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CONJUGACY CRITERIA FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation properties of linear differential equations of the second order

$$(*) \qquad (r(x)y')' + p(x)y = 0,$$

$x \in I = (a, b)$, $-\infty \le a < b \le \infty$, are viewed as a perturbation of the disconjugate equation

$$(**) \qquad (r(x)y')' = 0.$$

Sufficient conditions on the coefficients $r(x), p(x)$ ensuring that $(*)$ possesses a nontrivial solution having at least two zeros on I are obtained. It is shown that conjugacy criteria for $(*)$ are different in the case where the principal solutions y_a, y_b of $(**)$ at a and b are linearly independent or linearly dependent.

1. Introduction. In the present paper we deal with the second order differential equation of the form

$$(1.1) \qquad (r(x)y')' + p(x)y = 0,$$

where $x \in I = (a, b)$, $-\infty \le a < b \le \infty$, $r \in C^1(I)$, $r(x) > 0$ for $x \in I$.

Recall that two points x_1, $x_2 \in I$ are said to be conjugate relative to (1.1) if there exists a nontrivial solution y of this equation for which $y(x_1) = 0 = y(x_2)$. Equation (1.1) is said to be conjugate on I whenever there exists at least one pair of points of I which is conjugate relative to (1.1); in the opposite case equation (1.1) is said to be disconjugate on I.

The problem of disconjugacy of (1.1) on a given interval has a long history and disconjugacy results are exhibited in any monograph on

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linear differential equations, cf., e.g., [**6, 12, 14**]. On the other hand, much less attention has been paid to conjugacy criteria for (1.1).

Tipler [**15**] proved that the equation

$$y'' + p(x)y = 0 \tag{1.2}$$

is conjugate on $\mathbf{R} = (-\infty, \infty)$ if

$$\lim_{t_1 \downarrow -\infty} \inf_{t_2 \uparrow \infty} \int_{t_1}^{t_2} p(x)\, dx > 0. \tag{1.3}$$

This criterion was improved by Müller-Pfeiffer [**9, 10**] which showed that (1.1) is conjugate on $I = (a, b)$ if for some (and hence for every) $c \in I$,

$$\int_c^b r^{-1}(x)\, dx = \infty = \int_a^c r^{-1}(x)\, dx, \tag{1.4}$$

$$\lim_{t_1 \downarrow a} \inf_{t_2 \uparrow b} \int_{t_1}^{t_2} p(x)\, dx \geq 0 \tag{1.5}$$

and $p(x) \not\equiv 0$ on I. The result of Tipler was proved using some properties of the Riccati differential equation corresponding to (1.2), and the result of Müller-Pfeiffer is based on the application of the variational principle of Courant to the quadratic functional corresponding to (1.1). The conjugacy criteria for (1.1) and (1.2) has been extended to higher order equations and to linear Hamiltonian systems in [**3**] and [**4**].

To make some analysis of the above given criteria, further definitions and auxiliary results are needed. Let (1.1) be disconjugate on I. There exists a unique (up to a multiple by a nonzero real constant) solution y_b of this equation such that $\int^b r^{-1}(x) y_b^{-2}(x)\, dx = \infty$. This solution is said to be principal at b. The principal solution of (1.1) at a is defined in a similar manner, cf. [**6**]. If the principal solutions of (1.1) at a and b are linearly independent, equation (1.1) is said to be 1-general on I; in the opposite case (i.e., $y_a = k y_b$, k being a nonzero real constant) (1.1) is said to be 1-special on I, cf. [**2**] (the number 1 in these definitions reflects the fact that (1.1) is disconjugate on I, i.e., every solution of this equation has at most one zero on I).

It will be shown below (in Remark 4) that the one-term equation

$$(r(x)y')' = 0 \tag{1.6}$$

is 1-special on $I = (a, b)$ if and only if (1.4) holds. The result of Müller-Pfeiffer states that an addition of the term $p(x)y$ to (1.6), with p having essentially nonnegative mean value on (a, b) (i.e., (1.5) holds) and $p(x) \not\equiv 0$, makes the perturbed equation (1.1) be conjugate on (a, b). Going through the proof of this statement, one may easily see that the method used there does not apply to the case when (1.6) is 1-general on I.

The aim of the present paper is to prove a sufficient condition for conjugacy of (1.1) which is weaker than (1.4) and (1.5) (with strict inequality). This criterion is based on the combination of the Riccati technique and the transformation method and offers a unified approach to the investigation of conjugacy of (1.1) regardless of whether (1.6) is 1-special or 1-general on I. We also introduce the method of investigation of conjugacy of (1.1) in the case when (1.6) is 1-general via certain associated equations of the form (1.1) where (1.6) is 1-special.

2. Main result.

Theorem 1. *Suppose that there exist $c \in I$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that*

$$(2.1)_1 \quad \varepsilon_1 \int_c^b r^{-1}(x) \exp\left\{2\int_c^x r^{-1}(t)\left[\int_c^t p(s)\,ds - \varepsilon_1\right] dt\right\} dx > \pi/2,$$

$$(2.1)_2 \quad \varepsilon_2 \int_a^c r^{-1}(x) \exp\left\{2\int_c^x r^{-1}(t)\left[\int_c^t p(s)\,ds + \varepsilon_2\right] dt\right\} dx > \pi/2.$$

Then (1.1) *is conjugate on* I.

Proof. Suppose that $(2.1)_{1,2}$ hold and (1.1) is disconjugate on I. Denote by $u(x)$ the solution of (1.1) given by the conditions $u(c) = 1$, $u'(c) = 0$. Since (1.1) is disconjugate on I, $u(x)$ does not vanish at least on one of the intervals (a, c) and (c, b). Suppose that this interval is (c, b). Let $v(x)$ be the solution of (1.1) given by the initial conditions

$v(c) = 1$, $r(c)v'(c) = \varepsilon_1$. Then $v(x) > u(x)$ for $x \in (c,b)$ since the existence of $x_0 \in (c,b)$ such that $v(x_0) = u(x_0)$ contradicts the Sturm separation theorem for zeros of linearly independent solutions. The function $w = rv'/v$ is defined on (c,b) and satisfies the Riccati equation $w' + r^{-1}(x)w^2 + p(x) = 0$, i.e., $w(x) = \varepsilon_1 - \int_c^x r^{-1}(s)w^2(s)\,ds - \int_c^x p(x)\,ds$, hence $v(x) = v(c)\exp\{\int_c^x r^{-1}(t)[-\int_c^t (r^{-1}(s)w^2(s)+p(s))\,ds+\varepsilon_1]\,dt\} \le \exp\{\int_c^x r^{-1}(t)[-\int_c^t p(s)\,ds + \varepsilon_1]\,dt\}$. Let $\alpha(x) = \arctan v(x)/u(x)$. Since $u(x) \ne 0$ on (c,b), this function is well defined on this interval, $\alpha' = [1+(v/u)^2]^{-1} \cdot r^{-1}u^{-2}(rv'u - ru'v) = \varepsilon_1 r^{-1}(u^2+v^2)^{-1}$ and $\alpha(b-) \le \pi/2$. We have $u^2(x)+v^2(x) \le 2v^2(x) \le 2\exp\{2\int_c^x r^{-1}(t)[\varepsilon_1 - \int_c^t p(s)\,ds]\,dt\}$, hence $\alpha(x) = \pi/4 + \int_c^x \varepsilon_1 r^{-1}(t)(u^2(t)+v^2(t))^{-1}\,dt \ge \pi/4 + (\varepsilon_1/2).\ \int_c^x r^{-1}(t)\exp\{2\int_c^t r^{-1}(s)[\int_c^s p(t_1)\,dt_1 - \varepsilon_1]\,ds\}\,dt$ and, according to $(2.1)_1$, $\alpha(b-) > \pi/2$, a contradiction. A similar contradiction is obtained supposing $u(x) > 0$ on (a,c). The proof is complete. □

Corollary 1. *Suppose that* (1.4) *holds and there exists* $c_i \in I$, $i = 1, 2$, *such that*

$$(2.2)_1 \qquad \liminf_{x\uparrow b} \frac{\int_c^x r^{-1}(t)(\int_c^t p(s)\,ds)\,dt}{\int_c^x r^{-1}(t)\,dt} = c_1 > 0$$

$$(2.2)_2 \qquad \limsup_{x\downarrow a} \frac{\int_c^x r^{-1}(t)(\int_c^t p(s)\,ds)\,dt}{\int_c^x r^{-1}(t)\,dt} = c_2 < 0.$$

Then (1.1) *is conjugate on* I.

Proof. Suppose that $(2.2)_1$ holds. There exists $T_1 \in (c,b)$ such that $\int_c^x r^{-1}(t)(\int_c^t p(s)\,ds)\,dt/\int_c^x r^{-1}(t)\,dt > (3/4)c_1$ whenever $x \in (T_1,b)$, hence $\int_c^x r^{-1}(t)[\int_c^t p(s)\,ds - (3/4)c_1]\,dt > 0$. Now let $\varepsilon_1 = (1/4)c_1$ and $d \in (T_1,b)$. We have $\varepsilon_1\int_c^d r^{-1}(x)\exp\{2\int_c^x r^{-1}(t)[\int_c^t p(s)\,ds - \varepsilon_1]\,dt\}\,dx = \varepsilon_1\int_c^{T_1} r^{-1}(x)\exp\{2\int_c^x r^{-1}(t)\,dt[\int_c^t p(s)\,ds - \varepsilon_1]\,dt\}\,dx + \varepsilon_1\int_{T_1}^d r^{-1}(x)\exp\{2\int_c^x r^{-1}(t)[\int_c^t p(s)\,ds - (3/4)c_1 + (1/2)c_1]\,dt\}\,dx \ge K + \varepsilon_1\int_{T_1}^d r^{-1}(x)\exp\{4\varepsilon_1\int_c^x r^{-1}(t)\,dt\} = K + \varepsilon_1\int_{T_1}^{\tilde d}\exp\{4\varepsilon_1 t\}\,dt \to \infty$

as $d \to b+$, $K = \int_c^{T_1} r^{-1}(x) \exp\{2\int_c^x r^{-1}(t)[\int_c^t p(s)\,ds - \varepsilon_1]\,dt\}\,dx$, $\tilde{d} = \int_c^d r^{-1}(t)\,dt$. Similarly, we prove that $\varepsilon_2 \int_d^c r^{-1}(x) \exp\{2\int_c^x r^{-1}(t)[\int_c^t p(s)\,ds + \varepsilon_2]\,dt\}\,dx$ tends to infinity if $d \to a+$ and $\varepsilon_2 = -(1/4)c_2$. $\square$

Remark 1. The conjugacy criterion (1.4), (1.5), with strict inequality, (in this form it was proved in [**9**]) is a special case of Corollary 1. Indeed, let (1.5)—with strict inequality—hold. There exists a real-valued function $\tilde{p}(x) \in C(I)$, $\tilde{p}(x) \le p(x)$ on I, such that $\int_a^b \tilde{p}(x)\,dx = \lim_{t_1 \downarrow a} \inf_{t_2 \uparrow b} \int_{t_1}^{t_2} p(x)\,dx$.

Since $\int_a^b \tilde{p}(x)\,dx > 0$, there exists $c \in I$ such that $\int_c^b \tilde{p}(x)\,dx > 0$, $\int_a^c \tilde{p}(x)\,dx > 0$, cf. [**15**]. Now, using (1.4) and the L'Hopital rule, we get $\lim_{x\uparrow b}(\int_c^x r^{-1}(t)\,dt(\int_c^t(\tilde{p}(s)\,ds)\,dt/\int_c^x r^{-1}(t)\,dt = \lim_{x\uparrow b}\int_c^x \tilde{p}(t)\,dt > 0$. Hence, $\liminf_{x\uparrow b} \int_c^x r^{-1}(t)(\int_c^t p(s)\,ds)\,dt/\int_c^x r^{-1}(t)\,dt \ge \lim_{x\uparrow b} \int_c^x r^{-1}(t)(\int_c^t \tilde{p}(s)\,ds)\,dt/\int_c^t r^{-1}(t)\,dt > 0$. Similarly, $\limsup_{x\downarrow a} \int_c^x r^{-1}(t)(\int_c^t p(s)\,ds)\,dt/\int_c^x r^{-1}(t)\,dt \le \lim_{x\downarrow a}\int_c^x r^{-1}(t)(\int_c^t \tilde{p}(s)\,ds)\,dt/\int_c^x r^{-1}(t)\,dt < 0$.

Remark 2. Note that the disconjugacy criterion given by Theorem 1 is—similar to the Tipler criterion for (1.2)—really a focal point criterion. Indeed, the proof of Theorem 1 establishes that there is a focal point to c in (c, b) under condition $(2.1)_1$ and, analogously, $(2.1)_2$ implies the existence of a focal point to c in (a, c). Theorem 1 then follows from these two focal point results.

Remark 3. In [**3**] and [**4**] we studied similarity between oscillation and conjugacy criteria for self-adjoint differential equations and systems. This similarity consists of the fact that replacing a condition requiring some integrals of the coefficients of the equation to be divergent (which is sufficient for oscillation) by a condition requiring these integrals to satisfy some inequalities (usually to be positive), gives sufficient condition for conjugacy of the investigated equation. For example, condition (1.3) is dual to the Leighton-Wintner oscillation criterion which states that (1.2) is oscillatory at infinity if $\int^\infty p(x)\,dx = \infty$. In the sense of this similarity, the conjugacy criterion given in Theorem 1 is dual to the oscillation criterion of Ráb [**11**] and Corollary 1 corresponds to the criterion of Wintner [**16**].

3. Perturbations of 1-general equations. As we have already mentioned in Section 1, 1-special equations on a given interval are in a certain sense oscillatory unstable; an addition of the term $p(x)y$ to a 1-special equation (1.6) with the function $p(x)$ which is "only slightly positive" on I causes conjugacy of the "perturbed" equation (1.1). In the case where equation (1.6) is 1-general, one cannot expect a similar situation as is shown in the following example.

Example 1. Consider the equation

$$(e^{2x}y')' + \lambda e^{2x}y = 0, \qquad 0 < \lambda < 1 \tag{3.1}$$

as a perturbation of the equation $(e^{2x}y')' = 0$ which is 1-general on $\mathbf{R}$. Equation (3.1) is disconjugate on $\mathbf{R}$ (since the transformation $y = e^{-x}u$ transforms (3.1) into the equation $u'' - (1-\lambda)u = 0$), even if the integral of the potential λe^{2x} over $\mathbf{R}$ is infinity, i.e., $\int_{-\infty}^{\infty} \lambda e^{2x}\,dx = \infty$.

Remark 4. Observe that equation (1.6) is 1-general on I whenever at least one of the integrals $\int^b r^{-1}(x)\,dx$, $\int_a r^{-1}(x)\,dx$ is convergent. Particularly, the pair of functions

$$(y_a, y_b) = \begin{cases} (1, \int_x^b r^{-1}(t)\,dt), & \text{if } \int^b r^{-1} < \infty,\ \int_a r^{-1} = \infty, \\ (\int_a^x r^{-1}(t)\,dt, 1), & \text{if } \int_a r^{-1} < \infty,\ \int^b r^{-1} = \infty, \\ (\int_a^x r^{-1}(t)\,dt, \int_x^b r^{-1}(t)\,dt), & \text{if } \int_a r^{-1} < \infty,\ \int^b r^{-1} < \infty, \end{cases}$$

form the principal solutions of (1.6) at a and b, respectively.

In contrast to the criteria given by (1.4) and (1.5), Theorem 1 does not need any assumption concerning the kind of disconjugacy of (1.6) on I (1-special or 1-general). However, in some particular cases it may be difficult to find $\varepsilon_1, \varepsilon_2$ for which $(2.1)_{1,2}$ holds. For this reason, while investigating the conjugacy of (1.1), the criterion (1.4), (1.5) sometimes seems to be more convenient, but it requires (1.6) to be 1-special on I, i.e., (1.4) to hold.

In this section we introduce two methods of investigation of conjugacy of (1.1) on a given interval in the case where equation (1.6) is 1-general on this interval. The first method is based on the so-called hyperbolic transformation of the second order equations, cf. [**8**], and the second method depends on the computation of the least eigenvalue of a certain differential operator corresponding to (1.6).

Theorem 2. *Let equation* (1.6) *be* 1*-general on* I, *and let* y_a, y_b *be its positive principal solutions at* a *and* b *for which* $r(y_a' y_b - y_a y_b') = 1$. *If*

$$\liminf_{t_1 \downarrow a \;\; t_2 \uparrow b} \int_{t_1}^{t_2} [4p(x) y_a(x) y_b(x) - (r(x) y_a(x) y_b(x))^{-1}] \, dx \geq 0$$

and

$$4p(x) y_a(x) y_b(x) - (r(x) y_a(x) y_b(x))^{-1} \not\equiv 0 \quad \text{on } (a, b)$$

then (1.1) *is conjugate on* $I = (a, b)$.

Proof. Denote $h = (2y_a y_b)^{1/2}$. The transformation $y = hu$ transforms (1.1) into the equation $(q^{-1}u')' + (ph^2 - q)u = 0$, where $q = (rh^2)^{-1}$, and the equation

$$(q^{-1}(x) u')' = 0 \tag{3.2}$$

is 1-special on I. Indeed, the transformation $y = hu$ converts (1.1) into the equation $(h^2 r u')' + h((rh')' + ph)u = 0$, cf. [**1**], and one can directly verify that $h(rh')' = -(rh^2)^{-1}$. Moreover, $(y_b/y_a)' = -(ry_a^2)^{-1} = -(ry_a y_b)^{-1}(y_b/y_a)$, hence $y_b/y_a = \text{const}\exp\{-\int^x (ry_a y_b)^{-1} \, dt\}$. As y_b is the principal solution at b and y_a is linearly independent of y_b, $y_b/y_a \to 0$ as $x \to b^-$, cf. [**6**, p. 355]. It follows that $\int^b (ry_a y_b)^{-1} \, dx = 2\int^b q(x) \, dx = \infty$. Similarly, one can prove that $\int_a q(x) \, dx = \infty$. Now consider the equation $(q^{-1}u')' + (ph^2 - q)u = 0$ as a perturbation of 1-special equation (3.2) and apply (1.4) and (1.5). As the above transformation preserves oscillation behavior of the transformed equations, the proof is complete. □

Example 1 (continuation). Since $y_a = 1$, $y_b = e^{-2x}/2$ are principal solutions of the equation $(e^{2x}y')' = 0$ at $a = -\infty$ and $b = \infty$, by Theorem 2 the equation $(e^{2x}y')' + p(x)y = 0$ is conjugate on $R = (-\infty, \infty)$ if $\int_{-\infty}^{\infty} (p(x)e^{-2x} - 1) \, dx \geq 0$ and $p(x) \not\equiv e^{2x}$.

Theorem 3. *Let one of the following assumptions be satisfied.*

i) $\int_a^b r^{-1}(x)\,dx = A < \infty$, $c \in (a,b)$ *is such that* $\int_c^b r^{-1}(x)\,dx = \int_a^c r^{-1}(x)\,dx = A/2$,

$$\int_a^b \cos^2\left(\pi \int_c^x r^{-1}(s)\,ds/A\right)[p(x) - \pi^2/(A^2 r(x))]\,dx \geq 0$$

and $p(x) \not\equiv \pi^2/(A^2 r(x))$.

ii) $\int_a^c r^{-1}(x)\,dx < \infty$, $\int_c^b r^{-1}(x)\,dx = \infty$, $c \in (a,b)$,

$$\int_a^b \left[\left(\int_a^x r^{-1}\right)^2 \Big/ \left(1 + \left(\int_a^x r^{-1}\right)^2\right)\right]\left[p(x) - 3r^{-1}(x) \Big/ \left(1 + \left(\int_a^x r^{-1}\right)^2\right)^2\right] dx \geq 0$$

and $p(x) \not\equiv 3r^{-1}(x)/[1 + (\int_a^x r^{-1})^2]^2$.

Then (1.1) *is conjugate on* $I = (a,b)$.

Proof. The transformation of the independent variable $y(x) = z(t)$, $t = t(x) = \int_c^x r^{-1}(s)\,ds$ transforms (1.1) into the equation

$$\ddot{z} + r(x)p(x)z = 0, \quad \cdot = d/dt, \qquad t \in (-A/2, A/2),$$

$x = x(t)$ being the inverse function of $t = t(x)$. Since π^2/A^2 is the least eigenvalue of the differential operator $l(z) = -d^2/dt^2(z)$, $z(-A/2) = 0 = z(A/2)$, the equation

$$\ddot{z} + (\pi^2/A^2)z = 0$$

is 1-special on $(-A/2, A/2)$ (the function $y_{-A/2} = y_{A/2} = \cos(\pi/A)t$ is the principal solution of (3.5) both at $-A/2$ and $A/2$). The transformation $z = (\cos(\pi/A)t)u$ transforms the equation

$$\ddot{z} + (\pi/A)^2 z + [(r(x)p(x) - (\pi/A)^2)]z = 0$$

into the equation

$$(\cos^2(\pi t/A)\dot{u})^{\cdot} + \cos^2(\pi t/A)(r(x)p(x) - (\pi/A)^2)u = 0.$$

Now the statement follows from (1.4) and (1.5) using the substitution $dt = r^{-1}(x)\,dx$.

ii) The transformation $y(x) = z(t)$, $t = \int_a^x r^{-1}(s)\,ds$, transforms (1.1) into the equation $\ddot{z} + r(x)p(x)z = 0$ with $t \in (0,\infty)$. The last equation is equivalent to the equation

$$\ddot{z} + 3(1+t^2)^{-2}z + (r(x)p(x) - 3(1+t^2)^{-2})z = 0. \tag{3.3}$$

Since the equation $\ddot{z} + 3(1+t^2)^{-2}z = 0$ is 1-special on $(0,\infty)$ ($y_0 = y_\infty = t/(1+t^2)^{1/2}$), the transformation $z = t(1+t^2)^{-1/2}u$ transforms (3.3) into the equation

$$\left(\frac{t^2}{1+t^2}\dot{u}\right)^{\cdot} + \frac{t^2}{1+t^2}(r(x)p(x) - 3(1+t^2)^{-2})u = 0.$$

Now the statement follows using the same argument as in part i). $\square$

Remark 5. i) In part ii) of the previous proof, we used the fact that $\lambda_0 = 3$ is the least eigenvalue of the so-called Fridrich's extension of the minimal differential operator generated by the expression $l(z) = (1+t^2)^2\ddot{z}$, $t \in (0,\infty)$. The corresponding eigenfunction is $u_0 = t(1+t^2)^{-1/2}$.

ii) We get a similar result to ii) of Theorem 3 if $\int^b r^{-1}(x)\,dx < \infty$, $\int_a r^{-1}(x)\,dx = \infty$.

Example 1 (continuation). Using Theorem 3 one can directly verify that the equation $(e^{2x}y')' + p(x)y = 0$ is conjugate on $\mathbf{R}$ whenever

$$\int_{-\infty}^{\infty} \frac{1}{1+4e^{4x}}\left[p(x) - \frac{12e^{2x}}{(1+4e^{4x})^2}\right] dx \geq 0$$

and the integrated function does not vanish identically on $\mathbf{R}$.

4. Concluding remarks. i) Using the transformation theory of second order differential equations, one can produce various modifications of conjugacy criteria from the preceding sections. For the sake of simplicity, we restrict ourselves to equations of the form (1.2). The transformation

$$y(x) = \exp\{g(x)\}u(t), \qquad t = \int^x \exp\{-2g(s)\}\,ds$$

transforms equation (1.2) into the equation $\ddot{u} + \tilde{p}(t)u = 0$, where $\tilde{p}(t) = \exp\{4g(x)\}(p(x) + g''(x) + (g'(x))^2$ and $x = x(t)$ being the inverse function of $t = t(x) = \int^x \exp\{-2g(s)\}\,ds$. If $g(x) = (1/2)\ln f(x)$, $f \in C^2(R)$, $f(x) > 0$, $\int_0^\infty f^{-1}(x)\,dx = \infty = \int_{-\infty}^0 f^{-1}(x)\,dx$, then $t = \int_0^x f^{-1}(s)\,ds$ and $p(t) = f^2(x)(p(x) + (1/2)f''(x)f^{-1}(x) - (1/4)(f'(x))^2 f^{-2}(x))$, $x = x(t)$. Now, according to (1.5), equation (1.2) is conjugate on $\mathbf{R}$ if there exists a function f having the above given properties, such that

$$\lim_{t_1 \downarrow \infty} \inf_{t_2 \uparrow \infty} \int_{t_1}^{t_2} (f(x)(p(x) - \frac{1}{4}(f'(x))^2 f^{-1}(x) + \frac{1}{2} f''(x))\,dx \geq 0$$

and the integrated function does not vanish identically on $\mathbf{R}$.

ii) Until now we have studied perturbations of one-term differential equation (1.6). Again, using the transformation theory of second order equations, it is not difficult to extend the obtained results to two-term equations. Particularly, let equation (1.1) be 1-special on $I = (a, b)$ and consider the equation

$$(r(x)y')' + (p(x) + p_0(x))y = 0 \tag{4.1}$$

as a perturbation of (1.1). Let y_0 be the solution of (1.1) which is simultaneously principal at a and b. The transformation $y = y_0 u$ transforms (4.1) into the equation $(ry_0^2 u')' + y_0^2 p_0 u = 0$, whereby $\int_a (ry_0^2)^{-1} = \infty = \int^b (ry_0^2)^{-1}$. Now it suffices to replace r and p in the previous results by ry_0^2 and $p_0 y_0^2$, respectively. This idea has been used in the proof of Theorem 3.

iii) The problem of conjugacy of (1.1) on a given interval is closely related to the problem of existence of a nodal domain of the Schrödinger partial differential equation

$$\Delta_n u + p(x)u = 0, \tag*{$(4.2)_n$}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $\Delta_n = \sum_{i=1}^n (\partial^2/\partial x_i^2)$, $n \geq 2$, $p(x)$ is a locally Hölder continuous real-valued function defined on $\mathbf{R}^n$. It has been shown in [**13**] that the influence of the potential p on the existence of a nodal domain of $(4.2)_n$ is considerably different if $n = 2$ and $n \geq 3$. By transforming the Laplace operator into the spherical coordinates, we get $\Delta_n = (d/dr)(r^{n-1}(d/dr))$, $r = (\sum_{i=1}^n x_i^2)^{1/2}$. Now we can see the reason for this difference. If $n = 2$, the equation $(ru')' = 0$ $(' = d/dr)$ is 1-special on $(0, \infty)$, hence a "small" nonnegative function $\tilde{p}(r)$ (i.e., $\int_0^\infty \tilde{p}(r)\,dr \geq 0$, $\tilde{p}(r) \not\equiv 0$) suffices for the conjugacy of the equation $(ru')' + \tilde{p}(r)u = 0$ on $(0, \infty)$ and hence also for the existence of a nodal domain of $(4.2)_n$, where $p(x)$ and $\tilde{p}(r)$ are connected by the relation $\tilde{p}(r) = \int_{|x|=r} \tilde{p}(x)\,d\omega$, ω being the surface of the sphere $|x| = r$, cf. [**10**]. If $n \geq 3$, the equation $(r^{n-1}u')' = 0$ is 1-general on $(0, \infty)$, and in order to get conjugacy of the equation $(r^{n-1}u')' + \tilde{p}(r)u = 0$ on $(0, \infty)$, the potential $\tilde{p}(r)$ must be "much more positive" than for $n = 2$.

iv) Let us investigate the possibility of extending the results of Theorem 1 to the higher order self-adjoint differential equation

$$(r(x)y^{(n)})^{(n)} + p(x)y = 0, \tag{4.3}$$

$r \in C^n(I)$, $p \in C(I)$, $r(x) > 0$ on I. In the proof of Theorem 1, we have used the fact that the oscillation behavior of (1.1) depends on the behavior of the integral

$$\int_a^b r^{-1}(x)(u^2(x) + v^2(x))^{-1}\,dx, \tag{4.4}$$

where u and v are linearly independent solutions of (1.1).

Let $y_1, \dots, y_{2n}$ be linearly independent solutions of (4.3) satisfying certain additional assumptions (reflecting the fact that the "Wronskian" of these solutions equals 1) and let $U(x)$ and $V(x)$ be the Wronski matrices of $y_1, \dots, y_n$ and $y_{n+1}, \dots, y_{2n}$, respectively (i.e., $U = (u_{ij})_{i,j=1,\dots,n}$, $u_{ij} = y_j^{(i-1)}$, V is defined analogously). It was proved in [**5**] that (4.3) is conjugate on I (i.e., there exists a nontrivial solution y of (4.3) and $x_1, x_2 \in I$ such that $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$, $i = 0, \dots, n-1$) whenever

$$\int_a^b r^{-1}(x) e_n^T (U(x)U^T(x) + V(x)V^T(x))^{-1} e_n\,dx > n\pi, \tag{4.5}$$

where the superscript T denotes the transpose of the matrix indicated and $e_n = (0, \dots, 0, 1) \in \mathbf{R}^n$. The open problem is how to formulate condition (4.5) in terms of functions r and p, similar to the reformulation of the behavior of (4.4) in terms of r and p made in Theorem 1.

v) Similar to [**10, 13**], the results concerning conjugacy of (1.1) on a given interval (the existence of a nodal domain of $(4.2)_n$) can be used in order to study spectral properties of certain self-adjoint differential operators associated with these equations. Particularly, the existence of a pair of conjugate point (a nodal domain) implies the existence of at least one negative eigenvalue of these differential operators.

REFERENCES

1. C.D. Ahlbrandt, D.B. Hinton and R.T. Lewis, *The effect of variable change on oscillation and disconjugacy criteria with applications to spectral and asymptotic theory*, J. Math. Anal. Appl. **81** (1981), 234–277.

2. O. Boruvka, *Lineare Differentialtransformationen* 2. *Ordnung*, VEB, Deutscher Verlag der Wissenschaften, Berlin, 1971.

3. O. Došlý, *On the existence of conjugate points for linear differential systems*, Math. Slovaca **40** (1990), 87–99.

4. ———, *Existence of conjugate points for self-adjoint linear differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **113** (1989), 73–85.

5. ———, *On some problems in oscillation theory of self-adjoint linear differential equations*, Math. Slovaca **41** (1991), 101–111.

6. P. Hartman, *Ordinary differential equations*, John Wiley, New York, 1964.

7. V. Komkov, *A technique for detection of oscillation of second order ordinary differential equations*, Pacific J. Math. **42** (1972), 105–115.

8. J. Krbila, *The existence and unboundedness of the first hyperbolic phase of nonoscillatory differential equation* $y''+q(x)y=0$, Sborník prací VŠD a VÚD Žilina **25** (1969), 5–11 (in Czech).

9. E. Müller-Pfeiffer, *Existence of conjugate points for second and fourth order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **89** (1981), 281–291.

10. ———, *Nodal domains of one- or two-dimensional elliptic differential equations*, Z. Anal. Anwendungen **7** (1988), 135–139.

11. M. Ráb, *Kriterien für die Oszillation der Lösungen der Differentialgleichung* $(p(x)y')' + q(x)y = 0$, Čas. Pěst. Mat. **34** (1959), 335–370.

12. W.T. Reid, *Sturmian theory for ordinary differential equations*, Springer-Verlag, New York, Berlin, Heidelberg, 1980.

13. U.-W. Schminke, *On the lower spectrum of Schrödinger operator*, Arch. Rational Mech. Anal. **75** (1981), 147–155.

14. C.A. Swanson, *Comparison and oscillation theory of linear differential equations*, Academic Press, New York, 1968.

15. F.J. Tipler, *General relativity and conjugate differential equations*, J. Differential Equations **30** (1978), 165–174.

16. A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 115–117.

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