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DEGREES OF CLOSED CURVES IN THE PLANE

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ABSTRACT. In the present article we extend the notion of degree from regular closed curves to closed locally one-to-one curves and prove that the extended notion has analogous properties. In particular, a natural generalization of Whitney-Graustein's theorem is still true. A proof of a mean value theorem for nonstop curves is given using only the elementary ideas of this paper.

1. Introduction. Let us first recall some important definitions.

A curve $C : I \to \mathbf{R}^2$ is *regular* if it is continuously differentiable and if $C'(t) \neq 0$ for all $t \in I$.

The map $H : I \times I \to \mathbf{R}^2$ is a *regular homotopy* if the curve $H_u(t) = H(u,t)$ is regular for each u and if both H_u and its derivative vary continuously with u. If H is a regular homotopy, then H_0 and H_1 are said to be regularly homotopic.

If $C : I \to \mathbf{R}^2$ is a (continuous) curve such that $C(t) \neq 0$ for all $t \in I$, then the *winding number* $W(C)$ of C around 0 is defined as follows. Identify $\mathbf{R}^2$ with the complex plane, and write C as $C(t) = r(t)e^{2\pi i a(t)}$ where both r and a are continuous functions and r is positive. Let $W(C)$ be the difference $a(1) - a(0)$. If C is a closed curve $W(C)$ is clearly an integer. W is also homotopy invariant in the following sense: if two curves $C_1, C_2 : I \to \mathbf{R}^2$ are homotopic by a homotopy $H : I \times I \to \mathbf{R}^2 - \{0\}$ such that H_u, defined by $H_u(t) = H(u,t)$, is a closed curve for all $u \in I$, then $W(C_1) = W(C_2)$. The winding number of a curve counts the algebraic number of times the curve goes around the origin. If C is a closed curve it follows from the definition of the winding number that the vector $C(t)$ points in every direction for at least $|W(C)|$ different values of t. A very readable discussion of the winding number is given in [**1**].

If C Is a regular closed curve, then C' is a closed curve in $\mathbf{R}^2$ missing the origin. Therefore, $W(C')$ can be defined. $D(C) = W(C')$ is called

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the *degree* of C and measures the number of turns of C. $C'(t)$ points in every direction for at least $|D(C)|$ different values of t.

In 1937, H. Whitney classified regular homotopy classes of regular closed curves in the plane using the notion of degree (see [**5**]). He proved that two regular closed curves in the plane are regularly homotopic if and only if their degrees are equal (Whitney-Graustein's theorem). He also showed that the degree and the self-intersection number have different parity.

In this paper we introduce the notion of a "gentle homotopy" and extend the definition of degree to (continuous) locally one-to-one curves by defining it to be the winding number of short enough secant vectors. The main results are the following generalizations of Whitney's results.

Theorem 1. *Suppose $C_0, C_1 : I \to \mathbf{R}^2$ are closed locally 1–1 curves. Then*

$$D(C_0) = D(C_1)$$

if and only if C_0 is gently homotopic to C_1.

Theorem 2. *If $C : I \to \mathbf{R}^2$ is a normal curve, then*

$$D(C) \equiv 1 + \text{self}\,(C) \pmod 2.$$

We also discuss degrees of differentiable closed locally 1–1 curves which are not necessarily continuously differentiable but have nonvanishing derivative. We prove that in that case the tangent vector also points in every direction at least n times if n is the absolute value of the degree (Theorem 3). This is followed by mean-value theorems for nonstop curves (Corollaries 1 and 2) which generalize a result of [**4**]. In the final section we give a different proof of that result (Proposition 4).

The material is organized as follows. Section 2 contains a technical lemma needed in the proof of Theorem 1. Section 3 gives proofs of Theorems 1 and 2. In Section 4 we study degrees of differentiable curves and prove Theorem 3. Mean-value theorems are proved in Section 5, and a proof of Proposition 4 is given in the last section.

The only section where the tools go beyond the winding number is Section 2 in which we use Schönflies theorem. We use that section only for construction of gentle homotopies needed in the proof of Theorem 1. That material is not used in the rest of the paper. Therefore, all other results can be studied without reference to Section 2. In particular, this is the case with the proof of Proposition 4. Therefore, the proof given in the last section is more elementary than the original proof in [**4**] which uses the Jordan curve theorem.

2. Approximating locally 1–1 curves by regular curves. If $C : I \to \mathbf{R}^2$ ($I = [0, 1]$) is a closed curve let $\tilde{C} : \mathbf{R} \to \mathbf{R}^2$ be the periodic extension defined by $\tilde{C}(t) = C(t - [t])$ ($[t]$ is the integer part of t).

We shall say that a closed curve $C : I \to \mathbf{R}^2$ is locally 1–1 if its extension $\tilde{C} : \mathbf{R} \to \mathbf{R}^2$ is. In that case there exists a $\lambda > 0$ such that $\tilde{C}(s) \neq \tilde{C}(t)$ whenever $0 < |s - t| < \lambda$. A closed curve $C : I \to \mathbf{R}^2$ is differentiable if $\tilde{C} : \mathbf{R} \to \mathbf{R}^2$ is.

We shall say that a homotopy $H : I \times I \to \mathbf{R}^2$ is locally 1–1 if the function $\tilde{H} : I \times \mathbf{R} \to \mathbf{R}^2$ defined by $(u, t) \to H(u, t - [t])$ is locally 1–1. If $H : I \times I \to \mathbf{R}^2$ is a locally 1–1 homotopy, then we will say that H_0 and H_1 are *gently homotopic*. We require that H_u be a closed curve for all $u \in I$.

If a homotopy $H : I \times I \to \mathbf{R}^2$ is locally 1–1, then there exists a $\lambda > 0$ such that $\tilde{H}(u, s) \neq \tilde{H}(u, t)$ whenever $0 < |s - t| < \lambda$, for all $u \in I$. This follows from the compactness of I.

Clearly all regular curves and regular homotopies are locally 1–1.

The following lemma is the main result of this section.

Lemma 1. *Any closed locally* 1–1 *curve is gently homotopic to a regular curve.*

Proof. Let $C : I \to \mathbf{R}^2$ be a closed locally 1–1 curve. Choose a $\lambda > 0$ such that $\tilde{C}(s) \neq \tilde{C}(t)$ whenever $0 < |t - s| < \lambda$. Let $\mu = 3\lambda/4$. We shall construct a sequence of maps $h_i : [t_0, t_i] \times I \to \mathbf{R}^2$ such that

(i) $h_i(t, 0) = C(t)$.

(ii) $h_i(t, u) \neq h_i(s, u)$ whenever $0 < |t - s| < \mu/3$, $u \in I$,

(iii) the maps $[t_0 - 2\mu/3, t_0 + \mu/3] \times 0 \cup [t_0, t_0 + \mu/3] \times I \to \mathbf{R}^2$ and

$$(*) \qquad [t_i - \mu/3, t_i + 2\mu/3] \times 0 \cup [t_i - \mu/3, t_i] \times I \to \mathbf{R}^2,$$

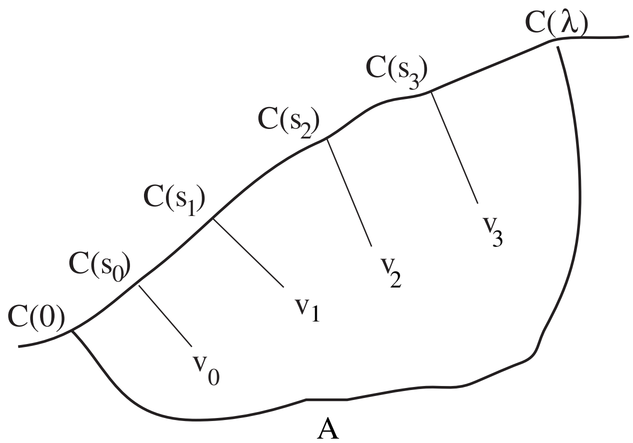

FIGURE 1.

defined to be $\tilde{C}$ on $[t_0-2\mu/3, t_0+\mu/3]\times 0$ and on $[t_i-\mu/3, t_i+2\mu/3]\times 0$, and of h_i on $[t_0, t_0+\mu/3]$ and on $[t_i-\mu/3, t_i]$, are embeddings.

(iv) $h_i \mid [t_0, t_i] \times 1$ is piecewise-linear.

We shall assume that the reader is familiar with a proof of Schönflies theorem as given, for example, in [**3**]. We shall also use the terminology of [**3**].

Construction of h_1. Connect the endpoints of $C([0,\lambda])$ by an arc A so that the curve $J = A \cup C([0,\lambda])$ is an embedded 1-sphere (see Figure 1). Choose points $s_0 \leq s_1 \leq s_2 \leq s_3$ in $(0,\lambda)$ such that

(i) $C(s_j)$, $j = 0,1,2,3$, are linearly accessible from the interior R of J (by the interior of J we mean the bounded component of $\mathbf{R}^2 \backslash J$),

(ii) $s_{j+1} - s_j > \mu/3$ for $j = 0,1,2$.

Let $v_j C(s_j)$, $j = 0,1,2,3$, be disjoint linear intervals such that $v_j C(s_j) - C(s_j)$ lies in R for all j. There exists an embedding

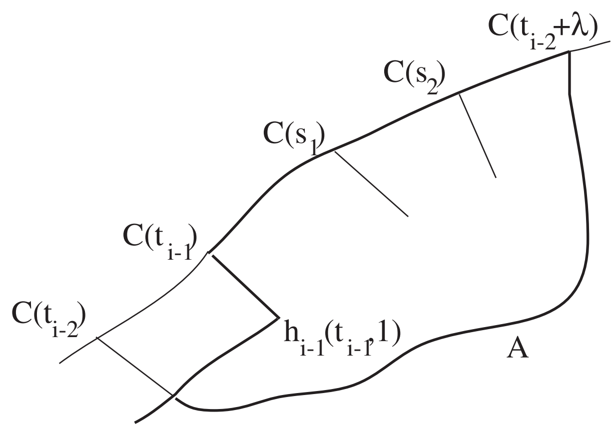

FIGURE 2.

$H : [s_0, s_3] \times I \to \mathbf{R}^2$ satisfying

$$(1)\qquad \begin{array}{rl} \text{(i)} & H(t,0) = C(t) \text{ for } t \in [s_0, s_3], \\ \text{(ii)} & \text{the curve } t \to H(t,1) \text{ is piecewise-linear}, \\ \text{(iii)} & H(s_j \times I) \text{ is contained in } v_j C(s_j), \text{ for each } j. \end{array}$$

Let $t_0 = s_1$, $t_1 = s_2$. By changing H, if necessary, the map $h_1 = H \mid [t_0, t_1] \times I$ will satisfy $(*)$ (i.e., in order to satisfy (iii), one might have to replace H by $H \circ k$ where $k : [t_0, t_1] \times I \to [t_0, t_1] \times I$ is given by $k(t, u) = (t, cu)$ for some small enough constant c).

Suppose now that $h_{i-1} : [t_0, t_{i-1}] \times I \to \mathbf{R}^2$, satisfying $(*)$, has been constructed. Connect $h_{i-1}(t_{i-2}, 1)$ to $C(t_{i-2} + \lambda)$ by an arc A so that the union

$$A \cup h_{i-1}([t_{i-2}, t_{i-1}] \times 1) \cup h_{i-1}(t_{i-1} \times I) \cup C([t_{i-1}, t_{i-2} + \lambda])$$

is an embedded 1-sphere J. Choose A so that $h_{i-1}([t_{i-2}, t_{i-1}] \times I)$ does not intersect the interior R of J (see Figure 2).

Choose points $s_0 \le s_1 \le s_2$ in $(t_{i-1}, t_{i-2} + \lambda)$ such that

(i) $s_0 = t_{i-1}$,

(ii) $\mu/3 < s_{j+1} - s_j \leq \lambda/3$, for $j = 0, 1$,

(iii) $C(s_j)$ are linearly accessible from R.

Let $v_j C(s_j)$, $j = 1, 2$, be disjoint linear intervals such that $v_j C(s_j) - C(s_j)$ lies in R, for $j = 1, 2$. There exists an embedding $H : [s_0, s_2] \times I \to \mathbf{R}^2$ satisfying (1). Let $t_i = s_1$ and extend h_{i-1} to h_i by letting

$$h_i \mid [t_{i-1}, t_i] \times I = H \mid [t_{i-1}, t_i] \times I.$$

Again, H might need an adjustment in order to have h_i satisfy property (iii) of $(*)$.

We shall now use the maps h_i to construct a mapping $h : [t_0, t_0+1] \times I \to \mathbf{R}^2$ satisfying properties (i), (ii) and (iv) of $(*)$ and such that the curve $t \to h(t, u)$ is closed for every $u \in I$.

From the sequence $t_0, t_1, t_2, \ldots$, choose the first t_i such that $1-\lambda/3 < t_i - t_0 < 1$ (thus $0 < (t_0+1) - t_i < \lambda/3$). If necessary, adjust h_i so that $h_i([t_{i-1}, t_i] \times I)$ and $h_i([t_0, t_1] \times I)$ do not intersect. Connect $h_i(t_{i-1}, 1)$ so $h_i(t_1, 1)$ by an arc A so that the union

$$A \cup h_i(\{t_{i-1}, t_1\} \times I) \cup C([t_{i-1}, t_1 + 1])$$

is an embedded 1-sphere J containing $h_i(t_i \times (0, 1])$ in its interior and such that $J \cap h_i(t_0 \times (0, 1]) = \varnothing$. Suppose $h_i(t_0, 1)$ does not lie in R. Let α be a curve in R from $h_i(t_i, 1)$ to $C(t_0)$ (see Figure 3). Then the closed curve defined by the union of $h_i([t_0, t_i] \times 1)$, $h_i(t_0 \times I)$ and α intersects C with nonzero intersection number. (Recall that the intersection number of two curves in the plane is the reduction modulo 2 of the number of intersections when the two curves are put in general position.) Since this is clearly impossible, $h_i(t_0, 1)$ also has to lie in R. There exists an embedding $H : [t_i, t_0 + 1] \times I \to \mathbf{R}^2$ satisfying (1). Using H we can extend h_i to a map $h : [t_0, t_0+1] \times I \to \mathbf{R}^2$ with desired properties. This map clearly defines a gentle homotopy from C to a piecewise-linear locally one-to-one closed curve. Since any piecewise-linear locally one-to-one closed curve can be homotoped to a regular curve by an arbitrarily small gentle homotopy (see the Appendix), the proof of Lemma 1 is completed. □

3. Degrees of curves. In this section we define the degree of a closed locally 1–1 curve and prove that it completely determines the

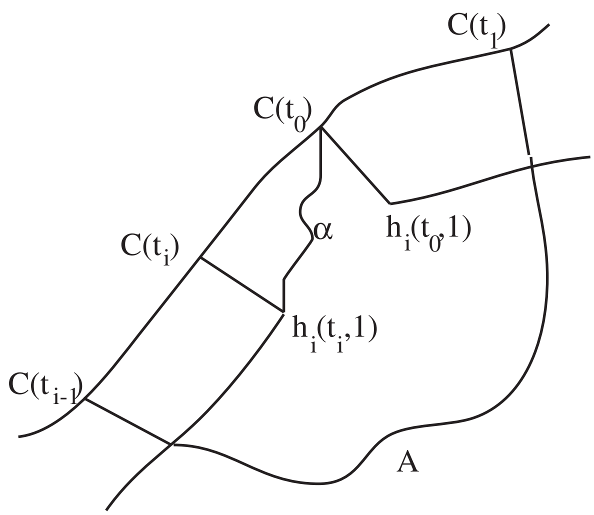

FIGURE 3.

gentle homotopy class of such a curve. We also prove that there is a relation between the degree and the self-intersection number.

Suppose that $C : I \to \mathbf{R}^2$ is a closed locally 1–1 curve. If $h > 0$, define $c^h : I \to \mathbf{R}^2$ by

$$c^h(t) = \frac{\tilde{C}(t+h) - \tilde{C}(t)}{h}.$$

If $k > 0$ is another number, then c^h and c^k are homotopic by the homotopy $u \to c^{(1-u)h+uk}$. Let $\lambda > 0$ be a number such that $\tilde{C}(t) \neq \tilde{C}(s)$ if $0 < |t-s| \leq \lambda$. Then $c^z(t) \neq 0$ for all $t \in I$, and for all z such that $0 < z \leq \lambda$. Therefore, $c^{(1-u)h+uk}(t)$ is also nonzero for all $u, t \in I$ and for all nonnegative numbers h, k which are less than or equal to λ. It follows from the homotopy invariance of the winding number that $W(c^h) = W(c^k)$ whenever $0 < h,\ k \leq \lambda$. (Recall that $W(f)$ denotes the winding number of f around 0.)

Definition. If $C : I \to \mathbf{R}^2$ is a closed locally 1–1 curve, let the *degree* $D(C)$ of C be equal to

$$\lim_{h \to 0^+} W(c^h).$$

By the preceding remarks, $D(C)$ exists.

Note that this definition coincides with the original definition when C is a regular closed curve. To see that, choose a positive number λ such that $\tilde{C}(t) \neq \tilde{C}(s)$ whenever $|t-s| \leq \lambda$ and define a homotopy $H : I \times I \to \mathbf{R}^2 - \{0\}$ as follows:

$$H(u,t) = \begin{cases} C'(t), & u = 0 \\ c^{u\lambda}(t), & u \neq 0. \end{cases}$$

By the homotopy invariance of winding numbers, we get $W(C') = W(c^\lambda) = D(C)$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. If C_0 is gently homotopic to C_1, then there exists a homotopy $H : I \times I \to \mathbf{R}^2$ from C_0 to C_1 and a $\lambda > 0$ such that $\tilde{H}_u(s) \neq \tilde{H}_u(t)$ whenever $0 < |s-t| \leq \lambda$ for all $u \in I$. Therefore, the mapping $h^\lambda : I \times I \to \mathbf{R}^2 - 0$ defined by

$$h_u^\lambda(t) = \frac{\tilde{H}_u(t+\lambda) - \tilde{H}_u(t)}{\lambda}$$

is a nonzero homotopy from c_0^λ to c_1^λ. By the homotopy invariance of the winding number we get

$$D(C_0) = W(h_0^\lambda) = W(h_1^\lambda) = D(C_1).$$

Suppose now that $D(C_0) = D(C_1)$. By Lemma 1 we can find locally 1–1 homotopies H^0 and H^1 such that $H_j^j = C_j$, $j = 0, 1$, and such that H_1^0 and H_0^1 are regular closed curves. By the first half of proof, we have $D(H_1^0) = D(C_0)$ and $D(H_0^1) = D(C_1)$. Since $D(C_0) = D(C_1)$, the degrees of H_1^0 and H_0^1 are equal. Therefore, the curves H_1^0 and H_0^1 are regularly homotopic by a regular homotopy F (see [**5**]). The following homotopy

$$H(u,t) = \begin{cases} H^0(3u,t), & u \in [0,1/3] \\ F(3u-1,t), & u \in [1/3,2/3] \\ H^1(3u-2,t), & u \in [2/3,1] \end{cases}$$

is a gentle homotopy from C_0 to C_1.

This finishes the proof of Theorem 1. $\square$

A point $P \in \mathbf{R}^2$ is a *singular point* of C if $C^{-1}(P)$ contains more than one point. P is a *double point* of C if $C^{-1}(P)$ contains two points.

Suppose P is a double point of C, say $P = C(t_1) = C(t_2)$, $t_1 < t_2$. P is a *normal crossing* if

(i) there exist disjoint closed intervals $I_1, I_2 \subset \mathbf{R}$ such that $t_j \in I_j$, $j = 1, 2$,

(ii) there exists a closed disc D with center at P such that $C \mid I_j : (I_j, \partial I_j) \to (D, \partial D)$ is an embedding, and such that $(C \mid I_j)^{-1}(\partial D) = \partial I_j$, $j = 1, 2$,

(iii) $C(I_1) \cap C(I_2) = \{P\}$,

(iv) $C(\partial I_1)$ and $C(\partial I_2)$ are linked in ∂D.

Definition. A closed curve $C : I \to \mathbf{R}^2$ is *normal* if it is locally 1–1 and if all its singularities are normal crossings.

A normal curve can have only finitely many double points because they form a discrete subset of a compact set.

If $C : I \to \mathbf{R}^2$ is a normal curve, let the *self-intersection number* self (C) of C be the number of double points of C.

Note. The self-intersection number of the curve $t \to C(t + x)$ is equal to self (C).

Proposition 1. *Suppose that $C : I \to \mathbf{R}^2$ is a locally* 1–1 *curve, and suppose that P is a double point of C, say $P = C(t_1) = C(t_2)$, $t_1 \neq t_2$. If $C'(t_1)$ and $C'(t_2)$ are linearly independent, then P is a normal crossing.*

Before proving this, we need the following proposition:

Proposition 2. *Suppose that $C : I \to \mathbf{R}^2$ is differentiable at $t \in I$, and suppose that $C'(t) \neq 0$. Given $\alpha > 0$, there exists a $\delta > 0$ such that if $|h| < \delta$, then*

(i) $|C(t+h)-C(t)| > (1/2)|C'(t)|\,|h|$

(ii) *the angle between* $C'(t)h$ *and* $C(t+h)-C(t)$ *is less than* α.

Proof. Let

$$\phi(h) = \begin{cases} \dfrac{(C(t+h)-C(t))\cdot C'(t)h}{|C(t+h)-C(t)|\,|C'(t)|\,|h|} & \text{if } h \neq 0 \\ 1 & \text{if } h = 0 \end{cases}$$

and

$$\Psi(h) = \begin{cases} \dfrac{|C(t+h)-C(t)|}{|h|} & \text{if } h \neq 0 \\ |C'(t)| & \text{if } h = 0 \end{cases}$$

Both ϕ and Ψ are continuous at 0; therefore, there exists a $\delta > 0$ such that

$$(-\delta, \delta) \subset \phi^{-1}((\cos\alpha, 1]) \cap \Psi^{-1}((|C'(t)|/2, \infty)).$$

Therefore, if $0 < |h| < \delta$ we have

$$\left| \frac{C(t+h)-C(t)}{h} \right| > \frac{1}{2}|C'(t)|$$

which implies (i) and

$$\phi(h) > \cos\alpha.$$

But $\phi(h)$ is the cosine of the angle between $C(t+h)-C(t)$ and $C'(t)h$ which proves (ii). $\square$

Proof of Proposition 1. Choose $\lambda > 0$ so that $\tilde{C}(s) \neq \tilde{C}(t)$ if $0 < |s-t| < \lambda$. If α_j is the angle between $C'(t_1)$ and $(-1)^j C'(t_2)$, for $j = 1, 2$, let $\alpha = \min\{\alpha_1, \alpha_2\}$. Since $C'(t_1)$ and $C'(t_2)$ are linearly independent, α is positive. By Proposition 2, there exists a positive number $\delta < \lambda$ such that if $|h| \leq \delta$, then

(a) $|C(t_i+h)-C(t_i)| \geq (1/2)|C'(t_i)|\,|h|$

(b) the angle between $C(t_i+h)-C(t_i)$ and $C'(t_i)h$ is less than $\alpha/3$ for $i = 1, 2$.

Let $r = (\delta/2)\min\{|C'(t_1)|, |C'(t_2)|\}$, and let D be the closed disc of radius r with center at P. Define functions ϕ_j^+ and ϕ_j^-, $j = 1, 2$, by

$$\phi_j^{\pm}(\tau) = |C(t_j \pm \tau) - C(t_j)|, \quad \text{for } \tau \in [0, \delta].$$

Since $|\phi_j^{\pm}(\delta)| \geq (1/2)|C'(t_j)| \cdot | \pm \delta| \geq r$, and since $\phi_j^{\pm}(0) = 0$, for $j = 1, 2$, the compact sets $A_j^{\pm} = (\phi_j^{\pm})^{-1}(r)$, $j = 1, 2$, are nonempty. Let I_j be the interval

$$[\min A_j^-, \min A_j^+], \qquad \text{for } j = 1, 2.$$

Then $C(I_j) \cap \partial D = C(\partial I_j)$, $j = 1, 2$. Let Z_j be the set of all points Q such that the vector from P to Q forms an angle smaller than $\alpha/3$ with the line which passes through P in the direction of $C'(t_j)$. Then $Z_1 \cap Z_2 = \{P\}$. By (b) the set $C(I_j)$ is contained in Z_j, $j = 1, 2$. Therefore, $C(I_1) \cap C(I_2) = \{P\}$. $C(\partial I_1)$ and $C(\partial I_2)$ are linked in ∂D because the components of $Z(I_j) \cap \partial D$ lie in different components of $\partial D - Z(I_{3-j})$ for $j = 1, 2$. This proves the proposition. $\square$

A proof from [**5**] can be modified to get the following proof of Theorem 2.

Proof of Theorem 2. Let S be the set of pairs (s, t) such that $0 \leq s < t < s + 1 \leq 2$. If $l(s, t) = \min\{t - s, 1 + s - t\}$ define $f_C : S \to \mathbf{R}^2$ by

$$f_C(s, t) = \frac{\tilde{C}(t) - \tilde{C}(s)}{l(s, t)}.$$

Clearly, f_C is continuous.

Whenever needed we shall assume that $C(0)$ is not a double point of C. For, if $C(0)$ is a double point of C, then we can choose a number x such that $C(x)$ is not a double point. Define a curve C_x by $C_x(t) = \tilde{C}(t+x)$. Let $H_u(t) = \tilde{C}(t+ux)$. Then H is a gentle homotopy from C to C_x: if $0 < |t - s| < \lambda$, then $0 < |(t + ux) - (s + ux)| < \lambda$, and therefore

$$H_u(s) = \tilde{C}(s + ux) \neq \tilde{C}(t + ux) = H_u(t).$$

Suppose that $P_1, \ldots, P_k$ are the double points of C, and let t_j, s_j, where $t_j < s_j$, be the elements of $C^{-1}(P_j)$, for $j = 1, \ldots, k$. For each j, choose a closed disc D_j with center at P_j and a pair of disjoint intervals $I_j = [u_j, v_j] \subset I$ and $J_j = [x_j, y_j] \subset I$, such that $t_j \in I_j$, $s_j \in J_j$, and which satisfy the four conditions in the definition of a normal crossing. Assume, furthermore, that the discs $D_1, \ldots, D_k$ are pairwise disjoint.

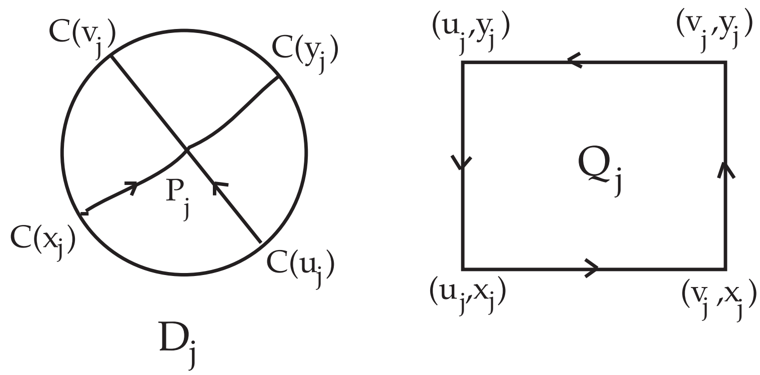

FIGURE 4.

Let $Q_j \subset S$ be the rectangle $I_j \times J_j$. Suppose that the pair $(C(y_j) - P_j, C(v_j) - P_j)$ determines the standard orientation of $\mathbf{R}^2$. We will show that the winding number of f_C as we go around ∂Q_j once in the positive direction is 1.

If $s \in I_j$, then $C(s) \in D_j$. Therefore, the vector $f_C(s, x_j) = (\tilde{C}(x_j) - \tilde{C}(s))/l(s, x_j)$ does not point in the direction of $P_j - C(x_j)$ if $s \in I_j$. It follows that the winding number ω_1 of f_C as we go from (u_j, x_j) to (v_j, x_j) along a straight line is equal to the angle between $f_C(u_j, x_j)$ and $f_C(v_j, x_j)$ divided by 2π. Clearly, $0 < \omega_1 < 1/2$. Similarly, the winding numbers $\omega_2, \omega_3, \omega_4$ of f_C as we go along the other three sides of ∂Q_j in the counterclockwise direction also satisfy the same condition. Therefore, the winding number $\omega_1 + \omega_2 + \omega_3 + \omega_4$ of f_C as we go once around ∂Q_j in the counterclockwise direction is between 0 and 2. Since it is an integer, it has to be $+1$.

In the same way we can see that the winding number of f_C as we go around ∂Q_j in the counterclockwise direction is -1 provided that the pair $(C(y_j) - P_j C(v_j) - P_j)$ determines the negative orientation of $\mathbf{R}^2$.

If necessary, reindex the quadrilaterals Q_j, $j = 1, \ldots, k$ in such a way that the winding number of f_C as we go once around ∂Q_j in the counterclockwise direction is -1 for $j \leq m$, and $+1$ if $j > m$.

Choose a number $0 < \lambda < 1/2$ such that $\tilde{C}(t) \neq \tilde{C}(s)$ whenever $0 < |t - s| \leq \lambda$. Let Q_λ be the quadrilateral with vertices $(0, \lambda)$, $(1 - \lambda, 1)$, $(\lambda, 1)$, and $(0, 1 - \lambda)$. Clearly, $Q_\lambda \subset S$ and all the zeros of f_C lie in $\operatorname{int}(Q_\lambda)$. Therefore, all the rectangles Q_i can be made small enough to be contained in $\operatorname{int}(Q_\lambda)$. Subdivide the rest of Q_λ into triangles. By the homotopy invariance of the winding number, the winding number of f_C as we go once around any one of these triangles is zero. If we add the winding numbers of f_C as we go once in the counterclockwise direction around each of these triangles and around $\partial Q_1, \dots, \partial Q_k$, we get $-m + (k - m) = k - 2m = \operatorname{self}(C) - 2m$. On the other hand, this sum is also equal to the winding number W of f_C as we go once around ∂Q_λ in the counterclockwise direction. This winding number is equal to the winding number of f_C as we follow the curves α_1, α_2, and α_3 where

$$
\begin{aligned}
\alpha_1(t) &= (t, t + \lambda), \quad t \in I \\
\alpha_2(t) &= \begin{cases} (1 - 2t\lambda, 1 + \lambda - 2t\lambda), & t \in [0, 1/2] \\ (2 - 3\lambda - 2t + 4t\lambda, 1), & t \in [1/2, 1] \end{cases} \\
\alpha_3(t) &= \begin{cases} (\lambda(1 - 2t), 1 - 2t\lambda), & t \in [0, 1/2] \\ (0, 2 - 3\lambda - 2t + 4t\lambda), & t \in [1/2, 1]. \end{cases}
\end{aligned}
$$

By definition of degree, the winding number of f_C along α_1 is equal to $D(C)$. It is easy to verify that $(f_C \circ \alpha_2)(t)$ and $(f_C \circ \alpha_3)(t)$ point in the opposite direction. Therefore, $W(f_C \circ \alpha_2) = W(f_C \circ \alpha_3)$. Since $(f_C \circ \alpha_2)(0) = (\tilde{C}(1 + \lambda) - \tilde{C}(1))/\lambda = -(\tilde{C}(1) - \tilde{C}(\lambda))/\lambda = -(f_C \circ \alpha_2)(1)$ the winding number of f_C along α_2 is equal to $1/2 + K$ for some integer K.

Therefore, we get

$$-2m + \operatorname{self}(C) = W(f_C \circ \alpha_1) + W(f_C \circ \alpha_2) + W(f_C \circ \alpha_3) = D(C) + 1 + 2K.$$

This proves Theorem 2. $\square$

Using the results of the next section, this theorem can be strengthened to get the same result as in [**5**] provided that $C'(0)$ exists and that $C(I)$ lies entirely on one side of the tangent line to C at $C(0)$. Instead of Q_λ, one has to take the triangle T_λ with vertices $(0, \lambda), (0, 1)$, and $(1 - \lambda, 1)$, and note that the winding number of f_C as we go once around ∂T_λ in the positive sense is the same as if we follow $\sigma_{\lambda,\Pi}$ (see the notation of the next section; Π is the partition $0 < 1$), and the line segments from (1,1) to (0,1), and from (0,1) to (0,0).

4. Degrees of differentiable curves. Suppose that $C : I \to \mathbf{R}^2$ is a closed locally 1–1 curve. If C is differentiable at $t \in I$ we can extend the definition of f_C (as defined in the proof of Theorem 2) to $S \cup \{(t,t)\}$ by setting $f_C(t,t) = C'(t)$. Note that this extension need no longer be continuous.

Let Q be a triangle with vertices $(t-x,t)$, (t,t), and $(t,t+x)$ such that $(t-x,t) \in S$, $t \in I$. Suppose that $C'(t)$ exists. We shall prove that the winding number of f_C, as we go around ∂Q once, is 0 provided f_C has no zeros on Q.

Suppose that f_C has no zeros in Q. In particular, $C'(t) \neq 0$. By Proposition 2, there exists a $\delta > 0$ such that if $|h| \leq \delta$, then the angle between $C(t+h) - C(t)$ and $C'(t)h$ is less than $\pi/4$. Take $\delta \leq x$, $\delta \leq 1/2$. Let Q_1 be the triangle with vertices $(t-\delta,t)$, (t,t) and $(t,t+\delta)$. Clearly, it is enough to show that the winding number of f_C as we go around the boundary of Q_1 is zero. The positively oriented boundary ∂Q_1 is the product $\alpha\beta\gamma$ of curves $\alpha,\beta,\gamma : I \to \mathbf{R}^2$ given by

$$\begin{aligned}
\alpha(s) &= ((1-s)(t-\delta) + st, t)\\
\beta(s) &= (t, (1-s)t + s(t+\delta))\\
\gamma(s) &= ((1-s)t + s(t-\delta), (1-s)(t+\delta) + st).
\end{aligned}$$

Since C is differentiable at t, the function $f_C \circ (\alpha\beta\gamma)$ is continuous. Therefore, $W(f_C \circ (\alpha\beta\gamma))$ is defined.

If $\tau \in [t-\delta, t)$, then the angle between $f_C(\tau,t) = (1/(t-\tau))(C(\tau) - C(t))$ and $-(1/(t-\tau))(C'(t)(\tau - t)) = C'(t) = f_C(t,t)$ is less than $\pi/4$. Therefore, the winding number ω_1 of $f_C \circ \alpha$ is less than 1/4 in absolute value.

Similarly, if $\tau \in (t, t+\delta]$, then the angle between $f_C(t,\tau) = (1/(\tau - t))(C(\tau) - C(t))$ and $(1/(\tau-t))C'(t)(\tau - t) = C'(t) = f_C(t,t)$ is less than $\pi/4$. Therefore, the winding number ω_2 of $f_C \circ \beta$ is also less than 1/4 in absolute value.

Let R_k be the region containing all points $P \in \mathbf{R}^2$ such that the angle between $P - C(t)$ and $(-1)^k C'(t)$ is less than $\pi/4$ for $k = 1$, or

$k = 2$. Clearly, $C([t-\delta, t)) \subset R_1$ and $C((t, t+\delta]) \subset R_2$. Since the angle between any vector starting in R_1 and ending in R_2 and $C'(t)$ is less than $\pi/4$, the winding number ω_3 of $f_C \circ \gamma$ is less than $1/4$ in absolute value.

The winding number of f_C as we move once around the boundary ∂Q_1 in the counterclockwise direction is equal to $\omega_1 + \omega_2 + \omega_3$. Since $|\omega_1 + \omega_2 + \omega_3| < 3/4$ and since $\omega_1 + \omega_2 + \omega_3$ is an integer, it has to be zero which proves our claim. $\square$

By a *partition* we mean a sequence of points $\Pi : 0 \le t_1 < t_2 < \cdots < t_k < 1$. Define the mesh $m(\Pi)$ of Π to be the maximum of $t_i - t_{i-1}$, $i = 2, \dots, k$, and $t_1 + 1 - t_k$.

Suppose λ is a positive number such that $\tilde{C}(t) \ne \tilde{C}(s)$ if $0 < |s-t| \le \lambda$. Let $\Pi : 0 \le t_1 < \cdots < t_k < 1$ be a partition of I, and suppose that $C'(t_i)$ exists and is nonzero for $i = 1, \dots, k$. Suppose $p \le \lambda$ is a positive number such that $p \le m(\Pi)$. Define a PL curve $\sigma_{p,\Pi} : I \to \mathbf{R}^2$ by

$$\sigma_{p,\Pi}(t) = \begin{cases} (t, t+p) & \text{if } t \in I - \cup_{i=1}^{k}(t_i - p, t_i) \\ (2t + p - t_i, t_i) & \text{if } t \in [t_i - p, t_i - p/2] \\ (t_i, 2t + p - t_i) & \text{if } t \in [t_i - p/2, t_i]. \end{cases}$$

Proposition 3. *Under the above conditions,*

$$D(C) = \lim_{p \to 0^+} W(f_C \circ \sigma_{p,\Pi}).$$

Proof. We shall divide the proof into two cases.

Case 1. $t_0 > 0$. Let $p > 0$ be smaller than $\lambda, t_1, 1 - t_k$, and $m(\Pi)$. Let T_i be the triangle with vertices $(t_i - p, t_i), (t_i, t_i)$, and $(t_i, t_i + p)$. As t goes from $t_i - p$ to t_i, $\sigma_{p,\Pi}(t)$ goes from $(t_i - p, t_i)$ to $(t_i, t_i + p)$ along the short sides of T_i. Since f_C does not vanish on T_i, the winding number of $f_C \circ \sigma_{p,\Pi}$ as t goes from $t_i - p$ to t_i is equal to the winding number of f_C as we go from $(t_i - p, t_i)$ to $(t_i, t_i + p)$ along a straight line. This shows that $W(f_C \circ \sigma_{p,\Pi}) = W(f_C \circ \sigma'_{p,\Pi})$ where Π' is obtained from Π by removing t_i. By successively removing points of Π, we get

$$W(f_C \circ \sigma_{p,\Pi}) = W(f_C \circ \sigma_{p,\varnothing}).$$

But $f_C \circ \sigma_{p,\varnothing} = c^p$; therefore,

$$W(f_C \circ \sigma_{p,\Pi}) = W(c^p) = D(C).$$

Since this is true for all such $p > 0$, the result is proved for this case.

Case 2. $t_0 = 0$. Let $0 < p \leq (1/2) \min\{m(\Pi), \lambda, 1 - t_k\}$. Denote by $T_x : \mathbf{R} \to \mathbf{R}$ the translation $t \to t + x$. If γ is a closed curve, let $\gamma_x = \gamma \circ T_{-x}$. The mapping $u \to \gamma \circ T_{-ux}$ defines a homotopy from γ to γ_x. Let $\Pi_p = \Pi + p$. Then we have the following equality (recall that $\tilde{C}_p(t) = \tilde{C}(t + p)$):

$$f_{C_p} \circ \sigma_{p,\Pi_p} \circ T_p = f_C \circ \sigma_{p,\Pi}.$$

Since $t_1 + p > 0$ it follows from Case 1 that $D(C_p) = W(f_{C_p} \circ \sigma_{p,\Pi_p})$. C and C_p are gently homotopic and $f_{C_p} \circ \sigma_{p,\Pi_p} \circ T_p$ is homotopic to $f_{C_p} \circ \sigma_{p,\Pi_p}$ in $\mathbf{R}^2 - \{0\}$; therefore,

$$D(C) = D(C_p) = W(f_{C_p} \circ \sigma_{p,\Pi_p}) = W(f_{C_p} \circ \sigma_{p,\Pi_p} \circ T_p) = W(f_C \circ \sigma_{p,\Pi}).$$

Since this is true for all small enough $p > 0$, the proposition is proved. $\square$

Suppose that $C : I \to \mathbf{R}^2$ is a differentiable closed curve such that $C'(t) \neq 0$ for all $t \in I$. Let $\phi_C(t) = C'(t)/|C'(t)|$. Define $N(C)$ by

$$N(C) = \min\{\#\phi_C^{-1}(p) \mid p \in S^1\}$$

if the right side exists, otherwise let $N(C) = \infty$. ($\#S$ denotes the number of elements in S.)

Theorem 3. *If $C : I \to \mathbf{R}^2$ is a differentiable closed curve such that*

(i) *$C'(t) \neq 0$ for all $t \in I$,*

(ii) *C is locally 1–1*

then $N(C) \geq |D(C)|$.

We shall need the following proposition.

Proposition 4. *If $A : I \to \mathbf{R}^2$ is a continuous embedding* ($I = [0,1]$) *such that*

(i) $A'(t)$ *exists and is nonzero on* $(0,1)$ *and*

(ii) $A(0) \neq A(1)$,

then there exists a $t \in (0,1)$ *such that* $A'(t)$ *points in the direction of* $A(1)$–$A(0)$.

Proposition 4 was proved in [**4**]. In Section 5 we give a more elementary proof.

Proof of Theorem 3. Let $\lambda > 0$ be chosen so that $\tilde{C}(t) \neq \tilde{C}(s)$ whenever $0 < |t-s| \leq \lambda$.

Suppose that $\#\phi_C^{-1}(q) < n = |D(C)|$ for some point $q \in S^1$. The elements of $\phi_C^{-1}(q)$ define a partition Π (possibly Π is empty). If $r > 0$ is small enough, then $D(C) = W(f_C \circ \sigma_{r,\Pi})$ by Proposition 3. Assume that $r \leq \lambda$. Since $|D(C)| = n$, $(f_C \circ \sigma_{r,\Pi})(t)$ points in the direction of q for at least n values of $t \in I$. Therefore, there has to be a $\tau \notin \phi_C^{-1}(q)$ such that $(f_C \circ \sigma_{r,\Pi})(\tau)$ points in the direction of q. But $(f_C \circ \sigma_{r,\Pi})(\tau)$ is of the form $(\tilde{C}(v) - \tilde{C}(u))/l(u,v)$ where $(u,v) \cap \phi_C^{-1}(q) = \varnothing$ and $v - u \leq \lambda$. Using Proposition 4, we get a $\xi \in (u,v)$ such that $C'(\xi)$ points in the direction of q. This shows that ξ is contained in $\phi_C^{-1}(q)$, which is a contradiction. $\square$

5. An application of degree to nonstop curves.

Definition. $C : I \to \mathbf{R}^2$ is a nonstop curve if

(i) $C'(t)$ exists and is nonzero for all $t \in (0,1)$

(note: we do not require C' to be continuous),

(ii) C is locally 1–1,

(iii) $C(0) \neq C(1)$.

Theorem 4. *Suppose that $C : I \to \mathbf{R}^2$ is a nonstop curve, and let $\hat{C}$ be the product of C and the straight path from $C(1)$ to $C(0)$, i.e.,*

$$\hat{C}(t) = \begin{cases} C(2t), & t \in [0,1/2] \\ C(1) - (2t-1)V & t \in [1/2,1] \end{cases}$$

where $V = C(1) - C(0)$.

If $\hat{C}$ *is locally* 1–1*, then* $C'(t)$ *points in the direction of* V *for at least* $M = |D(\hat{C})|$ *values* $t \in (0,1)$*. If* $\hat{C}$ *is not locally* 1–1*, then* $C'(t)$ *points in the direction of* V *for infinitely many values of* $t \in (0,1)$.

Proof. Without loss of generality, we can assume that $C(0) = (0,0)$, $C(1) = (0,1)$.

Since C is locally 1–1, there exists a $\lambda > 0$ such that $\tilde{C}(s) \neq \tilde{C}(t)$ whenever $0 < |s - t| < \lambda$. As before, we shall use the symbol $\tilde{C}$ to denote the periodic extension $\tilde{C} : \mathbf{R} \to \mathbf{R}^2$ of C.

Suppose that $\hat{C}(s) = \hat{C}(t)$ where $t < s$, $s - t < \lambda/2$ and $s, t \in [1/4, 5/4]$. Since $\hat{C}$ is an embedding on $((1-\lambda)/2, 1/2]$, on $(1/2, 1]$, and on $[1, 1 + \lambda/2)$, this is possible only if $t \in ((j-\lambda)/2, j/2)$, $s \in (j/2, (j+\lambda)/2)$ where $j \in \{1, 2\}$.

Suppose $\hat{C}$ is not locally 1–1. Then for $j \in \{1,2\}$ there exist two sequences $\{t_n\}, \{s_n\}$ such that

(i) $\{t_n\} \subset ((j-\lambda)/2, j/2)$, $\{s_n\} \subset (j/2, (j+\lambda)/2)$,

(ii) $\hat{C}(t_n) = \hat{C}(s_n)$ for all n,

(iii) $\lim_{n\to\infty}(t_n - s_n) = 0$.

Assume that $j = 1$. From (i) and (iii) we see that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = 1/2.$$

Since $\hat{C}(s) = (0, 2 - 2s)$ if $s \in [1/2, 1]$ it follows that $\lim_{n\to\infty} \hat{C}(t_n) = \lim_{n\to\infty} \hat{C}(s_n) = \hat{C}(1/2) = (0,1)$. If $C(t) = (x(t), y(t))$ this implies that

$$\lim_{n\to\infty} y(2t_n) = 1.$$

Since $t_n < 1/2$ and $y(2t_n) < 1$ for all n, we can choose a subsequence of $\{t_n\}$ which we shall again denote by $\{t_n\}$ such that $t_n < t_{n+1}$ and $y(2t_n) < y(2t_{n+1})$ for all n. Note that $x(2t_n) = 0$ for all n. Since $2t_{n+1} - 2t_n = 2(t_{n+1} - t_n) < 2(\lambda/2) = \lambda$, it follows that $C \mid [2t_n, 2t_{n+1}]$ is an embedding. Also, $C(2t_{n+1}) - C(2t_n)$ points in the direction of $(0,1) = V$. Therefore, by Proposition 4, there exists a $\tau_n \in (2t_n, 2t_{n+1})$ such that $C'(\tau_n)$ points in the direction of V.

Since the numbers τ_n lie in disjoint intervals, they are distinct; therefore, $C'(t)$ points in the direction of V for infinitely many $t \in (0,1)$.

The case $j = 2$ is treated in the same way.

If $\hat{C}$ is locally 1–1, then $D(\hat{C})$ is defined. There exists a $0 < \mu < 1/2$ such that $\tilde{\hat{C}} : \mathbf{R} \to \mathbf{R}^2$ restricts to an embedding on every interval of length less than or equal to μ.

Suppose that there are less than M points in (0,1) where the derivative C' points in the direction of V. Let Π be the partition which has these points for elements (possibly $\Pi = \phi$). Let $\hat{\Pi}$ be the corresponding partition of I such that $\hat{C}'$ points in the direction of V at the elements of $\hat{\Pi}$. Choose μ small enough so that $D(\hat{C}) = W(f_{\hat{C}} \circ \sigma_{\mu,\hat{\Pi}})$. Since there are less than M values $\tau \in (0,1/2)$ such that $\hat{C}'(\tau)$ points in the direction of V, there exists a point $t \in I$ such that $\hat{C}(t+\nu) - \hat{C}(t)$ points in the direction of V for some ν such that $\nu \leq \mu$, and such that $(t, t+\nu) \cap \hat{\Pi} = \phi$. Clearly, $[t, t+\nu]$ is not contained in $[1/2, 1]$. If $[t, t+\nu] \subset [0, 1/2]$, then, since $\hat{C} \mid [t, t+\nu]$ is an embedding, there exists a $\tau \in (t, t+\nu)$ such that $C'(\tau) = (1/2)\hat{C}'(\tau/2)$ points in the direction of V (see Proposition 4), a contradiction. Suppose now that $(t, t+\nu)$ intersects both $[0, 1/2]$ and $[1/2, 1]$. We shall treat the case $t \in (1/2 - \nu, 1/2)$, the case $t \in (1-\nu, 1)$ being analogous. Since $\hat{C}(t+\nu) - \hat{C}(t)$ points in the direction of V, there exists a constant $k > 0$ such that $\hat{C}(t+\nu) - \hat{C}(t) = kV$. Using the definition of $\hat{C}$, we get $\hat{C}(t) = (0, 2-2t-2\nu) - (0,k)$. If $\hat{C}(t) = (x,y)$, we therefore get

$$x = 0, \qquad y = 2 - 2t - 2\nu - k < 1.$$

Thus, the vector $\hat{C}(1/2) - \hat{C}(t)$ points in the direction of V. Since $\hat{C} \mid [t, 1/2]$ is an embedding, there exists a $\tau/2 \in (t, 1/2)$ such that $\hat{C}'(\tau/2) = 2C'(\tau)$ points in the direction of V. Since $\tau \notin \Pi$, this is a contradiction. $\square$

If $C : I \to \mathbf{R}^2$ is a nonstop curve let Y_C be the set of pairs (s,t) such that $s < t$ and such that $C(s) = (1-\mu)C(0) + \mu C(1)$, $C(t) = (1-\nu)C(0) + \nu(C(1))$ where $\mu < \nu$. For $(s,t) \in Y_C$, let $\hat{C}_{s,t} : I \to \mathbf{R}^2$ be the closed curve

$$\hat{C}_{s,t}(\tau) = \begin{cases} C((1-2\tau)s + 2\tau t), & \tau \in [0,1/2] \\ (2\tau - 1)C(s) + (2-2\tau)C(t), & \tau \in [1/2, 1]. \end{cases}$$

Define the order $\mathbf{0}(C)$ of C by

$$\mathbf{0}(C) = \max\{|D(\hat{C}_{s,t})|; (s,t) \in Y_C\}$$

if the right side exists; otherwise, let $\mathbf{0}(C) = \infty$. $\square$

The following generalization of Proposition 4 follows from Theorem 4:

Corollary 1. *Suppose that $C : I \to \mathbf{R}^2$ is a nonstop curve. If X is the set of points $t \in I$ such that $C'(t)$ points in the direction of $C(1) - C(0)$, then*

$$\#X \geq \mathbf{0}(C).$$

Proof. We shall distinguish three cases.

Case 1. For some $(s,t) \in Y_C$, the degree $D(\hat{C}_{s,t})$ does not exist. In this case it follows from Theorem 4 that $C'(t)$ points in the direction of $V = C(1) - C(0)$ for infinitely many values $\tau \in (s,t)$.

Case 2. $D(\hat{C}_{s,t})$ exists for all $(s,t) \in Y_C$, but $\mathbf{0}(C) = \infty$. Suppose that X is finite, say $\#X = n$. Then there exists $(s,t) \in Y_C$ such that $|D(\hat{C}_{s,t})| = m > n$. By Theorem 4, we get m points $\tau \in (s,t)$ such that $C'(\tau)$ points in the direction of V, a contradiction.

Case 3. $\mathbf{0}(C) = n < \infty$. Then there exists $(s,t) \in Y_C$ such that $n = |D(\hat{C}_{s,t})|$. The conclusion follows again from Theorem 4. $\square$

Corollary 2. *If $C : I \to \mathbf{R}^2$ is a nonstop curve and if, for some $(s,t) \in Y_C$, the curve $\hat{C}_{s,t}$ is normal and has even self-intersection number, then $C'(\tau)$ points in the direction of $C(1) - C(0)$ for some $\tau \in (0,1)$.*

Proof. See Theorems 4 and 2. $\square$

6. A proof of Proposition 4. First we prove the following

Proposition 5. *If $A : [a, b] \to \mathbf{R}^2$ is a nonstop arc (i.e., a nonstop curve which is 1–1), then there exists an interval $[a_1, b_1] \subset (a, b)$ of length less than $(1/2)(b - a)$ such that $A(b_1) - A(a_1)$ points in the direction of the vector $V = A(b) - A(a)$.*

Proof. Let R be rotation of the plane which maps V to a positive multiple of the vector $(0, 1)$. Let C_1 be defined by $C_1(t) = R(A(t) - A(a))$. If $C_1(t) = (x_1(t), y_1(t))$, let

$$y_0 = \max\{y_1(t); t \in x_1^{-1}(0)\},$$

and let t_0 be equal to $C_1^{-1}(y_0)$. Define $C : I \to \mathbf{R}^2$ by

$$C(t) = \frac{C_1((1 - t)a + t \cdot t_0)}{y_0}.$$

Clearly $C(0) = (0, 0)$, $C(1) = (0, 1)$ and, if $C(t) = (x(t), y(t))$, then $y(1) = \sup\{y(t); t \in x^{-1}(0)\}$.

Case 1. $x^{-1}(0) = \{0, 1\}$. In this case $x(t)$ does not change the sign on I, say $x(t) < 0$ for $t \in (0, 1)$. Let $\hat{C} : I \to \mathbf{R}^2$ be the closed curve obtained from C and the line segment from (0,1) to (0,0), i.e.,

$$\hat{C}(t) = \begin{cases} C(2t), & t \in [0, 1/2], \\ (0, 2 - 2t), & t \in [1/2, 1]. \end{cases}$$

Since $\hat{C}$ is a normal curve with no self-intersections, it follows from Theorem 2 that $D(\hat{C}) \neq 0$. By definition of degree, $D(\hat{C})$ is equal to $W(\hat{c}^{1/6})$. Therefore, there exists a point $\tau \in I$ such that $\hat{C}(\tau + 1/6) - \hat{C}(\tau)$ points in the direction of V. Clearly, τ is not contained in $[1/2, 5/6]$. If $\tau \in [2/6, 1/2) \cup (5/6, 1]$, then one of the points $\hat{C}(\tau + 1/6)$, $\hat{C}(\tau)$ lies on $C((0, 1))$ while the other lies on the line $x = 0$. Since $x(t) < 0$ for $t \in (0, 1)$, it follows that $\hat{C}(\tau + 1/6) - \hat{C}(\tau)$ cannot have the first coordinate equal to zero. Therefore, it cannot point in the direction V. Thus $\tau \in (0, 2/6)$ which implies that $[\tau, \tau + 1/6] \subset (0, 1/2)$. Let $a_1 = (1 - 2\tau) \cdot a + 2\tau \cdot t_0$, $b_1 = (1 - (2\tau + 1/3)) \cdot a + (2\tau + 1/3) \cdot t_0$.

Then $b_1 - a_1 = (1/3)(t_0 - a) < (1/2)(b - a)$ and

$$\begin{aligned} A(b_1) - A(a_1) &= R^{-1}(C_1(b_1) - C_1(a_1)) \\ &= R^{-1}(C_1((1 - (2\tau + 1/3)) \cdot a + (2\tau + 1/3) \cdot t_0) \\ &\qquad - C_1((1 - 2\tau) \cdot a + 2\tau \cdot t_0)) \\ &= y_0 \cdot R^{-1}(C(2\tau + 1/3) - C(2\tau)) \\ &= y_0 \cdot R^{-1}(\hat{C}(\tau + 1/6) - \hat{C}(\tau)) \end{aligned}$$

points in the direction of V.

Case 2. There exists a sequence $\{t_n\}$ of distinct points in $x^{-1}(0)$ which converges to 1. Since $y(1) = \max\{y(t); t \in x^{-1}(0)\}$, we can assume (by choosing a subsequence, if necessary) that $\{t_n\}$ and $\{y(t_n)\}$ are strictly increasing sequences. There exists a positive integer m such that $t_{m+1} - t_m < 1/2$.

Let $a_1 = (1 - t_m)a + t_m t_0$, $b_1 = (1 - t_{m+1})a + t_{m+1} t_0$.

Case 3. 1 is an isolated point of $x^{-1}(0)$. Let $t_0 = \max\{[0, 1) \cap x^{-1}(0)\}$. Then $C \mid [t_0, 1]$ intersects the line $x = 0$ only at $t = t_0$ and $t = 1$. This reduces the situation to Case 1. □

Proof of Proposition 4. Let R be a rotation of the plane which maps $A(1) - A(0)$ to a positive multiple of the vector $(0, 1)$. Define $C : I \to \mathbf{R}^2$ by

$$C(t) = \frac{R(A(t) - A(0))}{|A(1) - A(0)|}.$$

Then $C(0) = (0, 0)$ and $C(1) = (0, 1)$. Suppose that $C(t) = (x(t), y(t))$. By Proposition 5, there exists a sequence of intervals $[s_n, t_n]$ such that:

(i) $[s_1, t_1] = I$,

(ii) $[s_{n+1}, t_{n+1}] \subset (s_n, t_n)$,

(iii) $t_{n+1} - s_{n+1} < (1/2)(t_n - s_n)$, and

(iv) $C(t_n) - C(s_n)$ points in the direction of vector $(0, 1)$, i.e., $y(t_n) - y(s_n) > 0$ and $x(t_n) - x(s_n) = 0$.

Let $\{\tau\} = \cap_{n=1}^{\infty} [s_n, t_n]$. Clearly $\tau \neq t_n, s_n$ for all n. Since $x(t_n) - x(s_n) = 0$, the product $(x(t_n) - x(\tau))(x(\tau) - x(s_n))$ is non-positive.

Therefore,

$$\lim_{n\to\infty}\left(\frac{x(t_n)-x(\tau)}{t_n-\tau}\cdot\frac{x(\tau)-x(s_n)}{\tau-s_n}\right)=(x'(\tau))^2$$

is nonpositive. Thus, $x'(\tau)=0$. Since $y(t_n)-y(s_n)>0$ for all n, it follows that either $y(t_n)-y(\tau)$ or $y(\tau)-y(s_n)$ is positive for infinitely many values of n. We shall assume that $y(t_n)-y(\tau)$ is positive for infinitely many values on n, say $n=k_1,k_2,\dots$, the other possibility being treated similarly. Then we have

$$y'(\tau)=\lim_{j\to\infty}((y(t_{k_j})-y(\tau))/(t_{k_j}-\tau))\geq 0.$$

Since $x'(\tau)=0$ and $C'(\tau)\neq 0$, it follows that $y'(\tau)>0$. Therefore, $C'(\tau)$ points in the direction of the vector $(0,1)$. Since $C'(\tau)=R(A'(\tau))/|A(1)-A(0)|$, we have $A'(\tau)=|A(1)-A(0)|\cdot R^{-1}(C'(\tau))$. This shows that $A'(\tau)$ points in the direction of $A(1)-A(0)$. $\square$

Appendix

In this section we are going to prove that every piecewise-linear locally 1–1 curve is gently homotopic to a regular curve.

Suppose that C is a closed piecewise-linear curve such that for some $\lambda>0$ the points $\tilde{C}(t)$ and $\tilde{C}(s)$ are distinct whenever $|s-t|<\lambda$. We shall assume that there exist numbers $0<t_1<\cdots<t_n<1$ in the unit interval $[0,1]$ such that $\tilde{C}$ restricted to each of the intervals $[t_n-1,t_1]$ and $[t_{i-1},t_i]$, $i=2,\dots,n$, is linear. If this is not the case we can use a gentle homotopy of the type $(x,t)\to\tilde{C}(t+x)$ to homotope C to a curve having this property (compare with Section 3).

For every integer k define points t_k as follows. If $k=qn+j$, where $0<j\leq n$, let $t_k=t_j+q$. Then $\tilde{C}$ restricted to $[t_i,t_{i+1}]$ is defined by

$$\tilde{C}(t)=\tilde{C}(t_i)+(t-t_i)a_i=\tilde{C}(t_{i+1})-(t_{i+1}-t)a_i$$

where

$$a_i=\frac{\tilde{C}(t_{i+1})-\tilde{C}(t_i)}{t_{i+1}-t_i}.$$

Let s be a positive number smaller than or equal to $M = (1/4)\min\{t_i - t_{i-1}; i \in \mathbf{Z}\}$. Define the curve $h_s : \mathbf{R} \to \mathbf{R}^2$ as follows:

$$h_s(t) = \begin{cases} \tilde{C}(t), & t \notin \cup\{[t_i - s, t_i + s]; i \in \mathbf{Z}\}, \\ \alpha_i t^2 + \beta_i + \delta_i, & t \in [t_i - s, t_i + s], \end{cases}$$

where the constants α_i, β_i and δ_i are given by the following expressions

$$\alpha_i = A_i/4s, \qquad \beta_i = B_i/2 - (A_i/2s)t_i$$

$$\delta_i = \tilde{C}(t_i) + (A_i/4s)(t_i - s)^2 - t_i a_{i-1}.$$

By A_i we denote the difference $a_i - a_{i-1}$ and by B_i the sum $a_i + a_{i-1}$. Thus $A_i + B_i = 2a_i$ and $A_i - B_i = -2a_{i-1}$.

The function h_s will be used to define a gentle homotopy from C to a regular curve. First we are going to study the properties of h_s.

Let $x_i(\xi)$ be the function $\alpha_i(t_i + \xi)^2 + \beta_i(t_i + \xi) + \delta_i$. By a straightforward computation, one can show that $x_i(\xi)$ is equal to

$$(A_i/4s)(\xi^2 + s^2) + (B_i/2)\xi + \tilde{C}(t_i).$$

Clearly $x_i(-\xi) = x_i(\xi) - B_i\xi$.

Using the expression for $x_i(\xi)$, we get

$$\begin{aligned} h_s(t_i+s) &= x_i(s) = A_i s/2 + B_i s/2 + \tilde{C}(t_i) = a_i s + \tilde{C}(t_i) = \tilde{C}(t_i+s) \\ h_s(t_i-s) &= x_i(-s) = a_i s + \tilde{C}(t_i) - B_i s = \tilde{C}(t_i) - a_{i-1}s = \tilde{C}(t_i-s), \end{aligned}$$

which shows that h_s is continuous.

Next we calculate the derivatives of h_s at $t_i + s$ and $t_i - s$. We have

$$\begin{aligned} (h_s)'(t_i \pm s) &= 2\alpha_i(t_i \pm s) + \beta_i = (A_i/2s)(t_i \pm s) + B_i/2 - (A_i/2s)t_i \\ &= \pm A_i/2 + B_i/2. \end{aligned}$$

This gives us

$$(h_s)'(t_i + s) = a_i \quad \text{and} \quad (h_s)'(t_i - s) = a_{i-1},$$

which implies that h_s is continuously differentiable.

Claim. *The derivative of h_s does not vanish.*

We only need to verify this on $[t_i - s, t_i + s]$ where i is an arbitrary integer. If a_i and a_{i-1} are linearly independent, then α_i and β_i are also linearly independent which implies that $(h_s)'(t)$ cannot be equal to zero. If a_i and a_{i-1} are linearly dependent, then $a_{i-1} = ua_i$ for some positive number u (u has to be positive because otherwise $\tilde{C}$ cannot be locally 1–1). A straightforward calculation gives us the following expression for $(h_s)'(t_i + t)$:

$$(h_s)'(t_i + t) = (a_i/2)[(t/s)(1-u) + (u+1)].$$

The expression vanishes when t/s is equal to $(u+1)/(u-1)$. The last quotient, however, is greater than one by absolute value if u is positive; therefore, the equation $(h_s)'(t_i + t) = 0$ has no solution when $t_i + t$ lies in $[t_i - s, t_i + s]$ because in this case t/s cannot be greater than one by absolute value.

Since h_s is a regular curve, it is locally 1–1. We show next that h_s is 1–1 on $[t_{i-1} + s, t_{i+1} - s]$ for all i. It suffices to show that the function $\alpha_i t^2 + \beta_i t + \delta_i = x_i(t - t_i)$, restricted to $[t_i - s, t_i + s]$, is 1–1 and that it intersects the line segments from $\tilde{C}(t_{i-1})$ to $\tilde{C}(t_i) - s \cdot a_{i-1}$ and from $\tilde{C}(t_i) + s \cdot a_i$ to $\tilde{C}(t_{i+1})$, only in the points $\tilde{C}(t_i - s)$ and $\tilde{C}(t_i + s)$, respectively. Suppose first that a_i and a_{i-1} are linearly independent. Denote by b_j, $j = i-1, i$, a unit vector perpendicular to a_j. Then

$$\begin{aligned}
(x_i(\tau) - \tilde{C}(t_i)) \cdot b_j &= \left(\frac{A_i}{4s}(\tau^2 + s^2) + \frac{B_i}{2}\tau\right) \cdot b_j \\
&= \frac{(a_{2i-j-1} \cdot b_j)}{4s}((-1)^{i+j-1}(\tau^2 + s^2) + 2s\tau) \\
&= \frac{(a_{2i-j-1} \cdot b_j)}{4s}(-1)^{i+j-1}(\tau + (-1)^{i+j-1}s)^2.
\end{aligned}$$

If $j = i-1$, the above expression is equal to zero only if τ is equal to $-s$; therefore, $\alpha_i t^2 + \beta_i t + \delta_i$ intersects the line through $\tilde{C}(t_i)$ in the direction of a_{i-1} only at $t = t_i - s$. Similarly, if $j = i$, the above expression is equal to zero only if τ is equal to s; therefore, $\alpha_i t^2 + \beta_i t + \delta_i$ intersects the line through $\tilde{C}(t_i)$ in the direction of a_i only at $t = t_i + s$. Assume now that a_i and a_{i-1} are linearly dependent. Then, as before, $a_{i-1} = ua_i$

for some positive number u. Suppose that $x_i(\tau) = \tilde{C}(t_i) - z \cdot a_{i-1}$. Then

$$\begin{aligned} 0 = x_i(\tau) - \tilde{C}(t_i) + z \cdot a_{i-1} &= \frac{A_i}{4s}(\tau^2 + s^2) + \frac{B_i}{2}\tau + z \cdot a_{i-1} \\ &= \frac{a_i}{4s}((1-u)(\tau^2 + s^2) + 2s(1+u)\tau + 4s \cdot u \cdot z). \end{aligned}$$

Thus, $z = -(1/(4s \cdot u))[(1-u)(\tau^2+s^2)+2s(1+u)\tau] = -(1/(4s \cdot u))[(\tau+s)^2 - u(\tau - s)^2]$. If $|\tau| \leq s$, we get the following estimate for z:

$$z \leq -(1/(4s \cdot u))[-u(2s)^2] = s,$$

which shows that $x_i(\tau)$ restricted to $[-s, s]$ intersects the line segment from $\tilde{C}(t_{i-1})$ to $\tilde{C}(t_i) - s \cdot a_{i-1}$ only at $\tilde{C}(t_i) - s \cdot a_{i-1}$. If $x_i(\tau) = \tilde{C}(t_i) + z \cdot a_i$, then

$$\begin{aligned} 0 = x_i(\tau) - \tilde{C}(t_i) - z \cdot a_i &= \frac{A_i}{4s}(\tau^2 + s^2) + \frac{B_i}{2}\tau - z \cdot a_i \\ &= \frac{a_i}{4s}((1-u)(\tau^2 + s^2) + 2s(1+u)\tau - 4s \cdot z). \end{aligned}$$

Thus $z = (1/(4s))[(1-u)(\tau^2 + s^2) + 2s(1+u)\tau] = (1/(4s))[(\tau + s)^2 - u(\tau - s)^2]$. If $|\tau| \leq s$, we get the following estimate for z:

$$z \leq (1/(4s))[(2s)^2] = s$$

which shows that $x_i(\tau)$ restricted to $[-s, s]$ intersects the line segment from $\tilde{C}(t_i)+s \cdot a_i$ to $\tilde{C}(t_{i+1})$ only in the point $\tilde{C}(t_{i+s})$. Now we show that the restriction of h_s to $[t_i - s, t_i + s]$ is 1–1. This is equivalent to showing that x_i is 1–1 on $[-s, s]$. If $x_i(\tau) = x_i(\sigma)$, for some $\tau, \sigma \in [-s, s]$, then we get

$$\frac{A_i}{4s}(\tau^2 + s^2) + \frac{B_i}{2}\tau = \frac{A_i}{4s}(\sigma^2 + s^2) + \frac{B_i}{2}\sigma,$$

which implies that

$$\frac{A_i}{4s}(\tau^2 - \sigma^2) = \frac{B_i}{2}(\sigma - \tau)$$

and, therefore,

$$(\sigma - \tau)\left[\frac{B_i}{2} + \frac{A_i}{4s}(\tau + \sigma)\right] = 0.$$

If a_i and a_{i-1} are linearly independent, then so are A_i and B_i. Therefore the second factor in the above expression cannot vanish which implies that σ is equal to τ and, therefore, x_i is 1–1 on $[-s, s]$. Assume now that a_i and a_{i-1} are linearly dependent. Then, again, $a_{i-1} = ua_i$ for some positive number u. Therefore, the above equation becomes

$$(a_i/(4s))(\sigma - \tau)[2s(1+u) + (1-u)(\tau + \sigma)] = 0.$$

Since τ and σ are smaller than s by absolute value, $\tau + \sigma$ is less than $2s$ by absolute value. Also, $(u-1)/(u+1)$ is less than 1 by absolute value because u is positive. If the last factor was zero, then we would have the following estimate for $|2s|$:

$$|2s| = \left|\frac{u-1}{u+1}(\tau + \sigma)\right| < 2s,$$

which is a contradiction. Thus, as before, we see that σ is equal to τ.

From the above it is clear now that $h_s(t_1)$ is different from $h_s(t_2)$ whenever $|t_1 - t_2|$ is less than M.

Define a homotopy $H : I \times I \to \mathbf{R}^2$ by

$$H(u,t) = \begin{cases} h_{Mu}(t), & u \in (0,1] \\ C(t), & u = 0. \end{cases}$$

From the definition of h_s it follows that H_u is a regular closed curve for all $u \in (0,1]$. We show next that H is continuous. It is clear from the definition of h_s that H is continuous on $(0,1] \times I$. Suppose $(0,t)$ is a point in $\{0\} \times I$ such that $t \neq t_i$ for all i. Choose δ to be a positive number which is less than $(1/2)\min\{|t_i - t|; i \in \mathbf{Z}\}$. Then $H(u,\tau)$ is equal to $C(\tau)$ for all (u,τ) such that $|t-\tau|$ is less than δ and u is less than δ/M. Therefore, H is continuous at $(0,t)$. We still have to check that H is continuous at every point $(0,t_i)$. To do that it is enough to estimate the value $|C(t_i) - h_{Mu}(t)|$. Let $s = Mu$. Using the notation from above, we get

$$\begin{aligned} |C(t_i) - h_{Mu}(t)| &= |C(t_i) - x_i(t - t_i)| \\ &= \left|\frac{A_i}{4s}((t-t_i)^2 + s^2) + \frac{B_i}{2}(t - t_i)\right| \\ &\leq \frac{|B_i|}{2}|t - t_i| + \frac{|A_i|}{4s}((t-t_i)^2 + (Mu)^2). \end{aligned}$$

Since A_i, B_i and M do not depend on u and t, the last expression goes to zero as (u, t) approaches $(0, t_i)$. Thus, H is continuous. Since $H(u, t_1)$ is different from $H(u, t_2)$ whenever $|t_1 - t_2|$ is less than M, the homotopy H is a gentle homotopy from C to the regular curve H_M.

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