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NEUTRAL STRUCTURES ON EVEN-DIMENSIONAL MANIFOLDS

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ABSTRACT. The notion of a neutral structure on an even-dimensional manifold M, defined to be a $\mathbf{G}$-structure on M for which the structure group is the neutral orthogonal group $\mathbf{NO(n)}$ of all isometries and anti-isometries of the pseudo-Euclidean space $\mathbf{R}^{n,n}$, is introduced. A neutral structure is weaker than an $\mathbf{O(n,n)}$-structure, though it may reduce to such. The obstruction to such a reduction is shown to be an element of $H^1(M,\mathbf{Z}_2)$ and there is a close analogy with the notion of orientability. The basic differential geometry of neutral structures is presented, including a Gauss-Bonnet-Chern theorem. Results concerning neutral Einstein structures in four dimensions are obtained.

1. Introduction. The existence of anti-isometries for the pseudo-Euclidean space $\mathbf{R}^{n,n}$ allows one to enlarge the orthogonal group $\mathbf{O(n,n)}$ by including anti-isometries and so obtain a group I call the neutral orthogonal group and denote $\mathbf{NO(n)}$. In [**15**], $\mathbf{NO(n)}$ was shown to be the appropriate symmetry group for defining a notion of angle between any two non-null vectors in the Lorentz plane. Because of the neutrality of the signature of the pseudo-Euclidean metric for $\mathbf{R}^{n,n}$, and explicitly due to the existence of anti-isometries, it seems natural to regard all non-null vectors as on an equal footing independently of their "character," i.e., whether they be time-like or space-like. In other words, it appears natural to regard $\mathbf{NO(n)}$ as a symmetry group on $\mathbf{R}^{2n}$ and study that geometry of $\mathbf{R}^{n,n}$ which is invariant under the action of $\mathbf{NO(n)}$.

More generally, $\mathbf{NO(n)}$ may be employed to define a $\mathbf{G}$-structure on even-dimensional manifolds. Such a "neutral structure" generalizes the notion of a global neutral metric, i.e., a metric of signature type (n,n). Since $\mathbf{O(n,n)}$ is of index two in $\mathbf{NO(n)}$, the question of whether a reduction to $\mathbf{NO(n)}$ reduces further to $\mathbf{O(n,n)}$ is analogous to the question of orientability of manifolds.

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In Section 2, the relevant theory of **G**-structures is presented in a form suitable for exposing the basic features of neutral structures. Section 3 contains some non-existence results and a simple example. In Section 4 I discuss the elementary differential geometry of neutral structures and establish a Gauss-Bonnet-Chern formula, while in Section 5 I consider neutral Einstein structures in four dimensions, thereby providing a sequel to my previous paper [**16**]. In particular, an appropriate analogue of the Thorpe-Hitchin inequality of the Riemannian case is studied.

As regards notation, $\mathbf{R}^{p,q}$ denotes the pseudo-Euclidean space consisting of $\mathbf{R}^n$, $n = p + q$, equipped with the inner product

$$g(u, v) = u^1 v^1 + \cdots + u^p v^p - \cdots - u^n v^n$$

where (u^i) and (v^i) are components of the vectors $\mathbf{u}$ and $\mathbf{v}$ with respect to the standard basis of $\mathbf{R}^n$. In any $\mathbf{R}^{p,q}$, a vector $\mathbf{u}$ for which the squared norm $g(\mathbf{u}, \mathbf{u})$ is positive, negative, or zero, is called time-like, space-like, or null, respectively. A unit vector is a vector of squared norm plus or minus one. A pseudo-orthonormal basis is a basis of unit, mutually orthogonal vectors. Such a basis is said to have standard configuration if the time-like elements of the basis are listed first. For signature of type (n, n), a pseudo-orthonormal basis for which the first n elements are of like character, i.e., all time-like or all space-like, (whence the remaining are of like character also) is called configured. The notion of standard and non-standard configuration should then be obvious from the previous sentence.

2. Orientation structures and neutral structures. The theory of **G**-structures outlined below not only suffices for the treatment of neutral structures but also incorporates the **G**-structure formulation of orientability (cf. [**11**, p. 5]) and other notions of orientability such as time-orientability (cf. [**19**, pp. 240–242]).

Definition 2.1. Let M be a connected smooth n-manifold, $L(M)$ its bundle of linear frames, and $\mathbf{G}$ a Lie subgroup of $\mathbf{GL}\,(\mathbf{N}; \mathbf{R})$. Suppose M has a $\mathbf{G}$-structure, i.e., there is s a subbundle $F(M)$ of $L(M)$ with structure group $\mathbf{G}$. (One also says $F(M)$ is a reduction of $L(M)$, and that the structure group $\mathbf{GL}\,(\mathbf{n}; \mathbf{R})$ has been reduced to $\mathbf{G}$.) Let $\mathbf{H}$ be a normal, Lie subgroup of $\mathbf{G}$ with finite index m. Define M to be

H-orientable relative to $F(M)$ if there is a subbundle $B(M)$ of $F(M)$ with structure group **H**.

If M is **H**-orientable it is not difficult to see that $F(M)$ partitions, under the action of **H**, into m distinct subbundles, each with structure group **H**. The selection of a particular subbundle constitutes a choice of **H**-orientation (relative to $F(M)$) for M. See also (2.6) below.

Proposition 2.2. *With notation as above, if* $\mathbf{G}/\mathbf{H}$ *is Abelian* (*no restriction of course for* $m \leq 5$) *then the* **G**-*structure defines an element* w *of* $H^1(M, \mathbf{G}/\mathbf{H})$ *which is the obstruction to* **H**-*orientability (relative to the given* **G**-*structure*).

Proof. The argument is fairly standard. With respect to a simple cover $\mathcal{U} := \{U_i\}$ for M (in the sense of [**13**, pp. 167–168]), let f_{ij} be the transition functions on $U_i \cap U_j$ for the trivialization of $F(M)$ with respect to $\mathcal{U}$. Define a Čech 1-cochain with respect to this cover by $\nu_{ij}(U_i \cap U_j) :=$ the coset of **H** in **G** to which f_{ij} belongs on $U_i \cap U_j$. This is well defined (i.e., the coset $f_{ij}(x)\mathbf{H}$ is independent of x in $U_i \cap U_j$ and the cocycle condition for the transition functions implies that ν_{ij} is indeed a 1-cocycle). This assignment is well defined under the process of restriction to refinements.

Moreover, if $\mathcal{V} := \{V_I\}$ is another simple cover of M, by passing to a simple cover $\mathcal{W} := \{W_\alpha\}$ which is a common refinement of $\mathcal{U}$ and $\mathcal{V}$, one finds that the 1-cocycles induced on $\mathcal{W}$ from the trivializations with respect to $\mathcal{U}$ and $\mathcal{V}$ differ by a coboundary: if W_α is contained in $U_{\phi(\alpha)}$ and in $V_{\psi(\alpha)}$ say (where ϕ and ψ are mappings from the indexing set of $\mathcal{W}$ to those of $\mathcal{U}$ and $\mathcal{V}$, respectively), restrict the trivializing sections $\sigma_{\phi(\alpha)} : U_{\phi(\alpha)} \to F(M)$ and $\sigma_{\psi(\alpha)} : V_{\psi(\alpha)} \to F(M)$ to W_α so that $\sigma_{\phi(\alpha)}(z) = \sigma_{\psi(\alpha)}(z) f_\alpha(z)$ for some $f_\alpha(z)$ in **G** but with the coset $\nu_\alpha := f_\alpha(z)\mathbf{H}$ independent of z in W_α, then on a nonempty intersection $W_\alpha \cap W_\beta$,

$$\nu_{\psi(\alpha)\psi(\beta)} = \nu_\beta \nu_{\phi(\alpha)\phi(\beta)} (\nu_\alpha)^{-1}.$$

Thus, an element w of $H^1(M, \mathbf{G}/\mathbf{H})$ is determined independently of the simple cover.

Obviously, if M is **H**-orientable with respect to $F(M)$, then by standard theory one can find a simple cover of M such that the transition functions for the trivializations of $F(M)$ take values in **H** and w will be trivial in $H^1(M, \mathbf{G}/\mathbf{H})$. Conversely, if w is trivial, there exists a simple cover $\mathcal{U}$ with respect to which ν_{ij} is a coboundary: $\nu_{ij} = \nu_j(\nu_i)^{-1}$ for some 0-cochain ν_i. If $\sigma_i : U_i \to F(M)$ are the trivializing sections and f_{ij} the transition functions, writing ν_i as $f_i\mathbf{H}$, and using the fact that $\mathbf{G}/\mathbf{H}$ is Abelian, the coboundary condition amounts to

$$f_{ij}(z) = (f_i)^{-1} h_{ij}(z) f_j,$$

$h_{ij}(z)$ in $\mathbf{H}$, for each z in $U_i \cap U_j$. From $\sigma_j(z) = \sigma_i(z) f_{ij}(z)$, one then deduces

$$\sigma_j(z)(f_j)^{-1} = [\sigma_i(z)(f_i)^{-1}] h_{ij}(z),$$

i.e., there are trivializing sections for $F(M)$ with respect to which the transition functions lie in $\mathbf{H}$. Thus, the required reduction of $F(M)$ exists. □

The following simple results from group theory will be needed.

Lemma 2.3. *Let* $\mathbf{G}$ *be any group,* $\mathbf{A}$ *any Abelian group, and* $\mathbf{C}$ *the commutator subgroup of* $\mathbf{G}$*. Then*:

(i) $\mathrm{Hom}\,(\mathbf{G}, \mathbf{A}) \simeq \mathrm{Hom}\,(\mathbf{G}/\mathbf{C}, \mathbf{A})$ *as groups.*

(ii) *The set of subgroups of* $\mathbf{G}$ *of index two are in bijective correspondence with the nontrivial elements of* $\mathrm{Hom}\,(\mathbf{G}, \mathbf{Z}_2)$.

Proof. (i) follows easily from the fact that $\mathbf{C}$ is contained in the kernel of each element of $\mathrm{Hom}\,(\mathbf{G}, \mathbf{A})$. The correspondence of (ii) is the assignment $\phi \mapsto \ker(\phi)$ where ϕ is a nontrivial element of $\mathrm{Hom}\,(\mathbf{G}, \mathbf{Z}_2)$.

Lemma 2.4. *If* M *is a connected smooth* n*-manifold and* $\mathbf{A}$ *is any Abelian group,*

$$H^1(M, \mathbf{A}) \simeq \mathrm{Hom}\,(H_1(M, \mathbf{Z}), \mathbf{A}) \simeq \mathrm{Hom}\,(\pi_1(M), \mathbf{A}).$$

Proof. Since M is a connected manifold, $H_0(M, \mathbf{Z}) = \mathbf{Z}$. $\mathrm{Ext}\,(\mathbf{Z}, \mathbf{A}) = 0$, since $\mathbf{Z}$ is free, and so the first isomorphism follows from the universal coefficient theorem of algebraic topology. The second isomorphism follows from (2.3)(i). □

Proposition 2.5. *With notation as in* (2.1), *and supposing* $\mathbf{G}/\mathbf{H}$ *is Abelian,* M *is* $\mathbf{H}$*-orientable if* M *is simply connected or if only* $H_1(M, \mathbf{Z})$ *is trivial. If* $m = 2$*, whence* $\mathbf{G}/\mathbf{H} = \mathbf{Z}_2$*,* M *is* $\mathbf{H}$*-orientable if* $\pi_1(M)$ (*or* $H_1(M, \mathbf{Z})$) *has no subgroup of index two.*

Proof. Follows immediately from (2.2)–(2.4). □

Being a subgroup of $\mathbf{G}$, $\mathbf{H}$ acts on $F(M)$. The quotient E of $F(M)$ under this action of $\mathbf{H}$ is naturally identifiable with the bundle with fiber $\mathbf{G}/\mathbf{H}$ associated to $F(M)$ [**13**, p. 57]. Since $\mathbf{H}$ is normal in $\mathbf{G}$, E is in fact a principal $\mathbf{G}/\mathbf{H}$-bundle.

Proposition 2.6. *With notation as above, the bundle* E *is an* m*-fold covering space of* M *which is itself* $\mathbf{H}$*-orientable* (*in a sense made clear in the proof*)*.* M *is* $\mathbf{H}$*-orientable if and only if* E *is trivial as a bundle over* M*, in which case the possible* $\mathbf{H}$*-orientations of* M *correspond to the* m *distinct global sections of* E *over* M*.*

Proof. Let $\pi_E : E \to M$ be the projection. As E is locally diffeomorphic to M, one can identify the bundle $L(E)$ of linear frames over E with the pullback bundle $(\pi_E)^{-1}(L(M))$. The bundle $F(E) := (\pi_E)^{-1}(F(M))$ is a reduction of $L(E)$ with group $\mathbf{G}$. The assertion of the proposition is that E is $\mathbf{H}$-orientable with respect to $F(E)$. By standard theory [**13**, p. 57], $F(E)$ admits such a reduction if and only if the associated bundle with fiber $\mathbf{G}/\mathbf{H}$ admits a global section. As noted above, this associated bundle is identifiable with $F(E)/\mathbf{H} = (\pi_E)^{-1}(F(M))/\mathbf{H} = (\pi_E)^{-1}(F(M)/\mathbf{H}) = (\pi_E)^{-1}(E)$. But $e \mapsto (e, e)$ is a global section of the last bundle.

Finally, by the same standard theory, M is $\mathbf{H}$-orientable if and only if $E = F(M)/\mathbf{H}$ admits a global section, i.e., if and only if E is trivial, whence $E = M \times \mathbf{G}/\mathbf{H}$.

Examples 2.7. The following examples are well-known.

(i) $G = \mathbf{GL}(\mathbf{n}; \mathbf{R})$ and $\mathbf{H} = \mathbf{GL}^+(\mathbf{n}; \mathbf{R})$ give rise to the usual notion of orientability. The cohomology class w in $H^1(M, \mathbf{Z}_2)$ is identifiable with the first Stiefel-Whitney class of M.

(ii) $\mathbf{G} = \mathbf{O(p,q)}$. Let R and I be the linear transformations of $\mathbf{R}^n$, $n = p + q$, whose matrix representations with respect to the standard basis of $\mathbf{R}^n$ are

$$\underline{\mathrm{R}} = \begin{pmatrix} \underline{\mathrm{J}}_p & \underline{0} \\ \underline{0} & \underline{1}_q \end{pmatrix} \qquad \underline{\mathrm{I}} = \begin{pmatrix} \underline{\mathrm{J}}_p & \underline{0} \\ \underline{0} & \underline{\mathrm{J}}_q \end{pmatrix}$$

where $\underline{\mathrm{J}}_k$ is the $k \times k$ diagonal matrix $\underline{\mathrm{J}}_k := \mathrm{diag}\,(-1, 1, \dots, 1)$. With $\mathbf{SO^+(p,q)}$ the identity-connected component of $\mathbf{SO(p,q)}$, $\mathbf{O^+(p,q)} := \mathbf{SO^+(p,q)} \cup RI\mathbf{SO^+(p,q)}$ and $\mathbf{O_+(p,q)} := \mathbf{SO^+(p,q)} \cup R\mathbf{SO^+(p,q)}$ are subgroups of $\mathbf{O(p,q)}$. $\mathbf{H} = \mathbf{O^+(p,q)}$ gives rise to the notion of "time orientability" while $\mathbf{H} = \mathbf{O_+(p,q)}$ gives rise to the notion of "space orientability."

The existence of an $\mathbf{O(p,q)}$-structure entails the decomposition of the tangent bundle TM as an orthogonal sum of a time-like, rank p subbundle T and a space-like, rank q subbundle S. For the first Stiefel-Whitney classes, one has, in $H^1(M, \mathbf{Z}_2)$, $w_1(M) = w_1(T) + w_1(S)(*)$. For both time-orientability and space-orientability, $\mathbf{G/H} = \mathbf{Z}_2$, and the associated cohomology class may be identified with $w_1(T)$ and $w_1(S)$, respectively. The decomposition of TM is not, of course, unique, but if $\hat{T} \oplus \hat{S}$ is another such decomposition of TM, then so are $\hat{T} \oplus S$ and $T \oplus \hat{S}$. Applying the Whitney product formula to $\hat{T} \oplus S = T \oplus \hat{S}$ shows $w_1(\hat{T}) = w_1(T)$. Similarly, $w_1(\hat{S}) = w_1(S)$. From $(*)$, one easily deduces that any two of ordinary orientability, time orientability, and space orientability entail the third.

Taking $\mathbf{H} = \mathbf{SO^+(p,q)}$ gives rise to a notion I shall refer to as "semi-orientability." Note that $\mathbf{G/H} = \mathbf{Z}_2 \times \mathbf{Z}_2$ in this case and there are four orientation classes on an $\mathbf{SO^+(p,q)}$-orientable manifold equipped with a metric of type (p,q). The $\mathbf{Z}_2$ factors of $\mathbf{G/H}$ are naturally identifiable with $\mathbf{O^+(p,q)/H}$ and $\mathbf{O_+(p,q)/H}$ and by elementary group theory may then be identified with $\mathbf{G/O^+(p,q)}$ and $\mathbf{G/O_+(p,q)}$, respectively. Thus,

$$(**) \quad H^1(M, \mathbf{G/H}) \simeq H^1(M, \mathbf{G/O^+(p,q)}) \times H^1(M, \mathbf{G/O_+(p,q)}).$$

If μ, τ, and σ are, respectively, the obstructions to $\mathbf{H}-$, $\mathbf{O^+(p,q)}-$, and $\mathbf{O_+(p,q)}$-orientability, it follows from their definition and $(**)$ that $\mu = \tau + \sigma$. From this follows the relation between semi-orientability and the other three kinds of orientability. One can also think directly in terms of the bundle $F(M)$ and its components of course.

(iii) An almost complex structure on a $2n$-dimensional manifold M is a reduction to $\mathbf{GL(n;C)}$ (more precisely, a reduction to the real representation of $\mathbf{GL}(n;\mathbf{C})$ in $\mathbf{GL(2n;R)}$). Let $\mathbf{G} = \mathbf{O(n;C)}$ and $\mathbf{H} = \mathbf{SO(n;C)}$. Then, once again $\mathbf{G}/\mathbf{H} = \mathbf{Z}_2$. $\mathbf{H}$-orientability may be referred to as complex Riemannian orientability. Such $\mathbf{G}$-structures are actually intimately related to neutral geometry as will be discussed elsewhere.

I turn now to the notion of neutral structures. For the remainder of this paper, M shall denote a connected, $2n$-dimensional smooth real manifold.

Definition 2.8. A *neutral structure* on M is a reduction $N(M)$ of the frame bundle $L(M)$ with structure group $\mathbf{NO(n)}$. A neutral structure will be called *reducible* if there is a further reduction of $N(M)$ to a subbundle with structure group $\mathbf{O(n,n)}$ and *irreducible* otherwise.

Given a simple cover $\mathcal{U} = \{U_i\}$ of M together with trivializing sections $\sigma_i : U_i \to N(M)$, a neutral metric g_i may be defined on U_i by regarding $\sigma_i(x)$ as constituting a pseudo-orthonormal basis appropriate to a metric of type (n,n) with standard configuration. The fiber $N_x(M)$ then consists of the configured pseudo-orthonormal frames of $g_i(x)$. Note that with these conventions for the construction of g_i, the cover $\{U_i\}$ only determines g_i up to sign. Moreover, on a nonempty intersection $U_i \cap U_j$, either $g_i = g_j$ or $g_i = -g_j$. (This procedure is completely analogous to the construction of a metric of type (p,q) from a reduction of the frame bundle with structure group $\mathbf{O(p,q)}$.) A collection $\{U_i, g_i\}$ constructed as just described will be called a *representation* of the given neutral structure. A neutral structure is reducible if and only if it may be represented by a global neutral metric, i.e., $g_i = g_j$ on each nonempty intersection $U_i \cap U_j$.

A manifold equipped with a neutral structure will be called a *neutral manifold*.

Proposition 2.9. *Suppose M carries a neutral structure $N(M)$. Then there exists a connected two-fold covering $\pi : S \to M$ such that S is equipped with a global neutral metric g which is, up to sign, locally* *equal to the pullback via π of any local representative g_i of $N(M)$ on M.*

In particular, let $\{U_i\}$ be a simple cover of M which is also admissible for the two-fold covering (i.e., $\pi^{-1}(U_i)$ is a disjoint union of two open sets, U_i^+ and U_i^- say, each of which is diffeomorphic to U_i via π) and let $\{U_i, g_i\}$ be a representation of the neutral structure of M. Then the restriction of g to each of U_i^+ and U_i^- equals, up to sign, $\pi^* g_i$.

Notice that, given (S, g), $(S, -g)$ satisfies the above conditions equally well. The neutral structure is reducible if and only if S is disconnected, each component being either isometric or anti-isometric to M.

Proof. This result is just (2.6) applied to the case of neutral structures. Hence, S is $N(M)$ factored out by the right action of $\mathbf{O}(\mathbf{n}, \mathbf{n})$. If $\sigma_i : U_i \to N(M)$ is the section used to construct the representation g_i, let U_i^+ be the equivalence class of $[N(M)|_U]/\mathbf{O}(\mathbf{n}, \mathbf{n}) = [N(M)/\mathbf{O}(\mathbf{n}, \mathbf{n})]|_U$ which contains the image of σ_i. Defining $g_i^+ := g_i$ on U_i^+ and $g_i^- := -g_i$ on U_i^-, one may readily check that these local neutral metrics defined on S agree on intersections of the elements of the covering $\{U_i^+, U_i^-\}$ and thus constitute a global neutral metric g. □

3. Examples? With $\mathbf{G} = \mathbf{NO}(\mathbf{n})$ and $\mathbf{H} = \mathbf{O}(\mathbf{n}, \mathbf{n})$, reducibility of a neutral structure $N(M)$ is just $\mathbf{O}(\mathbf{n}, \mathbf{n})$-orientability. The cohomology class w in $H^1(M, \mathbf{Z}_2)$ of (2.2) constitutes the obstruction to reducibility. The formalism of Section 2 therefore provides an approach to determining the impossibility of irreducible neutral structures.

Examples 3.1. By (2.5), if M is simply connected it cannot admit an irreducible neutral structure. This includes $\mathbf{R}^{2n}$, $\mathbf{S}^{2n}$, (which cannot admit a global neutral metric either [**21**, (27.18) and p. 207] and so admits no neutral structure whatsoever), and $\mathbf{CP}^n$. Furthermore, any complex submanifold of complex codimension one in $\mathbf{CP}^n$, $n \geq 3$, is connected and simply connected according to the Lefschetz theorem [**8**, p. 159]. There are, of course, a variety of theorems guaranteeing simple-connectedness, e.g., [**19**, p. 321], [**14**, pp. 365, 368, and 370], and [**5**, p. 325].

Let M and N be connected, simple connected n-dimensional manifolds equipped with Riemannian metrics g and h, respectively. Then the product metric $g + (-h)$ on $M \times N$ is a neutral metric and so defines a reducible neutral structure. But $\pi_1(M \times N) = \pi_1(M) \times \pi_1(N)$, and so $M \times N$ is simply connected and admits no irreducible neutral structure. More generally, consider manifolds M and N of dimension p and q, respectively, with $p + q = 2n$, both connected and such that $H^1(M, \mathbf{Z}_2) = H^1(N, \mathbf{Z}_2) = 0$. By the Künneth formula, $H^1(M \times N, \mathbf{Z}_2) = 0$, and thus $M \times N$ admits no irreducible neutral structure. This includes $\mathbf{S}^p \times \mathbf{S}^q$ where $p + q = 2n$ and both p and q are greater than one.

As $H^1(\mathbf{S}^1 \times \mathbf{S}^{2n-1}, \mathbf{Z}_2) = \mathbf{Z}_2$, this leaves open the possibility that $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$ admits an irreducible neutral structure. On the other hand, with $n \geq 2$, suppose $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$ does admit an irreducible neutral structure and let (S, g) be the two-fold covering space of (2.9). Then $\pi_1(S)$ injects into $\pi_1(\mathbf{S}^1 \times \mathbf{S}^{2n-1}) = \mathbf{Z}$ as a subgroup of index two. It follows that $\pi_1(S) = \mathbf{Z}$ and is injected into $\pi_1(\mathbf{S}^1 \times \mathbf{S}^{2n-1}) = \mathbf{Z}$ as $2\mathbf{Z}$. If $A : \mathbf{S}^1 \to \mathbf{S}^1$ is the antipodal mapping, then $(A \times \mathbf{identity}) : \mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^1 \times \mathbf{S}^{2n-1}$ is a two-fold cover with the same induced mapping of the fundamental group as just described for $S \to \mathbf{S}^1 \times \mathbf{S}^{2n-1}$. Thus, from covering space theory, S must be diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$; whence the latter space admits an irreducible neutral structure only if it admits a reducible neutral structure.

Can $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$ admit a global neutral metric? Since $\mathbf{S}^1, \mathbf{S}^3$ and $\mathbf{S}^7$ are parallelizable, $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$ has trivial tangent bundle for $n = 1, 2$ and 4 and thus may be split as the sum of equal rank subbundles. In these cases, at least, $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$ does admit global neutral metrics. For the case $\mathbf{S}^1 \times \mathbf{S}^1$, a neutral metric is also Lorentzian and the result is, of course, well known. A simple irreducible neutral structure on $\mathbf{S}^1 \times \mathbf{S}^1$ will be exhibited below.

Example 3.2. The even-dimensional real projective spaces $\mathbf{RP}^{2n}$ have $H^1(\mathbf{RP}^{2n}, \mathbf{Z}_2) = \mathbf{Z}_2$ whence the obstruction to irreducible neutral structures is lacking. If, however, $\mathbf{RP}^{2n}$ had a neutral structure, then by (2.9) there exists a two-fold covering S carrying a global neutral metric. As $\pi_1(S)$ injects into $\pi_1(\mathbf{RP}^{2n}) = \mathbf{Z}_2$ as a subgroup of index two, $\pi_1(S)$ must be trivial and S is diffeomorphic to the universal covering space of $\mathbf{RP}^{2n}$, viz., $\mathbf{S}^{2n}$. As already noted above, however, $\mathbf{S}^{2n}$ does not admit a global neutral metric whence $\mathbf{RP}^{2n}$ does not admit any neutral structure. $\mathbf{RP}^n \times \mathbf{RP}^n$ does not appear to be barred as a candidate, however.

Example 3.3. Consider $\mathbf{R} \times \mathbf{S}^1$ with coordinates (t, x) where x is the natural angular coordinate. Let $\mathbf{S}^1$ be covered by two open sets U and V which when parametrized by x take the form $U = (0, \pi)$ and $V = (\pi - \delta, 2\pi + \delta)$ and such that $U \cap V$ has the form $(0, \delta) \cup (\pi - \delta, \pi)$ with respect to the parametrization on U. Let $\sigma : [0, \pi] \to [0, 1]$ be a monotonically increasing smooth function which is identically zero on a neighborhood of 0 containing $(0, \delta)$ and identically one on a neighborhood of π containing $(\pi - \delta, \pi)$.

Now, with respect to the relevant coordinates, define metrics

$$g_1(t, x) := \begin{pmatrix} \cos y & \sin y \\ \sin y & -\cos y \end{pmatrix},$$

where $y := \sigma(x)\pi$, on $\mathbf{R} \times U$, and

$$g_2(t, x) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $\mathbf{R} \times V$. Then g_1 and g_2 are Lorentz metrics such that $g_1 = g_2$ on $(0, \delta)$ in U and $g_1 = -g_2$ on $(\pi - \delta, \pi)$ in U. Although $\{U, V\}$ is not a simple cover of $\mathbf{S}^1$, it is obvious that the above defines an irreducible neutral structure on $\mathbf{R} \times \mathbf{S}^1$. Since the metrics are independent of t, passing from $\mathbf{R}$ to $\mathbf{S}^1$ via $t \mapsto \exp(it)$ yields an irreducible neutral structure on $\mathbf{S}^1 \times \mathbf{S}^1$.

By unwinding the second factor once (by regarding it as the image of $\mathbf{S}^1$ under the antipodal mapping) one obtains a global Lorentz metric on $\mathbf{S}^1 \times \mathbf{S}^1$ as required by (2.6) (cf. (3.1) also). This global metric is not time-orientable, but time-orientability can be achieved by unwinding the second factor of $\mathbf{S}^1 \times \mathbf{S}^1$ one more time in the same fashion (again, as required by (2.6), but as applied to time-orientability).

Problem. Find an example of a manifold admitting an irreducible neutral structure that does not admit a neutral metric.

4. Differential geometry.

Let M be a neutral manifold of dimension $2n$, and let $\{U_i, g_i\}$ be a representation of the neutral structure. Obviously, every point of M has a neighborhood on which the differential geometry is just that of a neutral metric. Hence, it is the global geometry of M that is of interest. Since, in effect, there is a neutral metric defined on M up to sign, a simple approach to discovering which of the basic notions of differential geometry remain valid for a neutral manifold is to determine which are independent of a change of sign of the metric. It is convenient to employ the abstract index notation of Penrose [**20**, Chapter 2].

Proposition 4.1. *M admits a unique, torsion-free connection which preserves all local metric representatives of the neutral structure. Call it the Levi-Civita connection of the neutral structure.*

Proof. The classical construction [**13**, p. 160] of the Levi-Civita connection is independent of the sign of g. □

Corollary 4.2. *The Levi-Civita connection defines a Riemann curvature tensor $R^a{}_{bcd}$ on M, which locally equals the Riemann curvature tensor of any local metric representative. Consequently, the Ricci curvature tensor $R_{bd} := R^a{}_{bad}$ and the Weyl conformal curvature tensor C^a_{bcd} are globally defined.*

Remarks 4.3. (i) In terms of local coordinates, (4.1) may be viewed as the fact that Christoffel symbols of a metric are independent of its sign. At a more abstract level, the assertion of (4.1) follows from a result of Weyl (refined by É. Cartan, Klingenberg, and Kobayashi & Nagano) by virtue of the fact that the Lie algebra $\mathbf{no(n)}$ is of course just the Lie algebra $\mathbf{so(n,n)}$ [**12**, p. 86].

(ii) The various quantities in (4.2) have their usual symmetry properties since these are valid locally. Those that are independent of metrics may, of course, be stated for the tensors in the forms that are globally valid on M (the same is true of the Bianchi identity). Although R_{abcd} is defined only up to sign, $R_{(ab)cd} = 0$ is unambiguous.

(iii) On U_i one can define $R_i := (g_i)^{ab} R_{ab}$, a local Ricci scalar curvature, but it is defined only up to sign as far as the neutral structure is concerned. Nevertheless, the quantity $R_i g_i$ is globally defined whence the notion of an Einstein neutral manifold makes sense. This topic will be pursued in the next section.

(iv) The Killing equation is well defined on M, and its solutions are indeed the infinitesimal automorphisms of the neutral structure.

Now suppose M is oriented. Then there exists a nowhere-vanishing, $2n$-form ω on M which is positive on frames with the prescribed orientation. This defines a function $f : N(M) \to \mathbf{R}^*$ by $f(\{F_1, \dots, F_{2n}\}) = \omega(F_1, \dots, F_{2n})$. Now $f^{-1}(\mathbf{R}^+)$ defines a subbundle $SN(M)$ of $N(M)$ with structure group $SNO(n)$. By the determinant formula, f is constant on the fibers of $SN(M)$ and thus induces a strictly positive function on M, also denoted f. Define the volume form $\eta := \omega/f$. With respect to local oriented coordinates $(x^1, \dots, x^{2n})$ on U_i, $\eta = \sqrt{|\det(g_i)|}\, dx^1 \wedge \cdots \wedge dx^{2n}$ and this is independent of the sign of g_i.

I now wish to establish a Gauss-Bonnet-Chern formula for neutral manifolds. If N is a $2n$-dimensional manifold with a global neutral metric g, then for n even the Gauss-Bonnet-Chern formula states (cf. [**16**]):

$$\chi(M) = \frac{(-1)^{n/2}}{2^{3n}\pi^n n!} \int_M \gamma_n, \tag{4.4}$$

where

$$\gamma_n := (\eta_{i_1 \dots i_{2n}} \eta_{j_1 \dots j_{2n}} R^{i_1 i_2 j_1 j_2} \dots R^{i_{2n-1} i_{2n} j_{2n-1} j_{2n}})\eta$$

and R^{abcd} and η (with and without indices) are the Riemann curvature tensor and volume element, respectively, of the given metric g. If n is odd, both γ_n and $\chi(M)$ vanish.

On a neutral manifold M, the form γ_n may be constructed onU_i with respect to g_i. For n even, one observes that γ_n is invariant under a change of sign in g_i. Hence, for n even, γ_n is actually well defined locally and yields a globally defined $2n$-form on M. Does it represent the Euler class of M?

One can adapt Chern's [**6**] argument to the present context, and I shall just point out the underlying reason for this even though I shall provide a simpler proof below. As $\mathbf{O(p)} \times \mathbf{O(q)}$ is a maximal compact subgroup of $\mathbf{O(p,q)}$, an $\mathbf{O(p,q)}$-structure on a manifold may always be reduced to an $\mathbf{O(p)} \times \mathbf{O(q)}$-structure. Since $\mathbf{O(p)} \times \mathbf{O(q)}$ is a subgroup of $\mathbf{O(p+q)}$, to any $\mathbf{O(p,q)}$-structure there is an associated Riemannian structure. This is a crucial aspect of Chern's argument. Although an irreducible neutral structure does not have a neutral metric, it still has an associated Riemannian structure. This follows by observing that $\mathbf{NO(n)}$ has a natural maximal compact subgroup which is also a subgroup of $\mathbf{O(2n)}$.

Proposition 4.5. *Let M be a compact, oriented, $2n$-dimensional, neutral manifold and S a two-fold covering of M as in* (2.9). *S is a compact, orientable, $2n$-dimensional manifold with a neutral metric. The tangent bundle TS on S may be regarded as the pullback, by the projection $\pi : S \to M$, of the tangent bundle TM of M. Specify an orientation on S by requiring $\pi^*\eta$ to be the volume form on S. Then π has degree two. The following diagram is a commutative diagram of linear isomorphisms*:

$$\begin{array}{ccc} H^{2n}(M) & \xrightarrow{\pi^*} & H^{2n}(S) \\ {\scriptstyle\int_M}\big\downarrow & & {\scriptstyle\int_S}\big\downarrow \\ \mathbf{R} & \xrightarrow{\times 2} & \mathbf{R} \end{array}$$

where $H^{2n}(M)$ and $H^{2n}(S)$ are de Rham cohomology groups.

Proof. It is well known that integration on a compact, orientable manifold is a linear isomorphism of the top de Rham cohomology group to $\mathbf{R}$. Commutativity of the diagram follows from the degree theorem of integration theory in differential topology. □

Theorem 4.6. *Let M be a compact, orientable, $2n$-dimensional neutral manifold. If n is odd, $\chi(M)$ is zero. If n is even*

$$\chi(M) = \frac{(-1)^{n/2}}{2^{3n}\pi^n n!}\int_M \gamma_n,$$

where γ_n is defined as above and is a globally defined $2n$-form on M.

Proof. Only the case of an irreducible neutral structure need be considered. With the notation of (4.5), let $e(M)$ and $e(S)$ be the

Euler classes of M in $H^{2n}(M)$ and S in $H^{2n}(S)$, respectively. By naturality, $\pi^*(TM) = TS$ entails that $e(S) = \pi^*(e(M))$. From the diagram in (4.5), $\chi(S) = 2\chi(M)$. Thus, if n is odd, since S carries a metric g of signature type (n, n), then, by the Gauss-Bonnet-Chern theorem for pseudo-Riemannian manifolds cited earlier, $\chi(S) = 0$, whence $\chi(M) = 0$. For n even, let c_n denote the coefficient in front of the integral in the Gauss-Bonnet-Chern theorem for g on S and $\gamma_n(g)$ the integrand. Thus, $c_n\gamma_n(g)$ represents $e(S)$. But it is clear from (2.9) and (4.5) that $\gamma_n(g) = \pi^*(\gamma_n)$. From the diagram in (4.5) and the fact that the mappings are isomorphisms, it follows that $c_n\gamma_n$ represents $e(M)$. □

Corollary 4.7. *If M is a compact, orientable, two-dimensional neutral manifold, then M is homeomorphic to the torus.*

The Pontryagin classes of a manifold N may be represented by forms built out of the curvature of any connection. Avez [**4**] writes these forms explicitly in terms of the Weyl conformal curvature tensor of the Levi-Civita connection of an arbitrary metric on N to make manifest the conformal invariance discovered by Chern and Simons [**7**]. From (4.2), it is clear that the expressions given by Avez remain valid for the Levi-Civita connection of a neutral manifold. Thus, if a neutral manifold is conformally flat, its Pontryagin classes are zero.

If N is a compact, orientable, four-dimensional manifold equipped with a neutral metric g, then [**3**] the first Pontryagin class is represented by the four-form $c\Theta$ where

$$c := \frac{1}{32\pi^2} \qquad \Theta := (R^a{}_b{}^{cd} R_a{}^{bef} \eta_{cdef})\eta \tag{4.8}$$

and $R^a{}_{bcd}$ and η are the curvature tensor and volume form, respectively, of g. Consequently, the Hirzebruch signature theorem yields

$$\tau(M) = p_1/3 = \frac{1}{96\pi^2} \int_M \Theta. \tag{4.9}$$

It is evident that Θ is actually well defined locally on a neutral four-dimensional manifold by (4.2) and gives rise to a globally defined four-form.

Theorem 4.10. *Let M be a compact, oriented, four-dimensional neutral manifold. The form Θ in* (4.8) *is globally defined,* $[1/32\pi^2]\Theta$ *represents the first Pontryagin class of M and*

$$\tau(M) = \frac{1}{96\pi^2}\int_M \Theta.$$

Proof. The proof is similar to that of (4.6), noting that if $\Theta(g)$ is the form in (4.8) for (S, g) then $\Theta(g) = \pi^*\Theta$. $\square$

5. Neutral Einstein structures. Einstein manifolds are of considerable interest (cf. [**5**]). The four-dimensional case is special because of the characterization of the Einstein condition as the commutativity of the Hodge star operator $*$ and the curvature endomorphism on $\Lambda^2(\mathbf{R}^{2,2})$ (hereafter denoted Λ^2). In this section results of [**16**] are generalized from the case of Einstein neutral metrics (cf. [**16**]) to Einstein neutral structures.

First note that changing the sign of the metric on $\mathbf{R}^{2,2}$ does not alter the induced metric on Λ^2 or the Hodge star operator on the latter space. The curvature endomorphism $\mathcal{R}$ will change sign, but this does not affect whether it is Einstein or note, or its subsequent classification according to type (cf. [**16**]).

Rewriting the integrands of the integral formulae in (4.6) (with $n = 2$) and (4.10) in endomorphism language yields

$$\gamma_2 = 2^4 \mathrm{tr}\,(\mathcal{R} \circ * \circ \mathcal{R} \circ *)\eta \qquad \Theta = 2^3 \mathrm{tr}(\mathcal{R} \circ * \circ \mathcal{R})\eta$$

where tr denotes trace of the endomorphism. From Section 4, these 4-forms are globally defined on any compact, oriented, four-dimensional neutral manifold M (indeed, the independence of these formulae of the sign of a metric is evident). At a point of M, and with respect to a metric representative of the neutral structure, the curvature endomorphism for M Einstein has the form

$$\mathcal{R} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where A and D are self-adjoint endomorphisms of $\mathbf{R}^{1,2}$ with $\mathrm{tr}\,(A) = \mathrm{tr}\,(D) = R$, the Ricci scalar curvature of the metric representative (cf. [**16**]).

The penultimate equations therefore become

$$\gamma_2 = 2^4[\mathrm{tr}\,(A^2) + \mathrm{tr}\,(D^2)]\eta \qquad \Theta = 2^3[\mathrm{tr}\,(A^2) - \mathrm{tr}\,(D^2)]\eta$$

whence, from (4.6) and (4.10), one obtains

$$\begin{aligned} -\chi(N) + \frac{3}{2}\tau(N) &= \frac{1}{4\pi^2}\int_M \mathrm{tr}\,(A^2)\eta \\ -\chi(N) - \frac{3}{2}\tau(N) &= \frac{1}{4\pi^2}\int_M \mathrm{tr}\,(D^2)\eta. \end{aligned} \tag{5.1}$$

Writing $A =: W^+ + (R/12)\mathbf{1}$ and $D =: W^- + (R/12)\mathbf{1}$, W^+ and W^- are traceless. They are the curvature endomorphisms of the self-dual and anti-self-dual parts of the Weyl conformal curvature, respectively. Thus,

$$\mathrm{tr}\,(A^2) = \mathrm{tr}\,[(W^+)^2] + R^2/48, \qquad \mathrm{tr}\,(D^2) = \mathrm{tr}\,[(W^-)^2] + R^2/48. \tag{5.2}$$

These results are exactly as in [**16**] for neutral Einstein metrics and, as there, further deductions may be made based on the classification of curvature endomorphisms. As demonstrated in [**16**], there are 4 types (Ia, Ib, II, and III) of self-adjoint endomorphism of $\mathbf{R}^{1,2}$, and thus 16 types (of the form Ia $\times$ II, for example), for the curvature endomorphism corresponding to the types of A and D or, equivalently, W^+ and W^-. Consult [**16**] for the fact that for any self-adjoint endomorphism P of $\mathbf{R}^{1,2}$, $\mathrm{tr}\,(P^2)$ is nonnegative except perhaps when P is of type Ib. From these facts and (5.2), one deduces the following result.

Theorem 5.3. *Let M be a compact, oriented, four-dimensional, Einstein neutral manifold. Let W^+ and W^- be determined by any local metric representatives of the neutral structure.*

(a) *If W^+ is never of type* Ib*, then $-\chi(M) + (3/2)\tau(M) \geq 0$.*

(b) *If W^- is never of type* Ib*, then $-\chi(M) - (3/2)\tau(M) \geq 0$.*

(c) *If* (a) *and* (b) *hold, then $-\chi(M) \geq (3/2)|\tau(M)|$. In particular, $\chi(M) \leq 0$.*

As already noted in [**16**], the geometric significance of prohibiting type Ib in the curvature, or more generally requiring the relevant trace to be nonnegative if type Ib occurs, deserves investigation and will be pursued elsewhere. I end with some further simple deductions.

Corollary 5.4. *Let M be as in* (5.3).

(a) *If W^+ is of type* Ia *everywhere, then $\chi(M) = (3/2)\tau(M)$ if and only if R^2 and W^+ both vanish.*

(b) *If W^- is of type* Ia *everywhere, then $\chi(M) = -(3/2)\tau(M)$ if and only if R^2 and W^- both vanish.*

(c) *If M is of type* Ia $\times$ Ia *everywhere, then $\chi(M) = 0$ if and only if M is flat, in which case $\tau(M) = 0$.*

(d) *If M is of type* III $\times$ III *everywhere, then $\tau(M)$ vanishes and $\chi(M) = [-R^2/2^6 3\pi^2]\mathrm{Vol}\,(M)$.*

Proof. (a) follows from (5.1), (5.2) and (2.3)(a) of [**16**]. Similarly for (b). For type Ia $\times$ Ia, the integrands in (5.1) are nonnegative whence the vanishing of $\chi(M)$ entails the vanishing of $\tau(M)$ and so of the integrands, and (c) follows from (2.3)(a) of [**16**] again. For type III $\times$ III, tr $(A^2) = \mathrm{tr}\,(D^2) = R^2/48$ (cf. [**16**], (2.3)(d)) so (d) follows from the expression for γ_2 and Θ given just before (5.1) and (4.6) with $n = 2$. □

Corollary 5.5. *If M is a compact, oriented, four-dimensional, Ricci-flat, half-conformally-flat, neutral manifold, then $\chi(M) = (3/2)\tau(M)$ or $-(3/2)\tau(M)$ according to whether W^+ or W^- vanishes.*

Proof. For example, if $W^- = 0$, then $\mathrm{tr}\,(D^2) = 0$ by (5.2), whence $\chi(M) = -(3/2)\tau(M)$ from (5.1). □

Now specialize to the case in which M is compact, oriented, four-dimensional and carries a neutral metric. Suppose further that in addition to orientability, M is orientable in any other of the senses discussed in (2.7)(ii). Then, in fact, M is orientable in all those senses. In particular, M admits a nonsingular field of oriented two planes.

Such fields, with possibly finitely many singularities, have been studied by Hirzebruch and Hopf [**9**] (also described in [**22**, p. 659]). The existence of a field of orientable two-planes, on a compact, oriented, four-dimensional manifold, without singularities, requires [**9**, Satz (4.3)] the vanishing of the following two expressions:

$$\alpha - 3\tau(M) - 2\chi(M), \qquad \beta - 3\tau(M) + 2\chi(M)$$

where, by a result on page 90 of [**10**], $\alpha = \tau(M)+8j$ and $\beta = \tau(M)+8k$, for some integers j and k. One easily deduces that $\chi(M) \equiv \tau(M)_{\text{mod}4}$ and $\chi(M)$ and $\tau(M)$ are both even. This is a special case of a result due to Atiyah [**1**, Theorem (3.1)]. The following results are analogous to results in [**17, 18**] based upon corrections discussed in [**16**].

Corollary 5.6. *Let M be a compact, four-dimensional manifold with an $\mathbf{SO}^+(\mathbf{2},\mathbf{2})$-structure that is Einstein. Then*:

(a) $\chi(M) \equiv \tau(M)_{\text{mod}4}$, *and both are even as just shown.*

(b) *By Poincaré duality and connectedness, $\chi(M) = 2 - 2b_1 + b_2$, where the b_i are the Betti numbers. Thus, b_2 is even. If $\chi(M) = -2k \leq 0$, then $2b_1 = 2 + 2k + b_2$ so $b_1 > 0$, i.e., M is not simply connected.*

(c) *If $-\chi(M) \geq (3/2)|\tau(M)|$ (cf. (5.3)(c)), then $\tau(M) = \chi(M) + 4k$, where k is a positive integer and $k = 0$ if and only if $\tau(M) = \chi(M) = 0$.*

(d) *If the curvature endomorphism $\mathcal{R} = W^+$, then $\tau(M) = 8m$ and $\chi(M) = -12m$ for some integer m, while if $\mathcal{R} = W^-$, then $\tau(M) = -8m$ and $\chi(M) = -12m$, for some integer m. Except perhaps when W^+ or W^- is type* Ib, *m must be nonnegative.*

(e) *If $\mathcal{R}$ has type* III $\times$ III *and R is nonzero, then $\tau(M) = 0$ and $\chi(M) = -4k$, $k > 0$. Furthermore,* $\text{Vol}\,(M) = 2^8 3\pi^2 k/R^2$.

Proof. For (c), write $4k = \tau(M) - \chi(M) \geq \tau(M) + (3/2)|\tau(M)| \geq 0$. Putting $k = 0$ in this inequality forces $\tau(M) = 0$ and $\chi(M) = 0$. For (d), if $\mathcal{R} = W^+$, then $-\chi(M) = (3/2)\tau(M)$ by (5.5). By (a), $4k = \tau(M) - \chi(M) = (5/2)\tau(M)$, i.e., $\tau(M) = 8k/5$, whence k must be divisible by five and $\tau(M) = 8m$, for some integer m. Then $\chi(M) = -12m$. If w^+ is not type Ib, then m must be nonnegative by (5.3)(c). If $\mathcal{R} = W^-$, the appropriate assertion follows similarly, or simply from the result just established by reversal of orientation. Finally, (e) follows from (5.4)(d); in particular, $\chi(M)$ must be a strictly negative multiple of four. $\square$

In a rather different vein, I conclude with a variation of a result of Avez [**2**].

Proposition 5.7. *Let M be a compact, orientable, four-dimensional manifold equipped with a metric g of signature type (p,q) with both p and q nonzero. Suppose that at each point x of M, the sectional curvatures of nondegenerate two-planes in T_xM are bounded, either below or above. Then (M,g) is a space of constant curvature and $(-1)^{[p/2]}\chi(M) \geq 0$ with equality if and only if (M,g) is flat. If g is Lorentzian, $\chi(M)$ is zero so (M,g) is flat. If g is neutral, $\chi(M) \leq 0$.*

Proof. The condition on the sectional curvatures at x entails their constancy at x [**19**, pp. 229–230], whence by Schur's lemma M is a space of constant curvature and hence an Einstein space. For an Einstein space, the integrand γ_2 in the Gauss-Bonnet-Chern formula [**16**, (3.1)] is a positive multiple of $(R^{abcd}R_{abcd})$ (cf. the formula for γ_2 preceding (5.1)), which for a space of constant curvature is a positive multiple of R^2, where R is the Ricci scalar curvature. The results now follow from the Gauss-Bonnet-Chern theorem [**16**, (3.1)]. □

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