} \alpha_{m_i} & \text{if } \alpha_{m_i} > -\infty \\ \beta_{m_i} & \text{if } \beta_{m_i} < \infty \end{cases} \qquad i = 1, \dots, p+1.$$

Choose $u(x) = \sum_{1 \le j \le k} d_j u_{i_j}(x) \in U_k$ with $\alpha_{i_j} < d_j < \beta_{i_j}$, $j = 1, \dots, k$ and define

$$\tilde{f}(x) = f(x) + u(x) + \sum_{i \in I} \gamma_i u_i(x) + \sum_{1 \le i \le p+1} \gamma_{m_i} u_{m_i}.$$

Let

$$\tilde{u}_\lambda(x) = u(x) + \lambda Q(x) + \sum_{i \in I} \gamma_i u_i(x) + \sum_{1 \le i \le p+1} \gamma_{m_i} u_{m_i} \equiv \sum \tilde{a}_i u_i.$$

Then $\tilde{u}_\lambda \in U(\alpha, \beta)$ for all sufficiently small λ. We claim

$$(7) \qquad \sum_{s=1}^{k} c_s u_i(x_s) \begin{cases} \le 0 & \text{if } \tilde{a}_i = \alpha_i \\ = 0 & \text{if } \alpha_i < \tilde{a}_i < \beta_i \\ \ge 0 & \text{if } \tilde{a}_i = \beta_i \end{cases} \qquad i = 1, \dots, n,$$

and

$$(8) \qquad (\tilde{f} - \tilde{u}_\lambda)(x_s) = \operatorname{sgn}(c_s)||\tilde{f} - \tilde{u}_\lambda||, \qquad s = 1, \dots, k.$$

To prove this claim, first we note $\{1, 2, \dots, n\} = S \cup (J \backslash S) \cup I$. Since $(c_1, \dots, c_k) \in W$, the null space of $[u_{i_j}(x_s)]$, (7) is valid for $i \in S$. By (2), (3) and (6), (7) is valid for $i \in J \backslash S$. By (5), (7) is valid for $i \in I = \{1, \dots, n\} \backslash (J \cup S)$. (8) comes from (4). This proves the claim. Without loss of generality, we can assume $c_1, c_2, \dots, c_r$ are all nonzero numbers and $c_{r+1} = \cdots = c_k = 0$ for some $1 \le r \le k$. Then (7) and (8) are valid for $k = r$. Now, by Theorem 1.1 $\tilde{u}_\lambda$ are best approximants to f from $U(\alpha, \beta)$ for all sufficiently small λ, and this contradicts the unicity hypothesis. □

Corollary 2.2. *Let N and J be defined as in Theorem* 2.1, *and let $U(\alpha, \beta)$ be a unicity set. Then $\{u_{i_1}, \dots, u_{i_k}\}$ is a Haar system for every choice of $S = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ satisfying $N \subset S$ and $|J \backslash S| \le 1$.*

This improves Theorem 1.3 because in Theorem 1.3 $J = \varnothing$ is assumed, and hence $|J \backslash S| = 0$ for any choice of S.

The next example shows the sharpness of Corollary 2.2 and, therefore, the converse of Theorem 1.2 is not valid.

Example 2.3. Let $B = [0, 1]$ and $n = 3$. Let $u_1 = 1$, $u_2 = x$, $u_3 = 1 + x^2$, and $U = \text{span}\{u_1, u_2, u_3\}$. Let $\alpha_1 > -\infty$, $\alpha_2 = \alpha_3 = -\infty$, $\beta_1 = \beta_2 = \infty$, and $\beta_3 < \infty$. Then $N = \{2\}$ and $J = \{1, 3\}$. It is easy to check that $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$ and $\{u_1, u_2, u_3\}$ are all Haar systems, but $\{u_2\}$ is not a Haar system.

Now we show $U(\alpha, \beta)$ is a unicity set. Suppose it is not. Then there exist $f \in C[0, 1]$ and $\tilde{u}_1, \tilde{u}_2 \in U(\alpha, \beta)$ such that both $\tilde{u}_1$ and $\tilde{u}_2$ are best approximants to f from $U(\alpha, \beta)$. Then by Theorem 1.1 there exist nonzero numbers $c_1, \dots, c_r$ and $x_1 < \cdots < x_r \in [0, 1]$, $1 \le r \le 4$ (the reason that we can pick the same x_j and c_j, $j = 1, \dots, r$ is given in [**5**]), such that

$$(f - \tilde{u}_i)(x_j) = \text{sgn}\,(c_j) \| f - \tilde{u}_i \|, \qquad i = 1, 2,\ j = 1, \dots, r,$$
$$\sum_{j=1}^{r} c_j \le 0, \qquad \sum_{j=1}^{r} c_j x_j = 0,$$

and

$$\sum_{j=1}^{r} c_j (1 + x_j^2) \ge 0.$$

Since $c_i \ne 0$ we have $r > 1$. If $r \ge 3$, then $\tilde{u}_1 = \tilde{u}_2$ because U is a Haar system. So we assume $r = 2$. From the fact that both $\{u_1, u_2\}$ and

$\{u_2, u_3\}$ are Haar systems, we have

$$(9)\qquad \begin{cases} c_1 + c_2 < 0 \\ c_1 x_1 + c_2 x_2 = 0 \\ c_1(1 + x_1^2) + c_2(1 + x_2^2) > 0. \end{cases}$$

From the middle equation we have $c_2 = -(x_1/x_2)c_1$. Substitute this in the first inequality and get

$$c_1\left(1 - \frac{x_1}{x_2}\right) < 0.$$

Since $x_1 < x_2$ we obtain $c_1 < 0$. From the third inequality of (9), we have

$$\begin{aligned} 0 < c_1(1 + x_1^2) + c_2(1 + x_2^2) &= c_1(1 + x_1^2) - \frac{x_1}{x_2}(1 + x_2^2)c_1 \\ = \frac{c_1}{x_2}(x_2 - x_1 + x_1^2 x_2 - x_1 x_2^2) &= \frac{c_1}{x_2}(x_2 - x_1)(1 - x_1 x_2) < 0, \end{aligned}$$

which is a contradiction. So we proved $U(\alpha, \beta)$ is a unicity set.

The next theorem improves Theorem 1.4.

Theorem 2.4. *If $U(\alpha, \beta)$ is a unicity set, then for every $f \in C(B)$ there exists $\gamma = \gamma(f) > 0$ such that for all $u \in U(\alpha, \beta)$*

$$||f - u|| \geq ||f - \tilde{u}|| + \gamma ||\tilde{u} - u||,$$

where $\tilde{u}$ is the unique best approximant to f from $U(\alpha, \beta)$.

Proof. Part of the proof is the same as the proof of Theorem 1.4. If $f \in U(\alpha, \beta)$, then nothing needs to be proven. Assume $f \neq \tilde{u}$. Let $\{x_i, i = 1, \dots, r\}$ and $\{c_j, j = 1, \dots, r\}$ be as in Theorem 1.1. Let

$$\tilde{u} = \sum_{i=1}^{n} \tilde{a}_i u_i$$

and $I_i = \{i \mid \Sigma c_s u_i(x_s) = 0\}$ and $I_2 = \{1, \dots, n\} \backslash I_1$. For $u = \Sigma a_i u_i \in U(\alpha, \beta)$ with $u \neq \tilde{u}$ and $i \in I_2$, we have

$$(\tilde{a}_i - a_i) \sum_{s=1}^{r} c_s u_i(x_s) \geq 0.$$

Thus,

$$\sum_{s=1}^{r} c_s(\tilde{u}(x_s) - u(x_s)) = \sum_{i=1}^{n} (\tilde{a}_i - a_i) \sum_{s=1}^{r} c_s u_i(x_s) \geq 0.$$

If $\tilde{u}(x_s) = u(x_s)$, $s = 1, \dots, r$, we denote $v(x) = u(x) - \tilde{u}(x)$ and then $v(x_s) = 0$, $s = 1, \dots, r$, and $\tilde{u}(x) + \lambda v(x) = \lambda u(x) + (1 - \lambda)\tilde{u}(x) \in U(\alpha, \beta)$ for $0 \leq \lambda \leq 1$. Choose $0 < \lambda_0 < 1$ such that $\lambda_0 ||v|| \leq (1/3)||f - \tilde{u}||$. Denote $A_1 = \{x \mid |f(x) - \tilde{u}(x)| \geq (1/3)||f - \tilde{u}||\}$ and $A_2 = \{x \mid f(x) = \tilde{u}(x)\}$. Since both A_1 and A_2 are closed sets, by Tietze extension theorem we can find a $g \in C(B)$ such that

$$g(x) = \operatorname{sgn}(f(x) - \tilde{u}(x))(|f(x) - \tilde{u}(x)| - \lambda_0 |v(x)|) \qquad \text{if } x \in A_1,$$
$$g(x) = 0, \qquad \text{if } x \in A_2,$$

and

$$||g||_{B \backslash (A_1 \cup A_2)} \leq ||g||_{\partial(B \backslash (A_1 \cup A_2))} \leq (1/3)||f - \tilde{u}|| + \lambda_0 ||v||,$$

where $\partial(A)$ denotes the boundary of A.

Let $\tilde{g}(x) = g(x) + \tilde{u}(x)$. Then for $0 \leq \lambda \leq \lambda_0$,

$$|\tilde{g}(x) - \lambda v(x) - \tilde{u}(x)| \leq ||f - u|| - (\lambda_0 - \lambda)|v(x)| \leq ||f - \tilde{u}||, \qquad x \in A_1,$$

and

$$|\tilde{g}(x) - \lambda v(x) - \tilde{u}(x)| \leq (1/3)||f - \tilde{u}|| + (\lambda_0 + \lambda)||v|| \leq ||f - \tilde{u}||, \qquad x \notin A_1.$$

Therefore,

$$||\tilde{g} - \lambda v - \tilde{u}|| \leq ||f - \tilde{u}||$$

and

$$(\tilde{g} - \lambda v - \tilde{u})(x_s) = \operatorname{sgn}(f(x_s) - \tilde{u}(x_s))||f - \tilde{u}|| = \operatorname{sgn}(c_s)||\tilde{g} - \lambda v - \tilde{u}||, \quad s = 1, \dots, r.$$

Since $f \neq \tilde{u}$ and $c_s \neq 0$, we have $||f - \tilde{u}|| = ||\tilde{g} - \lambda v - \tilde{u}||$.

Now, by Theorem 1.1, $\tilde{u}(x) + \lambda v(x)$, $0 \leq \lambda \leq \lambda_0$ are all best approximants to $\tilde{g}$ from $U(\alpha, \beta)$. This is a contradiction. This contradiction shows that, for all $u \in U(\alpha, \beta)$ with $u \neq \tilde{u}$,

$$\max_{1 \leq j \leq r} (\operatorname{sgn}(c_j)(\tilde{u} - u)(x_j)) > 0.$$

By a standard compactness argument, we have

$$\max_{1 \leq j \leq r} (\operatorname{sgn}(c_j)(\tilde{u} - u)(x_j)) \geq \gamma ||\tilde{u} - u||$$

for all $u \in U(\alpha, \beta)$ and some $\gamma > 0$. Then by Theorem 1.1,

$$\begin{aligned} ||f - u|| &\geq \max_{1 \leq j \leq r} (\operatorname{sgn}(c_j)(f - u)(x_j)) \\ &= \max_{1 \leq j \leq r} [\operatorname{sgn}(c_j)(f - \tilde{u})(x_j) + \operatorname{sgn}(c_j)(\tilde{u} - u)(x_j)] \\ &= ||f - \tilde{u}|| + \max_{1 \leq j \leq r} (\operatorname{sgn}(c_j)(\tilde{u} - u)(x_j)) \\ &\geq ||f - \tilde{u}|| + \gamma ||\tilde{u} - u||. \quad \square \end{aligned}$$

As in the unconstrained case, we define the strong unicity constant in this setting.

Definition 2.5. Let $f \in C(B)$ have a strongly unique best approximant $\tilde{u}$ from $U(\alpha, \beta)$. The strong unicity constant $\gamma_0 = \gamma_0(f) > 0$ is defined by

$$\gamma_0 = \sup\{\gamma \mid ||f - u|| \geq ||f - \tilde{u}|| + \gamma ||u - \tilde{u}|| \quad \text{for all} \quad u \in U(\alpha, \beta)\}.$$

Theorem 2.6. *If $f \in C(B)$ has a strongly unique best approximant $\tilde{u}$ from $U(\alpha, \beta)$, then*

$$\gamma_0 = \min_{u \in U(\alpha, \beta)} \left\{ \max_{x \in E} \left[\operatorname{sgn}(f(x) - \tilde{u}(x)) \frac{(u(x) - \tilde{u}(x))}{||\tilde{u} - u||} \right] \right\},$$

where $E = \{x \mid |f(x) - \tilde{u}(x)| = ||f - \tilde{u}||\}$.

The proof of the above theorem is similar to that of the unconstrained case, so we omit it (see [**3, 4**]).

Theorem 2.7. *Let $f \in C(B)$ have a strongly unique best approximant $\tilde{u} = \Sigma \tilde{a}_i u_i$ from $U(\alpha,\beta)$. Set*

$$I = \{i \mid \tilde{a}_i = \alpha_i \text{ or } \tilde{a}_i = \beta_i\}$$

and

$$E = \{x \mid |f(x) - \tilde{u}(x)| = ||f - \tilde{u}||\}.$$

Then $|I| + |E| \geq n + 1$.

Proof. Suppose $|I| = n - r$ and $|E| = p + 1$. If $r > p$, then we can find r numbers $c_1, \dots, c_r$ and $u_{i_1}, \dots, u_{i_r}$ with $i_j \in \{1, \dots, n\} \backslash I$, $j = 1, \dots, r$, such that

$$\sum_{j=1}^{r} c_j u_{i_j}(x_k) \operatorname{sgn}(f(x_k) - \tilde{u}(x_k)) \leq 0, \qquad k = 1, \dots, p+1.$$

Let $u_\lambda = \tilde{u} + \lambda \Sigma c_j u_{i_j} \in U(\alpha,\beta)$ for small $\lambda > 0$. Then we have

$$\max_{x \in E} (\operatorname{sgn}(f(x) - \tilde{u}(x))(u_\lambda(x) - \tilde{u}(x))) \leq 0,$$

and this contradicts Theorem 2.6. $\square$

Combining Theorem 2.4 and Theorem 2.7, we have

Corollary 2.8. *If $U(\alpha,\beta)$ is a unicity set, then for any $f \in C(B)$, $|I| + |E| \geq n + 1$, where I and E are defined as in Theorem 2.7.*

Theorem 2.9. *Let $f \in C(B)$ have a strongly unique best approximant $\tilde{u}$ from $U(\alpha,\beta)$, and let I and E be defined as in Theorem 2.7. If $|I| + |E| = n + 1$ with $E = \{x_1, \dots, x_{k+1}\}$, then*

$$\lambda_0 \leq \min\{1/||g_i||, i = 1, \dots, k+1\}$$

where $g_i \in \operatorname{span}\{u_i, i \notin I\}$ satisfying $g_i(x_j) = \operatorname{sgn}(f(x_j) - \tilde{u}(x_j))$ for $j \neq i$.

Proof. Let $\{1,\dots,n\}\backslash I=\{i_1,\dots,i_k\}$. First we claim

$$\operatorname{rank}\begin{bmatrix} u_{i_1}(x_1) & \cdots & u_{i_k}(x_1) \\ \cdots & \cdots & \cdots \\ u_{i_1}(x_{j-1}) & \cdots & u_{i_k}(x_{j-1}) \\ u_{i_1}(x_{j+1}) & \cdots & u_{i_k}(x_{j+1}) \\ \cdots & \cdots & \cdots \\ u_{i_1}(x_{k+1}) & \cdots & u_{i_k}(x_{k+1}) \end{bmatrix}=k,$$

for $j=1,\dots,k+1$. If this is not true, we can find $v=\Sigma_{1\le j\le k}c_j u_{i_j}\neq 0$ such that

$$v(x_j)\operatorname{sgn}(f(x_j)-\tilde{u}(x_j))\le 0, \qquad j=1,\dots,k+1.$$

Also, by the definition of I, $\tilde{u}(x)+\lambda v(x)\in U(\alpha,\beta)$ for small λ. This contradicts Theorem 2.6. So we proved our claim and hence all g_i are well defined and $\lambda g_i\in U(\alpha,\beta)$ for sufficiently small λ. Now by Theorem 2.6

$$\begin{aligned}\gamma_0 &= \min_{u\in U(\alpha,\beta)}\left\{\max_{x\in E}\left[\operatorname{sgn}(f(x)-\tilde{u}(x))\frac{u(x)-\tilde{u}(x)}{||u-\tilde{u}||}\right]\right\} \\ &\le \min_{1\le i\le r}\left\{\max_{x\in E}\left[\operatorname{sgn}(f(x)-\tilde{u}(x))\frac{\lambda g_i(x)}{||\lambda g_i||}\right]\right\}=\frac{1}{||g_i||}.\end{aligned}$$

□

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