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FACTORIZABLE SEMIGROUP OF PARTIAL SYMMETRIES OF A REGULAR POLYGON

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ABSTRACT. The semigroup of partial symmetries of a convex polygon P is the inverse semigroup of all isometries of subpolygons of P, under composition. This semigroup is a natural generalization of the group of symmetries of a polygon, as well as a particular instance of an inverse semigroup formed by taking all isomorphisms between substructures of a given mathematical structure. The general properties of the semigroup of partial symmetries of any convex polygon were explored in [**5**]; in this paper we restrict consideration to regular polygons where, as in the situation with groups, much more structural information can be obtained. We show that every isometry between subpolygons of P can be extended to an isometry of P and use this to factorize these semigroups into a product of a semilattice and a group. If the number of vertices of the polygon is odd, a complete characterization is given in terms of the group of symmetries of P.

1. Preliminaries. In [**5**] we explored the inverse semigroup of partial symmetries of a convex polygon. We now restate the precise definition of these semigroups.

Let P be a convex polygon with a set of vertices $V = \{v_1, v_2, \dots, v_n\}$, $n \geq 3$, where v_i is adjacent to v_{i+1} for $i = 1, \dots, n-1$, and v_n is adjacent to v_1. A polygon A is a *subpolygon* of P if A has as its set of vertices $V(A) = \{v_{i(1)}, v_{i(2)}, \dots, v_{i(m)}\}$ contained in V with $i(1) < i(2) < \cdots < i(m)$, and the edges of A are between $v_{i(k)}$ and $v_{i(k+1)}$, $k = 1, \dots, m-1$, and between $v_{i(m)}$ and $v_{i(1)}$. The subpolygon A is the region of the plane enclosed by the edges of A. Note that, under our definition, subpolygons may have two vertices (line segment), one vertex (point), or no vertex (empty polygon, denoted by ϕ).

For two subpolygons A and B of P, an *isometry* of A onto B is any bijection α from $V(A)$ onto $V(B)$ which is distance-preserving under the usual Euclidean metric. The domain and range of α will be denoted

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as $\operatorname{dom}\alpha$, $\operatorname{rng}\alpha$, respectively; the rank of α is the cardinality of $\operatorname{dom}\alpha$. We will say that $\operatorname{dom}\alpha \cong \operatorname{rng}\alpha$ even though it is more precisely the subpolygons with these sets of vertices that are congruent. For the empty polygon ϕ, the isometry from ϕ to ϕ is the empty mapping which will be denoted by Φ.

For the convex polygon P with set of vertices V, we define $S(P)$ to be the set of all isometries between subpolygons of P. Under composition, $S(P)$ is an inverse semigroup, called the *semigroup of partial symmetries of* P. $S(P)$ is an inverse subsemigroup of the symmetric inverse semigroup of V, $I(V)$, and its group of units is the group of symmetries of P. An idempotent of $S(P)$ is just the identity on a subset A of V, denoted by ι_A, so that $E_S = E_{I(V)}$.

As was discussed in [**3**] and [**7**], for a given mathematical structure, the collection of isomorphisms between its substructures forms an inverse semigroup under composition. Our semigroup $S(P)$ is one such example. In [**5**] we explored the ideals and congruences of $S(P)$ for any convex polygon P. In this paper we will restrict ourselves to regular polygons in order to obtain stronger results concerning the structure of $S(P)$.

2. Extending isometries of subpolygons to isometries of P. For the rest of the paper, P will be a regular polygon with n vertices, $n \geq 3$, forming the set V, $S = S(P)$ will be the semigroup of partial symmetries of P, and G will be the group of symmetries of P, that is, the dihedral group of order $2n$.

The following result is central to the rest of the paper. We say that an isometry θ from subpolygon A onto B can be extended to an isometry of P if there is a θ' in G such that $\theta'|_{\operatorname{dom}\theta} = \theta$.

Theorem 2.1. *If A and B are subpolygons of P and $\theta : A \to B$ is an isometry, then there is an isometry of P which extends θ.*

We will divide this theorem into the following lemmas since we also wish to count the number of isometries of P which extend θ.

Lemma 2.2. *The empty mapping can be extended to $2n$ isometries of P.*

Proof. This is obvious since G has $2n$ elements. □

Lemma 2.3. *Every singleton mapping can be extended to exactly two isometries of P: a rotation and a reflection.*

Proof. Let a and b be vertices. Then there is a reflection which takes a onto b. This can be found by letting k be the minimum number of vertices between a and b. If k is odd, we pick the middle vertex between a and b and reflect about the line through it and the center. If k is even, we pick the midpoint of the edge between the middle two vertices and reflect about the line through this point and the center. Let γ be this reflection. Let ρ be the rotation counterclockwise by $2\pi/n$ radians. Then the dihedral group G can be written as

$$\{\rho^j\gamma^k | 1 \leq j \leq n, 0 \leq k \leq 1\}.$$

Clearly there is exactly one rotation ρ^m which takes a onto b. If $(a)\rho^j\gamma = b$, then $a\rho^j = a$ since $a\gamma = b$. Hence, $j = n$ and $\rho^j\gamma = \gamma$. Thus, γ and ρ^m are the only elements of G which take a onto b. □

For vertices a and a', the line segment joining them is written $\overline{aa}'$. We will say that a' is the vertex opposite a if $\overline{aa}'$ passes through the center of P. When we write $\theta : \overline{aa}' \to \overline{bb}'$, it is understood that $a \to b$, $a' \to b'$.

Lemma 2.4. *Let $\theta : \overline{aa}' \to \overline{bb}'$ be an isometry.*

1. *If a' is the vertex opposite a, then there are exactly two isometries of P which extend θ.*

2. *If a' is not a vertex opposite a, then there is exactly one isometry of P which extends θ.*

Proof. Assume first that a' is opposite a. From Lemma 2.3 there is a reflection which takes a onto b. But since a' is opposite a, then b' must be opposite b and this reflection must take a' onto b'. Also, let π be the rotation which sends a onto b. Then, since a is opposite a', $a\pi$ must be opposite $a'\pi$. That is, π extends θ. Since there are exactly two elements of G which take a onto b, then these are the only two which extend θ.

Now assume that a is not opposite a'. Let k be the number of vertices from a to a' in the clockwise direction. Then k is either the number of vertices from b to b' in the clockwise direction or the counterclockwise direction, but not both. If it is in the clockwise direction, then the rotation of P which takes a onto b will also take a' onto b'. If k is the number in the counterclockwise direction, then the reflection which takes a onto b will take a' onto b'. Thus, in either case, there is exactly one element of G which extends θ. □

For the general case, we will extend the isometry θ first to an isometry of the plane and show that this new isometry, in turn, induces an isometry of P.

Lemma 2.5. *If $\theta : A \to B$ is an isometry and $|A| > 2$, then there is a unique isometry of P which extends θ.*

Proof. Let θ be an isometry of A onto B. This isometry θ can be extended to an isometry of the plane, call it θ'. We will show that θ', when restricted to P, is an isometry of P. Let a, b, c be three adjacent vertices of A in clockwise order. Let

$$(a)\ \ \theta = a', \qquad (b)\ \ \theta = b', \qquad (c)\ \ \theta = c'.$$

Then a', b', c' must be adjacent vertices in B. Furthermore, since these sets of points are vertices of a regular polygon, a, b, and c are noncollinear as are a', b', c'. Now from properties of geometry (see, for example, p. 39 of [**8**]), we know that every isometry of the plane is an affine transformation of the plane, so θ' is an affine transformation. Further, the fundamental theorem of affine geometry states that for these two sets of noncollinear points, there is a unique affine transformation mapping a onto a', b onto b', and c onto c'. Thus, θ' is the only such affine transformation.

However, we can easily construct an isometry of the polygon P which takes a onto a', b onto b', and c onto c'. For, if we take a rotation of P which maps b onto b', then if c' is the adjacent vertex to the right of b' in B, then this rotation is such an isometry. If c' is not the adjacent vertex to the right, then a' is, so that we follow the rotation by a reflection which has b' on the axis of symmetry. In either case, there is an isometry of P which takes a onto a', b onto b', and c onto c', and, of course, this isometry can be extended to the plane. But then, this isometry is an affine transformation and, by uniqueness, it must be θ'. Thus, θ', restricted to P, is an isometry of P which extends θ, and it is the only one. □

The fact that every isometry of P can be extended to one in G now allows us to classify $S(P)$ as a factorizable semigroup.

Definition. A semigroup is said to be *factorizable* if there is a group G and a set of idempotents E such that $S = EG$.

A factorizable semigroup has been defined more generally as a product of two subsemigroups; however, for inverse semigroups the above definition seems most appropriate. Chen and Hsieh [**1**] showed that the symmetric inverse semigroup over a set X is factorizable if and only if the set X is finite. Tirasupa [**9**] extended this to the partial and full transformation semigroups on a set, and D'Alarcao [**2**] related this to certain finiteness conditions. In [**1**] it was shown that it is necessary that E be E_S and G be the group of units of S.

It is easy to see that, for a general convex polygon P, the semigroup of partial symmetries of P need not be factorizable. For example, if P is a polygon with no two sides of equal length, then the group of symmetries of P is the trivial group so that $E_S G = E_S$, while $S \neq E_S$. If P is an isosceles triangle with $\overline{ab} \cong \overline{bc}$, $\overline{ab} \ncong \overline{ac}$, then the group of symmetries has just two elements, but the isometry which takes the line segment $\overline{ab}$ onto $\overline{bc}$ (with $a \to b$, $b \to c$) cannot be extended to an isometry of P. Thus, a consequence of the next result is the fact that equilateral triangles are exactly those triangles which will produce a factorizable semigroup of partial symmetries.

Theorem 2.6. *The semigroup of partial symmetries of a regular polygon is factorizable.*

Proof. Let G be the group of symmetries of P and E_S be the set of all idempotents of S. Let θ be an isometry of A onto B. Then, by Theorem 2.1, θ can be extended to an isometry θ' of P; that is, $\theta = \iota_A\theta'$ and, therefore, $S = E_S G$. □

Regularity of the polygon is not necessary for the semigroup of partial symmetries of P to be factorizable. For example, if P is any parallelogram, it can easily be seen that S is factorizable.

We now use the lemmas to count the elements of S. Clearly, from Lemma 2.4, we will have to separate even and odd n.

Theorem 2.7. *Let S be the semigroup of partial symmetries of a regular polygon P with n vertices. Then*

1. *If n is odd, S has $n(2^{n+1} - n - 2) + 1$ elements*;
2. *if n is even, S has $n(2^{n+1} - (3/2)n - 2) + 1$ elements.*

Proof. Since, by Theorem 2.6, $S = E_S G$, with E_S having 2^n elements and G having $2n$ elements, then S has $2^n(2n)$ elements minus the number of repeats in the characterization $\iota_A\alpha$, $A \subseteq V$, α in G. By Lemma 2.5, every element of rank greater than 2 is uniquely described in this way.

Let n be odd. Then no two vertices are opposite one another so that the only elements which do not have a unique representation are those of rank ≤ 1. From Lemma 2.2, there are exactly $2n$ representations of the empty map, so there are $2n - 1$ repeats. From Lemma 2.3, each singleton is represented twice and there are n^2 singletons so there are n^2 repeats. Therefore, if n is odd, the number of elements of S is

$$2^{n+1}n - (n^2 + 2n - 1) = n(2^{n+1} - n - 2) + 1.$$

Let n be even. Then, in addition to the repeats of rank ≤ 1, those mappings of rank 2 which map a line of symmetry onto a line of symmetry appear twice in the representation, by Lemma 2.4. There are $n^2/2$ of these mappings, since there are $n/2$ lines of symmetry and each line of symmetry can map onto n others when taking into account where the individual elements go. Therefore, there are

$$2^{n+1}n - (n^2 + 2n - 1 + n^2/2) = n(2^{n+1} - (3/2)n - 2) + 1$$

elements in S. □

Corollary 2.8. *If n is odd, S has an even number of elements; if n is even, S has an odd number of elements.*

The inverse semigroup of partial symmetries of the equilateral triangle has 34 elements. There are 6 of rank 3, 18 of rank 2, 9 of rank 1 and 1 of rank 0. The semigroup of partial symmetries of the square has 97 elements: 8 of rank 4, 32 of rank 3, 40 of rank 2, 16 of rank 1, and 1 of rank 0.

We now look at the number of elements that are required to generate S as an inverse semigroup. The *rank of an inverse semigroup* is the smallest number of elements needed to generate S as an inverse semigroup. In [**4**, Theorem 3.1] it was shown that the symmetric inverse semigroup on a finite set has rank 3. With a modification of that proof, we obtain the same result for our semigroup. If S is generated as an inverse semigroup by $x_1, \dots, x_m$, we write $S = \langle x_1, \dots, x_m \rangle$.

Theorem 2.9. *The rank of the semigroup of partial symmetries of a regular polygon is* 3.

Proof. Since P is regular, G is a dihedral group generated by two elements, ρ and γ. Also, for a regular polygon, any two subpolygons with $n-1$ vertices are congruent, so that

$$\mathbf{J}' = \{\alpha \in S | \operatorname{rank} \alpha = n-1\}$$

is a $\mathcal{J}$-class [**5**, Lemma 2.1]. Let α be a fixed element in $\mathbf{J}'$ and β be any element of $\mathbf{J}'$. Then $\operatorname{dom} \alpha \cong \operatorname{dom} \beta$ so there is an isometry which takes $\operatorname{dom} \beta$ onto $\operatorname{dom} \alpha$. This isometry can be extended to an isometry, σ, of P. Let $\tau = \alpha^{-1}\sigma^{-1}\beta$. Thus, $\operatorname{dom} \tau = \operatorname{rng} \alpha$, $\operatorname{rng} \tau = \operatorname{rng} \beta$, so τ is $\mathbf{J}'$ and τ can be extended to an element τ' of G. Then σ and τ' are in G and

$$\begin{aligned} \sigma\alpha\tau' = \sigma\alpha(\alpha^{-1}\sigma^{-1}\beta) = \sigma\iota_{\operatorname{dom}\alpha}\sigma^{-1}\beta = \iota_{\operatorname{dom}\beta}\sigma\sigma^{-1}\beta \\ = \iota_{\operatorname{dom}\beta}\iota_P\beta = \beta. \end{aligned}$$

Therefore, β is in the inverse subsemigroup of S generated by ρ, γ, and α; i.e., $\beta \in \langle \rho, \gamma, \alpha \rangle$.

Now let $A \subseteq V$, $|A| < n$. Then $V - A = \{x_1, x_2, \dots, x_k\}$, and let $V - \{x_i\} = X(i)$. Then $\iota_A = \iota_{X(1)}\iota_{X(2)} \dots \iota_{X(k)}$, so that ι_A is in $\langle \rho, \gamma, \alpha \rangle$ for all A contained in V.

If δ is in S, then by Theorem 2.1, $\delta = \iota_A \delta'$ for some δ' in G, with $A = \operatorname{dom} \delta$. Thus, since ι_A is in $\langle \rho, \gamma, \alpha \rangle$ and δ' is in $\langle \rho, \gamma \rangle$, then δ is in $\langle \rho, \gamma, \alpha \rangle$. Hence, rank of $S \leq 3$. Moreover, 3 is the minimum since G requires two generators itself and at least one other is needed to obtain the rest of the elements of S. □

3. Structure of the semigroup of partial symmetries. In this section we will use the results of Section 2 to obtain a construction of $S(P)$ in the case where P has an odd number of vertices. The sets P, V, S, and G will be as in the last section, and $P(V)$ will be the set of all subsets of V.

Theorem 3.1. *On $T = P(V) \times G$, define $*$ by*

$$(A, \alpha) * (B, \beta) = ((A\alpha \cap B)\alpha^{-1}, \alpha\beta).$$

Then $(T, *)$ is an inverse semigroup with $E_T \cong E_S$. Moreover, the mapping $f : T \to S$ defined by $f : (A, \alpha) \to \iota_A \alpha$, is a homomorphism onto S.

Proof. It is a very direct element-chasing argument to show that * is associative. The inverse of (A, α) is $(A\alpha, \alpha^{-1})$ and the idempotents of T are (A, ι_P), where ι_P is the identity of G.

Since, in S, $(\iota_A \alpha)(\iota_B \beta) = (\iota_{(A_\alpha \cap B)_\alpha^{-1}})\alpha\beta$, f is a homomorphism. That f maps onto S is a direct result of the fact that S is factorizable (Theorem 2.6). □

If the number of vertices is odd, no two vertices are opposite one another, so that Lemmas 2.4 and 2.5 imply that every isometry of rank greater than 1 has a unique representation of the form given in Theorem 3.1, and those of rank ≤ 1 correspond to (A, α) where $|A| \leq 1$. Thus, we have the following stronger results for regular polygons with an odd number of vertices.

Corollary 3.2. *Let n be odd, with $S_1 = S/\{\theta | \operatorname{rank} \theta \leq 1\}$, and $T_1 = T/\{(A, \alpha) | \, |A| \leq 1\}$. Then $S_1 \cong T_1$.*

The semigroup T_1 an be described on the set $(V_1 \times G) \cup \{0\}$, where

$V_1 = \{A \subseteq V \,|\, |A| > 1\}$, and 0 acts as the zero of T_1, with operation $*$ defined by

$$(A,\alpha) * (B,\beta) = \begin{cases} ((A\alpha \cap B)\alpha^{-1}, \alpha\beta) & \text{if } |A\alpha \cap B| > 1, \\ 0, & \text{if } |A\alpha \cap B| \leq 1. \end{cases}$$

We know from Proposition 2.4 of [**5**] that $\{\theta | \operatorname{rank}\theta \leq 1\}$ is a Brandt semigroup over the trivial group, i.e., $B = M^0(V, 1, V; \Delta)$. Thus S can be considered an ideal extension of B by the inverse semigroup S_1. Ideal extensions of Brandt semigroups are discussed in Chapter 5 of [**6**]. In particular, since B has a trivial group, any ideal extension of B by S_1 is completely determined by a partial homomorphism of $S_1 \backslash \{0\}$ into $I(V)$ (see Theorems V.4.6 and V.4.7 of [**6**]).

We can give the multiplication in S explicitly by using the above results and [**6**; Theorem V.4.7], or we can extract the multiplication directly from the construction as follows. On $B = V \times V$, (v, w) in B corresponds to the singleton mapping: $v \to w$, and in $V_1 \times G$, (A, α) corresponds to the mapping $\iota_A\alpha$. Thus, for multiplication defined in $S = \{0\} \cup B \cup (V_1 \times G)$, $(v, w)^*(A, \alpha) = (v, w\alpha)$ if w is in A and 0 otherwise. For (A, α), (B, β) in $V_1 \times G$, if $|A\alpha \cap B| > 1$, then multiplication stays as in T_1; if $(A\alpha \cap B)\alpha^{-1} = \{u\}$, then $(A, \alpha) * (B, \beta) = (u, u\alpha\beta)$ since $(\iota_A\alpha))(i_B\beta)$ is the singleton map: $(u \to u\alpha\beta)$. That is,

$$\begin{cases} (A,\alpha) * (B,\beta) = ((A\alpha \cap B)\alpha^{-1}, \alpha\beta) & \text{if } |A\alpha \cap B| > 1, \\ (A,\alpha) * (B,\beta) = (u, u\alpha\beta) & \text{if } (A\alpha \cap B)\alpha^{-1} = \{u\}, \\ (A,\alpha) * (v,w) = (v\alpha^{-1}, w) & \text{if } v \text{ is in } A\alpha, \\ (v,w) * (A,\alpha) = (v, w\alpha) & \text{if } w \text{ is in } A, \\ (v,w) * (x,y) = (v,y) & \text{if } w = x, \end{cases}$$

and all other products are equal to 0.

With this we have given an explicit description of the semigroup of partial symmetries of an odd sided regular polygon, in terms of sets and the group of symmetries of the regular polygon.

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