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REMARKS ON FUNCTIONS WITH POSITIVE REAL PART ON THE BALL IN C^n

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Let P denote the convex set consisting of the holomorphic functions on the unit ball B of C^n $(n > 1)$ which have positive real part and take the value 1 at 0. In a recent paper McDonald [**4**] extended some of the earlier results of Forelli [**1, 2, 3**] which described the known extreme points of P. The purpose of this paper is to elaborate upon one of McDonald's main results, Theorem 3b given below. Let us first review some of the results of Forelli and McDonald which led to this theorem.

Given an n-tuple $\varphi = (\varphi_1, \dots, \varphi_n)$ of nonnegative integers, choose $c_\varphi > 0$ so that the monomial $h_\varphi(z) = c_\varphi z^\varphi$ satisfies $||h_\varphi|| \equiv \sup\{|h_\varphi(z)| : z \in B\} = 1$. Forelli had shown

Theorem 1. *Let $\varphi_j > 0$ for $j = 1, 2, \dots, n$. The function $(1 + h_\varphi)/(1 - h_\varphi)$ is an extreme point of P if and only if $gcd(\varphi_j) = 1$.*

McDonald observed that, when $n > 1$, the functions h_φ are extreme points of the unit ball A in $H^\infty(B)$ if and only if $\varphi_j > 0$ for $j = 1, 2, \dots, n$. This led to

Theorem 2. *The extreme points in P are images, under the mapping $f \to (1 + f)/(1 - f)$ of extreme points in A which satisfy $f(0) = 0$.*

Remark 1. We observe from the above that, if $gcd(\varphi_j) = k > 1$ and if $\theta = \varphi/k$, then both h_θ and $h_\varphi = (h_\theta)^k$ are extreme in A, but only the image of h_θ is extreme in P.

More generally, if f is an extreme point of A, and if $m > 1$ is a positive integer, then [**5**, Corollary 3.2] f^m is extreme in A, but its image $(1 + f^m)/(1 - f^m)$ is not an extreme point of P [**2**].

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On the other hand, given a homogeneous polynomial p of degree $k \geq 1$, which is also an extreme point of A, set

$$F(p) = \{f \in P : f_k = p\}$$

where $f = 1 + 2\sum_1^\infty f_j$ is the homogeneous expansion of f. We note that $F(p)$ is a face of P when p is extreme in A. By [**4**] there is a polynomial q of degree $\leq k-1$ such that the function $[(1+p+2q)/(1-p)]$ (which is in $F(p)$) is extreme in P. Theorem 3 below describes q when $p = h_\varphi$. Before stating this theorem we need a preliminary definition.

When $n = 1$, we will denote B by D and P by P_1 and define $P_{1,k}$ to be the set of all functions f^* in P_1 of the form,

$$f^*(u) = [1 + u^k + 2q^*(u)]/(1 - u^k), \qquad u \in D$$

where q^* is a polynomial of degree $\leq k - 1$. Thus, $q^*(0) = 0$.

Note that it follows from [**4**, Theorem 3] that $P_{1,k} = F(u^k)$. Thus, $P_{1,k}$ is a face of P_1.

Theorem 3. (a) *If* $gcd(\varphi_j) = 1$*, then* $F(h_\varphi)$ *reduces to a single point, and* $(1 + h_\varphi)/(1 - h_\varphi)$ *is an extreme point of* P.

(b) *If* $gcd(\varphi_j) = k > 1$*, then* $F(h_\varphi)$ *consists of all functions of the form* $f = f^*(h_\theta)$*, where* $f^* \in P_{1,k}$ *and* $\theta = \varphi/k$.

Remark 2. In [**3**] Forelli defines a subset H_φ of the homogeneous polynomials of degree $|\varphi| = \varphi_1 + \cdots + \varphi_n$. Given $f \in H_\varphi$ with $||f|| = 1$, let $X_f = \{z \in S : |f(z)| = 1\}$, where S denotes the unit sphere in C^n. He then shows that Theorem 3a holds for all functions f in H_φ satisfying X_f is "thick" in C^n; i.e., if g is a homogeneous polynomial, and if $g = 0$ on X_f, then $g = 0$. Since $h_\varphi \in H_\varphi$, and X_{h_φ} is thick in C^n, it is worth noting that a slight modification of the argument given in the proof of Lemma 2 of [**4**] shows

Proposition 1. *If* f *is a homogeneous polynomial with* $||f|| = 1$*, and if* X_f *is thick in* C^n*, then* f *is an extreme point of* A.

Remark 3. The functions h_φ with $\varphi_j > 0$ for $j = 1, 2, \ldots, n$, and the function $g(z) = z_1^2 + \cdots + z_n^2$ are extreme points in A [**4**]. Since the sets X_{h_φ} and X_g are thick in C^n, it is reasonable to conjecture that the converse to Proposition 1 must hold.

Conjecture. *If f is a homogeneous polynomial which is also an extreme point of A, then X_f is thick in C^n.*

As a consequence of Theorem 3b, the extreme points of the face $F(h_\varphi)$ must be of the form $f^*(h_\theta)$, where f^* is an extreme point of $P_{1,k}$. The following theorem describes the extreme points of $P_{1,k}$.

Theorem 4. *A function f^* in $P_{1,k}$ is an extreme point of $P_{1,k}$ if and only if $f^*(u) = (1+cu)/(1-cu)$ for some complex number c satisfying $c^k = 1$.*

Proof. If $f^*(u) = (1+cu)/(1-cu)$ with $|c| = 1$, then [**3**] f^* is an extreme point of P_1. The condition $c^k = 1$ guarantees that f^* is in $P_{1,k}$ (let $q^*(u) = cu + \cdots + (cu)^{k-1}$).

Conversely, let

$$f^*(u) = [1 + u^k + 2q^*(u)]/(1-u^k) = 1 + 2\sum_1^\infty a^m u^m$$

be an extreme point of $P_{1,k}$. Since q^* is a polynomial of degree $\le k-1$, we see that $a_k = 1$.

Let $g(u) = 2\sum_1^\infty b^m u^m$ be the Taylor series of some function holomorphic on the disk D that satisfies $\operatorname{Re}(f^* \pm g) > 0$ in D.

By Herglotz's theorem, $|a_k \pm b_k| \le 1$, which implies $b_k = 0$ (recall $a_k = 1$). Hence, $(f^* \pm g)_k(u) = u^k$ is an extreme point of the unit ball in $H^\infty(D)$; hence, $f^* \pm g$ is in $P_{1,k}$ as a consequence of [**4**, Theorem 3].

Since f^* is assumed to be an extreme point in $P_{1,k}$, we must have $g = 0$; consequently, f^* is an extreme point of P_1. Thus, $f^*(u) = (1+cu)/(1-cu) = 1+2\sum_1^\infty c^m u^m$ for some complex number c satisfying $|c| = 1$. Finally, $c^k = a_k = 1$. □

Corollary. *If $gcd(\varphi_j) = k > 1$, then the extreme points of P which lie in the face $F(h_\varphi)$ are of the form $(1+ch_\theta)/(1-ch_\theta)$, where $\theta = \varphi/k$ and $c^k = 1$.*

Proof. Follows directly from Theorem 3b and Theorem 4. □

Remark 4. In [**4**] it is claimed that the function

$$f^*(u) = (1 + u^3 + 1.5u + 1.5u^2)/(1 - u^3)$$

is an extreme point of $P_{1,3}$. That this is false follows from Theorem 4. In fact, it is easy to show that

$$f^*(u) = (1/12)[10g^*(u) + g^*(cu) + g^*(c^2u)]$$

where $g^*(u) = (1 + u)/(1 - u)$, and $c = e^{i(2\pi/3)}$.

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