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BOUNDEDNESS AND ASYMPTOTICS OF A MATRIX ITERATION

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ABSTRACT. The Generalized Theodorus Iteration, a matrix iteration on $\mathbf{R}^n$, is studied. Sufficient conditions for boundedness and asymptotic estimates of divergence are given. The limit sets, which are generally strange attractors, are also briefly discussed.

1. Introduction. In [**1**], Philip J. Davis studied a sequence of points $z_i \in \mathbf{C}$ defined by

$$z_{n+1} = z_n + iz_n/|z_n|$$

with $z_0 = 1$. These points form a spiral, called the Spiral of Theodorus. Davis then suggested the generalization

$$z_{n+1} = \alpha \cdot z_n + \beta \cdot z_n/|z_n|$$

for complex α, β, and z_0. This iteration, called the Complex Generalized Theodorus Iteration, displays a number of strange attractors and has been studied in [**1, 7**]. Davis then proposed the further generalization

$$V_{n+1} = A * V_n + B * V_n/||V_n|| \tag{1.1}$$

where A and B are real $m \times m$ matrices, V_0 is a given nonzero m-vector, and $||\cdot||$ is the Euclidean vector norm. This iteration, called the Generalized Theodorus Iteration (GTI), displays a great variety of strange attractors for appropriate choices of A and B and has been studied in [**2, 7, 9**]. We wish to present some further results on the iteration in this paper. We will make the assumption that V_0 is such that V_n is nonzero for all $n \geq 0$, so that the corresponding infinite sequence $\{V_n\}_{n=0}^{\infty}$ will always be well-defined. The case where $V_n = 0$ for some n is not of interest here, as we are concerned primarily with questions of boundedness and asymptotics.

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2. The linear case. When B is the zero matrix, the iteration given by (1.1) reduces to the linear system

$$V_{n+1} = A * V_n. \tag{2.1}$$

The properties of this iteration are well-known and depend primarily on the spectral radius of A. We summarize them briefly. If $\rho(A) < 1$, then the iteration is bounded for any V_0 and, in fact, $V_n \to 0$ as $n \to \infty$ for any V_0. If $\rho(A) > 1$, then there exists some V_0 such that $||V_n||$ is unbounded; there may also be choices of V_0 which lead to bounded orbits. If $\rho(A) \leq 1$ and any eigenvalues of A having unit modulus are simple (in which case A is said to be of bounded type), then the iteration is always bounded (though not, in general, independently of V_0).

The case where $A = 0$ in (1.1), which is related to the power method for numerically determining eigenvalues of a matrix, is also understood; all orbits eventually lie on a hyperellipse in a space of dimension equal to the number of nonzero eigenvalues of A (see [**8**]).

3. Green's function representation. Given an iteration of the form

$$z_t = A * z_{t-1} + \omega_{t-1}$$

with ω_t and z_0 given vectors and A a matrix, the solution can be written in the form [**10**]

$$z_{t+1} = \sum_{s=0}^{t+1} Y_{t+1} \cdot Y_s^{-1} \cdot \omega_s + Y_{t+1} \cdot Y_0^{-1} \cdot z_0$$

where $\{Y_t\}$ is a fundamental matrix set of solutions for

$$z_t = A * z_{t-1}.$$

For the iteration under consideration, this gives

$$V_{n+1} = A^{n+1} V_0 + \sum_{t=0}^{n} A^{n-1} * B * V_t / ||V_t||. \tag{3.1}$$

We will use this formula (the Green's function representation) to address the questions of boundedness and asymptotics.

4. Boundedness. We now seek a sufficient condition for the boundedness of the orbits of (1.1). Suppose that $\rho(A) < 1$. Then there exists some natural matrix norm $||\cdot||_\alpha$ such that $||A||_\alpha < 1$. Using $||\cdot||_\alpha$ to represent the vector norm which induces this matrix norm as well, we have from (3.2)

$$\begin{aligned} ||V_{n+1}||_\alpha &\le ||A^{n+1}V_0||_\alpha + \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t/||V_t|| \right\|_\alpha \\ &\le ||A||_\alpha^{n+1} \cdot ||V_0||_\alpha + \sum_{t=0}^{n} ||A||_\alpha^t \cdot ||B||_\alpha \cdot k \\ &= ||A||_\alpha^{n+1} \cdot ||V_0||_\alpha + k||B||_\alpha \sum_{t=0}^{n} ||A||_\alpha^t \end{aligned} \tag{4.1}$$

where k is a positive constant such that

$$||V||_\alpha/||V|| \le k$$

for all nonzero vectors V (such a k exists since all vector norms are equivalent on $\mathbf{R}^m$ [**4**]). Now, since $||A||_\alpha < 1$, we have that

$$||A||_\alpha^{n+1}||V_0||_\alpha < ||V_0||_\alpha$$

and, since the finite sum in (4.1) defines an increasing sequence, it is less than

$$k \cdot ||B||_\alpha/(1 - ||A||_\alpha)$$

and so

$$||V_{n+1}||_\alpha \le ||V_0||_\alpha + k||B||_\alpha/(1 - ||A||_\alpha)$$

for all $n \ge 1$. This proves that the solution is bounded.

We have shown that $\rho(A) < 1$ suffices for boundedness. In an analogous way we can show that if $\rho(A) > 1$ then there must exist some V_0 such that $||V_n||$ is unbounded. For, considering (3.1) (and using the spectral norm) we have

$$||V_{n+1}|| \ge \left| ||A^{n+1}V_0|| - \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t/||V_t|| \right\| \right|.$$

Let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$, and take V_0 to be an eigenvector of A associated with the eigenvalue λ. Then

$$\begin{aligned} ||V_{n+1}|| &\geq \left| ||\lambda^{n+1}V_0|| - \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t/||V_t|| \right\| \right| \\ &\geq \left| |\lambda|^{n+1}||V_0|| - \sum_{t=0}^{n} ||A||^t \cdot ||B|| \right| \\ &\geq \Big| ||V_0||\, |\lambda|^{n+1} - ||B|| \cdot (1 - ||A||^{n+1})/(1 - ||A||) \Big|. \end{aligned}$$

If we let

$$c = ||B||/(||A|| - 1),$$

then $c > 0$ and

$$||V_{n+1}|| \geq \Big| ||V_0||\, |\lambda|^{n+1} - c||A||^{n+1} + c \Big|.$$

Now $||A|| = |\lambda| + \varepsilon$ for some $\varepsilon \geq 0$, so

$$\begin{aligned} ||V_{n+1}|| &\geq \Big| ||V_0||\, |\lambda|^{n+1} - c(|\lambda| + \varepsilon)^{n+1} + c \Big| \\ &\geq \Big| |\lambda|^{n+1}[||V_0|| - c(1 + \varepsilon/|\lambda|)^{n+1}] + c \Big|. \end{aligned} \tag{4.2}$$

If ε is equal to zero we may now choose $||V_0||$ to be greater than c in order to have $||V_n|| \to \infty$; otherwise, the term involving ε will grow without bound, again giving $||V_n|| \to \infty$.

Thus, the GTI (1.1), like the linear system (2.1), has the property that if $\rho(A) < 1$ then all orbits are bounded, and if $\rho(A) > 1$ then there exist unbounded orbits. In the case $\rho(A) = 1$ the GTI may have all orbits bounded for one choice of the B matrix and all orbits unbounded for another choice of the B matrix (with the same A matrix). In the next section we will derive asymptotics for this indeterminate case. Before doing so, we collect the information above in a theorem.

Theorem 1. *Consider the iteration* (1.1). *If $\rho(A) < 1$, then the iteration is bounded for any V_0. If $\rho(A) > 1$, then there exists a V_0 such that the iteration is unbounded.*

5. Asymptotics. From (4.2) it is apparent that when $\rho(A) > 1$ the divergence will, in general, be exponential. This leaves the case $\rho(A) = 1$. Let us first look at the case where A is of bounded type. Using the spectral matrix norm, (3.1) gives

$$
\begin{aligned}
||V_{n+1}|| &\le ||A^{n+1}V_0|| + \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t/||V_t|| \right\| \\
&\le K_1 + \sum_{t=0}^{n} ||A^t|| \cdot ||B||
\end{aligned}
\tag{5.1}
$$

where $K_1 \ge 0$ is a bound on $||A^{n+1}V_0||$. Now since A is of bounded type, $||A^n||$ is also bounded, and so

$$
\begin{aligned}
||V_{n+1}|| &\le K_1 + ||B|| \sum_{t=0}^{n} K \\
&\le K_1 + K_2 n
\end{aligned}
$$

showing that the divergence is no worse than linear.

If $\rho(A) = 1$ but A is not of bounded type, then $||A^n|| = \mathcal{O}(n^{p-1})$, where ρ is the highest order of a nonlinear elementary divisor of A associated with an eigenvalue of unit modulus. Thus, from (5.1), we have

$$
||V_{n+1}|| \le ||A^{n+1}|| \cdot ||V_0|| + \sum_{t=0}^{n} ||A^t|| \cdot ||B||,
$$

and clearly $||V_n||$ diverges at most as $\mathcal{O}(n^p)$. Hence, when $\rho(A) = 1$, if the iteration diverges then it does so at most polynomially. We write this as a theorem.

Theorem 2. *Consider* (1.1) *and suppose that* $\rho(A) = 1$ *and the iteration is unbounded. Then the divergence is no worse than polynomial of order* p*, where* p *is the order of the largest elementary divisor of* A*.*

6. Limit sets. We now have a sufficient condition for boundedness of the GTI and some estimates of the asymptotics for divergent cases. When the iteration is bounded, it is of interest to consider its limit set which we denote $\Gamma(A, B, V_0)$ (also called the ω-limit set). A point Y is in $\Gamma(A, B, V_0)$ if there exists an increasing subsequence $n(i)$ such that $V_{n(i)} \to Y$ as $i \to \infty$ when the initial condition is V_0. A modification of results in [**5**] (where the mappings are assumed to be everywhere continuous, a condition not met here) gives:

Theorem 3. *Suppose that, for some V_0, the sequence $\{V_n\}$ is bounded. Then $\Gamma(A, B, V_0)$ is a nonempty compact set and $V_n \to \Gamma(A, B, V_0)$ as $n \to \infty$. If $0 \notin \Gamma(A, B, V_0)$, then $\Gamma(A, B, V_0)$ is positively invariant.*

Proof. Clearly the complement of $\Gamma = \Gamma(A, B, V_0)$ is open and so Γ is closed. The boundedness of $\{V_n\}$ implies that Γ is also bounded and, hence, it is compact. By the Bolzano-Weierstrass theorem, Γ contains at least one point. Now $V_n \to \Gamma$ means that $\inf(||V_n - Y|| : Y \in \Gamma) \to 0$ as $n \to \infty$. Suppose the contrary. Then there is some increasing subsequence $m(i)$ such that $V_{m(i)}$ does not tend to Γ. Suppose further that no subsequence of $V_{m(i)}$ tends to Γ. By the Bolzano-Weierstrass theorem, this sequence has a limit point; but then this limit point must be in Γ. By contradiction, then, $V_n \to \Gamma$.

Define

$$T(V) = A * V + B * V/||V||$$

so that $V_{n+1} = T(V_n)$ is the iteration under consideration. Since $0 \notin \Gamma$, T is continuous in some neighborhood of any point $Y \in \Gamma$. Fix Y and choose a subsequence $n(i)$ such that $V_{n(i)} \to Y$. By the continuity of T, we have that $T(V_{n(i)}) \to T(Y)$ as $i \to \infty$. But

$$T(V_{n(i)}) = V_{n(i)+1}$$

so that $V_{m(i)} \to T(Y)$ with $m(i) = n(i) + 1$. Hence, $T(Y) \in \Gamma$, and Γ is positively invariant. $\square$

In fact, it is clear from the above that $T(Y) \in \Gamma$ whenever $Y \in \Gamma$ and $T(Y)$ is defined, and we could simply say that $\Gamma \backslash \{0\}$ is positively invariant. Further topological results on discrete dynamical systems that can be extended to this iteration (with some care taken near the origin) are to be found in [**6**].

FIGURE 1. $A = \begin{pmatrix} .99 & 0 \\ 0 & -.45 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\rho(A) = .99$.

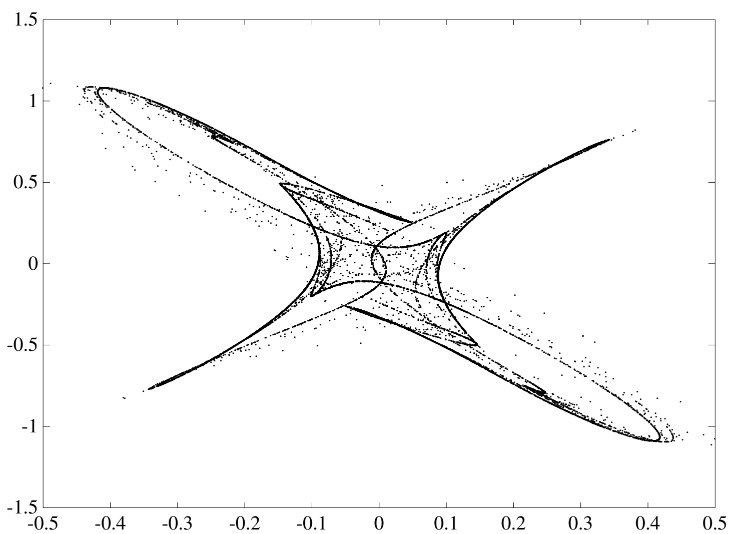

FIGURE 2. $A = \begin{pmatrix} 1 & .001 \\ -.001 & -.79 \end{pmatrix}$, $B = \begin{pmatrix} -.5 & -.1 \\ 1 & .5 \end{pmatrix}$; $\rho(A) = .99\ldots$

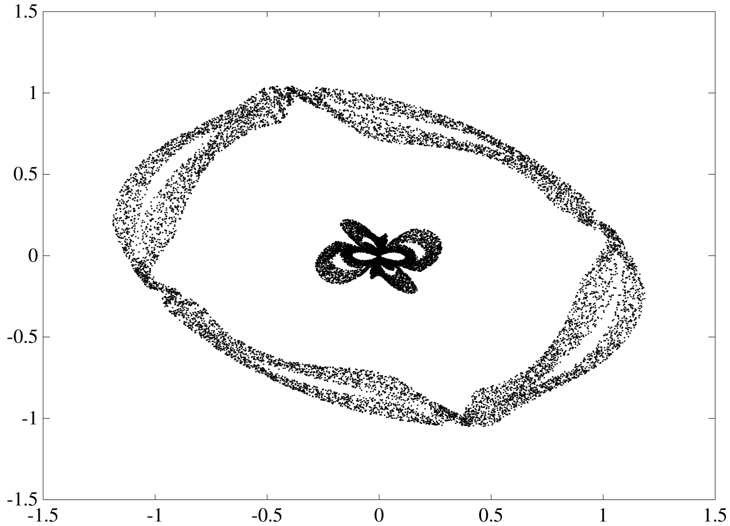

FIGURE 3. $A = \begin{pmatrix} .97 & .71 \\ -.87 & .58 \end{pmatrix}$, $B = -A$; $\rho(A) \cong 1.09$

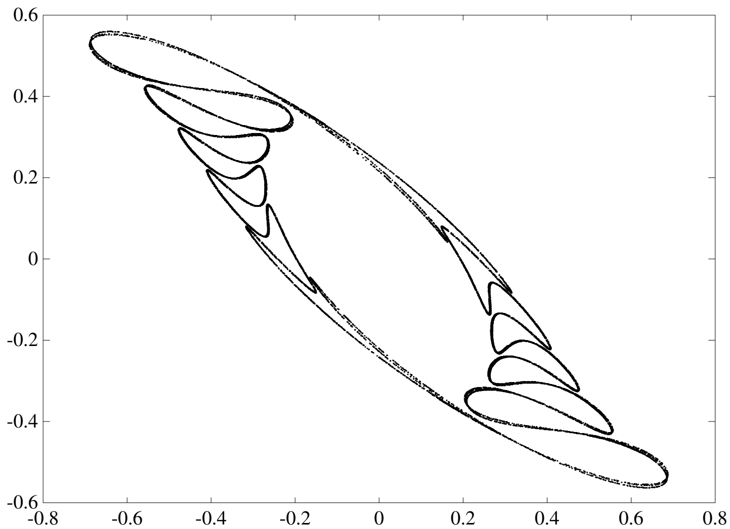

FIGURE 4. $A = \begin{pmatrix} .39 & -.1 \\ .08 & -.37 \end{pmatrix}$, $B = \begin{pmatrix} -.12 & .71 \\ -.22 & -.45 \end{pmatrix}$; $\rho(A) \cong .38$

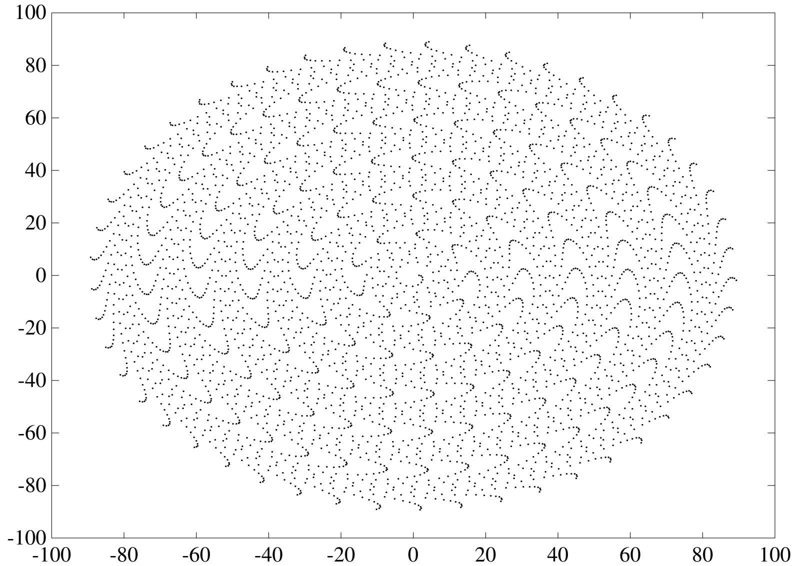

FIGURE 5. $A = (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $\rho(A) = 1$ (divergent)

Suppose that $Y \in \Gamma(A, B, V_0)$, so that $V_{n(i)} \to Y$ for some increasing subsequence $n(i)$. Since the transformation $T : \mathbf{R}^m \to \mathbf{R}^m$ of Theorem 3 has the property that

$$T(-V) = -T(V),$$

it follows that the sequence generated by $-V_0$ is $-V_{n(i)}$, which converges to $-Y$ so that $-Y \in \Gamma(A, B, -V_0)$. Thus, any global attractor (where $\Gamma(A, B, V_0)$ is independent of V_0 for the particular A and B under consideration) must be centrally symmetric. Experimenting with this iteration for varying choices of A and B produces a surprisingly wide variety of attractors, which are as yet not understood; see Figures 1–5 (produced using MATLAB®). We conjecture that the attractors are strange but not chaotic (in the sense of [**3**]) and hope to investigate them in detail in future work.

7. Discussion. The Generalized Theodorus Iteration is in some ways very similar to the simple linear system (Theorem 1) and yet it contains some very complicated structures (Figures 1–5). Iserles has found an analytical expression for the attractors for a certain set of cases of the form $A = \alpha B$, $\alpha > 0$ [**2**]. For the case $A = 0$, the behavior of the iteration is also known [**8**]. However, the iteration contains a vast number of interesting attractors which are as yet unanalyzed.

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