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QUASI-UNIFORM STRUCTURES IN LINEAR LATTICES

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ABSTRACT. The study of some quasi-uniform structures likely to be defined in the usual normed lattices R^n, l_1, l_2 and $C([0,1])$, prompted a generalization of this particular method to the class of all normed lattices, in such a way that every normed linear lattice can be decomposed into two quasi-pseudometric linear structures (quasi-normed spaces), which enable us to restrict in some aspects the study of such normed lattices to its positive cone. All linear spaces under consideration are assumed to be defined over the field of real numbers.

1. Introduction. The three sources of material listed at the end may be useful to clearly illustrate the purpose of this paper. The idea of comparing ordered structures with uniform ones is at least as old as L. Nachbin's now classical *Topology and order*, [**3**]. We intend to use some results and terminology of linear lattice theory as it is exposed in G.J.O. Jameson's *Topology and normed spaces*, [**2**], as well as the topological ordered space concepts that can be seen in P. Fletcher and W.F. Lindgren's *Quasi-uniform spaces*, [**1**] in order to show that every normed lattice is determined by a quasi-uniform structure compatible with the linear and order structures of the space.

The following definitions can be found in [**2**, p. 375], except for the notion of E-space which is introduced here for the first time.

Definition 1.1. A normed lattice $(E, ||\quad||, \leq)$ is said to be an L-space, M-space or E-space, provided it satisfies that, for positive x, y:

$$||x+y|| = ||x|| + ||y||, \qquad ||x \vee y|| = ||x|| \vee ||y||,$$

or

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2,$$

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respectively.

Mimicking the terminology used in [**1**], given a quasi-uniformity U in a set X, then $T(U)$ will stand for its induced topology; U^{-1} is its conjugate quasi-uniformity. Meanwhile, U^* represents the uniformity generated by $U \cup U^{-1}$ as a subbase. Also, the triple $(X, T, \leq)$ is a topological ordered space provided that T and $\leq$ are, respectively, a topology and order in X such that its graph $G(\leq)$ is closed (i.e., T is a Hausdorff topology). The next definition, taken from [**1**, p. 81], is fundamental for further results.

Definition 1.2. Given a topological ordered space $(X, T, \leq)$ and a quasi-uniformity U in X, then U is said to *determine* $(X, T, \leq)$ whenever $T = T(U^*)$ and $G(\leq) = \cap U$.

2. Quasi-normed spaces. We introduce now a concept which is only new as far as its name is concerned, since it is very close to what, in many texts, is known as a sublinear function. It extends the notion of a norm, and it plays an analogous role to that of the norm since it provides a relationship between linear spaces and quasi-uniformities.

Definition 2.1. A nonnegative real valued function q defined on a linear space E is said to be a *quasi-norm* provided it satisfies the following properties: for every $x, y \in E$ and $t \geq 0$,

(i) if $q(x) = q(-x) = 0$, then $x = 0$,

(ii) $q(tx) = tq(x)$,

(iii) $q(x + y) \leq q(x) + q(y)$.

Given a quasi-normed space (E, q), the function $Q(x, y) = q(x - y)$

defines a quasi-pseudometric in E whose induced quasi-uniform structure U_q is translation-invariant. Besides, the function $q'(x) = q(-x)$ defines another quasi-norm in E such that its induced quasi-pseudometric Q' and quasi-uniformity $U_{q'}$ are the conjugates of Q and U_q, respectively. The following result gives us a way to generate three different norms from a given quasi-norm with the common property that they all induce the uniformity U_q^*, i.e., the three norms are equivalent.

Proposition 2.2. *Given a quasi-normed space* (E, q), *the functions*

$$q_L^*(x) = q(x) + q(-x), \qquad q_M^*(x) = q(x) \vee q(-x),$$

and

$$q_E^*(x) = (q(x)^2 + q(-x)^2)^{1/2}$$

define equivalent norms in E *whose induced uniformity is* U_q^*.

Proof. The only nontrivial axioms to be verified are the scalar multiplicity and the triangle inequality. For every $x, y \in E$ and nonzero t,

$$\begin{aligned}
q_L^*(tx) &= |t|\bigg(q\bigg(\frac{t}{|t|}x\bigg) + q\bigg(-\frac{t}{|t|}x\bigg)\bigg) \\
&= |t|(q(x) + q(-x)) = |t|q_L^*(x). \\
q_M^*(tx) &= |t|\bigg(q\bigg(\frac{t}{|t|}x\bigg) \vee q\bigg(-\frac{t}{|t|}x\bigg)\bigg) \\
&= |t|(q(x) \vee q(-x)) = |t|q_M^*(x). \\
q_E^*(tx) &= |t|\bigg(q\bigg(\frac{t}{|t|}x\bigg)^2 + q\bigg(-\frac{t}{|t|}x\bigg)^2\bigg)^{1/2} \\
&= |t|(q(x)^2 + q(-x)^2)^{1/2} \\
&= |t|q_E^*(x). \\
q_L^*(x+y) &\le q(x) + q(y) + q(-x) + q(-y) = q_L^*(x) + q_L^*(y). \\
q_M^*(x+y) &\le (q(x) + q(y)) \vee (q(-x) + q(-y)) \\
&\le q_M^*(x) + q_M^*(y). \\
q_E^*(x+y) &\le ((q(x) + q(y))^2 + (q(-x) + q(-y))^2)^{1/2} \\
&\le q_E^*(x) + q_E^*(y).
\end{aligned}$$

Now, since $(1/2)\cdot q_L^*(x) \le q_M^*(x) \le q_E^*(x) \le q_L^*(x)$, we have that these norms are equivalent. Finally, for each $n \ge 1$, the basic entourages

$$V_n = \{(x, y) \in E \times E : q(x - y) < 1/n\}$$

and

$$V_n^* = \{(x,y) \in E \times E : q_L^*(x-y) < 1/n\}$$

satisfy the relation $V_{2n} \cap V_{2n}^{-1} \subset V_n^* \subset V_n \cap V_n^{-1}$, so that the uniformity induced by q_L^* is U_q^*. □

3. Quasi-uniform structures in normed lattices. If $(E, \leq)$ is a linear lattice and $x \in E$, then $x^+ = x \vee 0$ and $|x| = x \vee (-x)$. Also P will denote its positive cone, i.e.,

$$P = \{x \in E : x \geq 0\}.$$

Theorem 3.1. *In a normed lattice $(E, || \quad ||, \leq)$ the function $q(x) = ||x^+||$ defines a quasi-norm in E such that the norms q_L^*, q_M^*, q_E^* are equivalent to $|| \quad ||$. Furthermore, q_L^* (respectively, q_M^* and q_E^*) coincides with $|| \quad ||$ whenever E is an L-space (respectively, M-space and E-space).*

Proof. It is straightforward to verify the axioms of a quasi-norm considering the following relations that hold in a normed lattice: $x = x^+ - (-x)^+$, $(tx)^+ = tx^+$, for $t \geq 0$; $(x+y)^+ \leq x^+ + y^+$, and $|x| \leq |y|$ implies $||x|| \leq ||y||$.

Now, for every $x \in E$, we have that

$$(1/2) \cdot ||x|| \leq q_M^*(x) = ||x^+|| \vee ||(-x)^+|| \leq ||x||,$$

thus showing, in light of Proposition 2.2, that the norms q_L^*, q_M^*, q_E^* and $|| \quad ||$ are all equivalent.

If E is an L-space, then

$$\begin{aligned} q_L^*(x) &= q(x) + q(-x) = ||x^+|| + ||(-x)^+|| \\ &= ||x^+ + (-x)^+|| = ||\,|x|\,|| = ||x||. \end{aligned}$$

If E is an M-space, then

$$\begin{aligned} q_M^*(x) &= q(x) \vee q(-x) = ||x^+|| \vee ||(-x)^+|| \\ &= ||x^+ \vee (-x)^+|| = ||\,|x|\,|| = ||x||. \end{aligned}$$

If E is an E-space, then

$$\begin{aligned} q_E^*(x) &= (q(x)^2 + q(-x)^2)^{1/2} = (||x^+||^2 + ||(-x)^+||^2)^{1/2} \\ &= ((1/2)(||x^+ + (-x)^+||^2 + ||x^+ - (-x)^+||^2))^{1/2} \\ &= ((1/2)(||\,|x|\,||^2 + ||x||^2))^{1/2} \\ &= ||x||. \qquad \square \end{aligned}$$

Considering the normed lattice $(E, ||\quad||, \leq)$ as a topological ordered space (the cone P is closed), from Proposition 2.2 and the last theorem we have that the topologies induced by the norm $||\quad||$ and the uniformity U_q^* are the same. Also, since $q(x) = 0$ if and only if $x \in -P$, then $G(\leq) = \cap U_q$. Thus, we obtain the following result.

Corollary 3.2. *Every normed lattice $(E, ||\quad||, \leq)$ is determined by the quasi-uniformity U_q deduced from the quasi-norm $q(x) = ||x^+||$.*

Consequently, we have that the usual normed lattices R^n (with the usual norms: $||x||_L = \sum_{i=1}^n |x_i|$, $||x||_M = \max_{1\leq i\leq n} |x_i|$, and $||x||_E = (\sum_{i=1}^n x_1^2)^{1/2}$); $B(S) = \{x \in R^s : x \text{ is bounded}\}$ with the supremum norm, l_1, l_2 and $C([0,1])$ (with the integral norm), are determined by the respective quasi-norms:

$$||x^+||_L = \sum_{i=1}^n (x_i \vee 0); \qquad ||x^+||_M = \max_{1\leq i\leq n} (x_i \vee 0);$$

$$||x^+||_E = \left(\sum_{i=1}^n (x_i \vee 0)^2\right)^{1/2}; \qquad ||x^+|| = \sup_{t\in S}(x(t) \vee 0);$$

$$||x^+|| = \sum_{n=1}^\infty (x_n \vee 0); \qquad ||x^+|| = \left(\sum_{n=1}^\infty (x_n \vee 0)^2\right)^{1/2}$$

and

$$||x^+|| = \int_0^1 (x(t) \vee 0)\, dt.$$

4. Continuous linear functionals. As a particular case of the examples mentioned before, the usual normed lattice structure of the real line R is determined by the quasi-norm $q_0(x) = x \vee 0$, which induces the right-ray topology. The symbol R^0 will refer to this quasi-normed structure.

Now, given a normed lattice $(E, || \quad ||, \leq)$, let q denote its determining quasi-norm. Consider the collection of all linear functionals that are continuous with respect to the quasi-normed spaces (E, q) and R^0. Let this collection be represented by E'_q. We show that the elements of E'_q are precisely the positive continuous linear functionals; thus, if $(E', || \quad ||, \leq)$ is the dual Banach lattice (with the dual ordering), then E'_q is the cone of positive elements of E' such that the restriction of the dual norm to E'_q determines the dual lattice. The following two results are quite simple to prove.

Proposition 4.1. *Let (E, p) and (F, q) be two quasi-normed spaces and $f : E \to F$ a linear function. Then the following statements are equivalent*:

(i) *f is continuous*,

(ii) *f is continuous at* 0,

(iii) *there is a positive number M such that $q(f(x)) \leq Mp(x)$, for every $x \in E$,*

(iv) *f is quasi-uniformly continuous with respect to the quasi-uniform spaces (E, U_p) and (F, U_q).*

Proposition 4.2. *Let f be a linear functional in a quasi-normed space (E, q). Then f is continuous with respect to (E, q) and R^0 if and only if f is bounded below in the* 0*-neighborhood*

$$V_1(0) = \{x \in E : q(-x) < 1\}.$$

Proposition 4.3. *Let $(E, || \quad ||, \leq)$ be a normed lattice. For each $x \in E$, let $q(x) = ||x^+||$. Let f be a linear functional. Then f is continuous with respect to (E, q) and R^0 if and only if f is a positive continuous linear functional (i.e., $f \in E'$ and $f(x) \geq 0$ for $x \geq 0$).*

Proof. If $f : (E, q) \to R^0$ is continuous, by Proposition 4.2, there is a positive number M such that $f(V_1(0)) > -M$. Since $V_1(0) \cap (-V_1(0))$

is a 0-neighborhood with respect to the norm $|| \quad ||$, and

$$f(V_1(0) \cap (-V_1(0))) \subset]-M, M[\,,$$

we have that $f \in E'$. Now, suppose there is a nonzero positive vector x such that $f(x) < 0$.

By setting $y = (M/f(x))x$, then $y \in -P \subset -V_1(0)$ and $f(y) < M$, but this is a contradiction since $f(y) = M$.

If f is a positive continuous functional and

$$U_1 = \{x \in E : ||x|| < 1\},$$

there is a positive number M such that $f(U_1) \subset]-M, M[$. For every $x \in V_1(0)$, since $-x \leq (-x)^+$, f is positive and $||(-x)^+|| = q(-x) < 1$, then $-f(x) = f(-x) \leq f((-x)^+) < M$. Thus, $f(V_1(0)) > -M$, and f is continuous from (E, q) to R^0 by Proposition 4.2. □

As a consequence of last result, $E'_q \subset E'$; also it is interesting to notice that, for $f \in E'_q$, $||f|| = -\inf f(V_1(0))$. Now, if we consider that a quasi-norm satisfies the requirements of a sublinear real valued function as it appears in the general algebraic version of the Hahn-Banach theorem, and that by the former proposition the positive continuous linear functionals are exactly the elements of E'_q, we can give a somewhat different proof of Proposition 33.15 of [**2**, p. 371].

Corollary 4.4. *Let F be a linear sublattice of the normed lattice $(E, || \quad ||, \leq)$, and let g be a positive continuous linear functional defined on F. Then there is a positive continuous linear functional on E that extends g and has the same norm.*

Proof. By the last proposition, g is continuous with respect to (F, q) and R^0, thus, for every $x \in F$,

$$g(x) \leq g(x) \vee 0 \leq ||g||q(x).$$

Since $||g||q$ is sublinear, by the Hahn-Banach theorem there is a linear functional f defined on E such that it extends g and $f(x) \leq ||g||q(x)$, for every $x \in E$. But $q_0(f(x)) = f(x) \vee 0 \leq ||g||q(x)$, for every $x \in E$, proves f to be in E'_q, i.e., f is a positive continuous linear functional. Also, for every $x \in V_1(0)$, we have $-f(x) = f(-x) \leq ||g||q(-x) < ||g||$, thus

$$||f|| = -\inf f(V_1(0)) \leq ||g|| = -\inf f(V_1(0) \cap F) \leq ||f||. \qquad \square$$

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REFERENCES

1. P. Fletcher and W.F. Lindgren, *Quasi-uniform spaces*, Lect. Notes Pure Appl. Math. **77** (1982).

2. G.J.O. Jameson, *Topology and normed spaces*, Chapman and Hall, Ltd., London (1974).

3. L. Nachbin, *Topology and order*, Robert E. Kriegler Publishing Co., Huntington, New York (1976).

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