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MONOTONE OPEN IMAGES OF 0-SPACES

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ABSTRACT. A 0-*space* is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. The class of almost rimcompact spaces is intermediate between the class of rimcompact spaces and that of 0-spaces.

It is known that rimcompactness is preserved under monotone open maps. In this paper it is shown that the properties of almost rimcompactness and of being a 0-space are preserved under monotone open maps.

1. Introduction and known results. All spaces considered are completely regular and Hausdorff. Recall that a space is *rimcompact* if it possesses a base of open sets with compact boundaries [**8**]. Monotone maps, generally with some additional property, have appeared in the investigation of the preservation of rimcompactness. For example, if Y is the image of a rimcompact space under either a monotone open map or a monotone quotient map for which preimages of points have compact boundaries, then Y is rimcompact ([**6**] and [**1**, 3.4], respectively). The second result with "rimcompact" replaced either by "almost rimcompact" or "0-space" was proved in [**3**]. As mentioned in the abstract, the result for monotone open maps with "rimcompact" replaced by either "almost rimcompact" or "0-space" is proved in this paper.

The main results appear in Section 2. In the remainder of this section, we present some terminology and known results. A *map* is a continuous surjection. A function $f : X \to Y$ is *closed* (*open*) if whenever F is closed (open) in X, $f[F]$ is closed (open) in Y, and *monotone* if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.

The maximum compactification of a space X, the *Stone-Cĕch compactification* of X, is denoted by βX (where the partial ordering on the family of compactifications of X is the usual). If KX is a compactification of X, then $KX \backslash X$ is the *remainder* of KX. If $f : X \to Y$ is a

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map, the extension of f from βX onto βY will be denoted by f^β. The following is a special case of 4.7 of [**4**].

Theorem 1.1. *Let $f : X \to Y$ be a monotone quotient map. Then $f^\beta : \beta X \to \beta Y$ is monotone.*

An open subset U of X is *π-open* in X if $\mathrm{bd}_X U$ is compact. A subset V of βX is *clopen at infinity*, denoted by CI, if $V \cap (\beta X \backslash X)$ is *clopen* (that is, open and closed) in $\beta X \backslash X$. If U is open in X and KX is a compactification of X, the *extension* of U in KX, denoted by $\mathrm{Ex}_{KX} U$, is defined to be $KX \backslash \mathrm{cl}_{KX}(X \backslash U)$. It is easy to verify that $\mathrm{Ex}_{KX} U$ is the largest open set of KX whose intersection with X is U. A compactification KX of X is a *perfect* compactification of X if for each open subset U of X, $\mathrm{cl}_{KX} \mathrm{bd}_X U = \mathrm{bd}_{KX} \mathrm{Ex}_{KX} U$. It follows from Lemma 1 of [**8**] that βX is a perfect compactification of X. Then, for U π-open in X, $\mathrm{bd}_{\beta X} \mathrm{Ex}_{\beta X} U = \mathrm{cl}_{\beta X} \mathrm{bd}_X U = \mathrm{bd}_X U$, hence $\mathrm{Ex}_{\beta X} U \cap (\beta X \backslash X) = \mathrm{cl}_{\beta X} U \cap (\beta X \backslash X)$ and is clopen in $\beta X \backslash X$.

Definition 1.2. The decomposition of βX consisting of $\{\{x\} : x \in X\}$

$\cup \{C_p : C_p$ is the connected component in $\beta X \backslash X$ of $p \in \beta X \backslash X\}$ is denoted by $\mathbf{C}(\beta X)$.

A space X is a *0-space*, that is, has a compactification with zero-dimensional remainder, if and only if each connected component of $\beta X \backslash X$ is compact and a quasicomponent of $\beta X \backslash X$, $\mathbf{C}(\beta X)$ is an upper semicontinuous decomposition of βX, and each element of $\mathbf{C}(\beta X)$ contained in $\beta X \backslash X$ has a base of CI open sets of βX. If X is a 0-space, then X possesses a maximum compactification $F_0 X$ having zero-dimensional remainder; $F_0 X = \beta X / \mathbf{C}(\beta X)$. (See [**7, 8**] for a justification of these statements.)

If F is closed in X, U is open in X and $F \subseteq U$, then F is *nearly π-contained* in U if there is a compact subset K of F so that whenever F' is a closed subset of F and $F' \cap K = \phi$, there is a π-open subset V of X with $F' \subseteq V \subseteq \mathrm{cl}_X V \subseteq U$. A space X is *nearly rimcompact at* x if whenever $x \in U$, where U is open in X, there is an open set W of X with $x \in W \subseteq \mathrm{cl}_X W \subseteq U$ and $\mathrm{cl}_X W$ nearly π-contained in U;

X is *quasi-rimcompact at* x if there is a compact set K_x of X so that whenever F is closed in X and $F \cap K_x = \phi$, there is a π-open subset V of X with $x \in V \subseteq \mathrm{cl}_X V \subseteq X \backslash F$. Finally, X is *almost rimcompact at* x if X is nearly rimcompact and quasi-rimcompact at x, and *almost rimcompact* if X is almost rimcompact at each point. Each rimcompact space is almost rimcompact, each almost rimcompact space is a 0-space; neither converse holds [**2**].

2. The main results. The first result will provide the necessary clopen at infinity subsets of βY.

Lemma 2.1. *Suppose that X is a 0-space and that $f : X \to Y$ is a monotone open map. If U is a CI open set of βX, then $f^\beta[U]$ is a CI open set of βY.*

Proof. Suppose that $p \in (\beta Y \backslash Y) \cap f^\beta[U]$. Then $f^{\beta\leftarrow}(p) \subseteq \beta X \backslash X$ and $f^{\beta\leftarrow}(p) \cap U \neq \phi$. Since $U \cap (\beta X \backslash X)$ is clopen in $\beta X \backslash X$ and f^β is monotone, $f^{\beta\leftarrow}(p) \subseteq U$. Then $f^{\beta\leftarrow}[f^\beta[U] \cap (\beta Y \backslash Y)] = U \cap f^{\beta\leftarrow}[\beta Y \backslash Y]$, thus $f^\beta[U] \cap (\beta Y \backslash Y)$ is clopen in $\beta Y \backslash Y$. Also, since f^β is closed, $p \in \mathrm{int}_{\beta Y} f^\beta[U]$.

Suppose that $p \in f^\beta[U] \cap Y$. We first show that $p \in f[U \cap X]$. If $p \in [f^\beta[U] \cap Y] \backslash f[U \cap X]$, then $f^\leftarrow(p) \subseteq X \backslash U$ so that $\mathrm{cl}_{\beta X} f^\leftarrow(p) \cap U = \phi$. Since $U \cap (\beta X \backslash X)$ is clopen in $\beta X \backslash X$, there is at least one connected component C of $\beta X \backslash X$ such that $C \cap f^{\beta\leftarrow}(p) \neq \phi$ while $C \cap \mathrm{cl}_{\beta X} f^\leftarrow(p) = \phi$. The map f^β is monotone, hence $F[f^{\beta\leftarrow}(p)]$ is a connected subset of F_0X, where $F : \beta X \to F_0 X$ is the natural map. On the other hand, $F[f^{\beta\leftarrow}(p)] \backslash F[\mathrm{cl}_{\beta X} f^\leftarrow(p)]$ is a nonempty open subset of $F[f^{\beta\leftarrow}(p)]$ contained in the zero-dimensional set $F_0 X \backslash X$. This contradiction proves that $p \in f[U \cap X]$.

Choose $x \in f^\leftarrow(p) \cap U$ and W open in βX such that $x \in W \subseteq \mathrm{cl}_{\beta X} W \subseteq U$. Then $f[W \cap X]$ is an open neighborhood of p in Y. It follows that $p \in \mathrm{int}_{\beta Y} \mathrm{cl}_{\beta Y} f[W \cap X] \subseteq \mathrm{cl}_{\beta Y} f[W \cap X] \subseteq f^\beta[\mathrm{cl}_{\beta X} W] \subseteq f^\beta[U]$. Thus, $p \in \mathrm{int}_{\beta Y} f^\beta[U]$ and $f^\beta[U]$ is open in βY. □

For any space X and $p \in \beta X$, let $K_p = \cap\{\beta X \backslash U : U$ is CI open in βX, $p \notin U\}$. The next results provide a useful description of the connected components of the remainder of a 0-space as the sets K_p for $p \in \beta X \backslash X$.

Lemma 2.2. *For $p \in \beta X$, K_p is a compact connected subset of βX. If $K_p \subseteq \beta X \backslash X$, then K_p is the quasicomponent of p in $\beta X \backslash X$ and has a base of CI open sets in βX.*

Proof. The set K_p is clearly compact. Suppose that K_p is not connected. There are open sets U_1, U_2 of βX such that

$\mathrm{cl}_{\beta X} U_1 \cap \mathrm{cl}_{\beta X} U_2 = \phi$, $K_p \subseteq U_1 \cup U_2$ and $K_p \cap U_i \neq \phi$, $i = 1, 2$. Since the finite union of CI open sets is open and CI, by compactness there is a CI open set W of βX with $p \in \beta X \backslash W \subseteq U_1 \cup U_2$. Assume without loss of generality that $p \in U_1$, and consider $W' = W \cup U_2$. Since

$$\begin{aligned}
\mathrm{bd}_{\beta X \backslash X}&[W' \cap (\beta X \backslash X)] \\
&\subseteq \mathrm{bd}_{\beta X \backslash X}[W \cap (\beta X \backslash X)] \cup \mathrm{bd}_{\beta X \backslash X}[U_2 \cap (\beta X \backslash X)] \\
&\subseteq \mathrm{bd}_{\beta X \backslash X}[U_2 \cap (\beta X \backslash X)] \\
&\subseteq W \cap (\beta X \backslash X) \subseteq W' \cap (\beta X \backslash X),
\end{aligned}$$

W' is CI in βX. As $p \notin W'$, $K_p \subseteq U_1$, a contradiction.

If $K_p \subseteq \beta X \backslash X$, then

$$\begin{aligned}
K_p &= K_p \cap (\beta X \backslash X) \\
&= \cap\{\beta X \backslash U : U \text{ is } CI \text{ open in } \beta X, p \notin U\} \cap (\beta X \backslash X) \\
&= \cap\{(\beta X \backslash X) \backslash U : U \text{ is } CI \text{ open in } \beta X, p \notin U\};
\end{aligned}$$

that is, K_p is an intersection of clopen sets of $\beta X \backslash X$. Thus, the quasicomponent of p in $\beta X \backslash X$ is contained in the connected set K_p and therefore equals K_p.

Suppose that $K_p \subseteq V$, where V is open in βX. As above, there is a CI open set V' of βX with $K_p \subseteq \beta X \backslash V' \subseteq V$. Let W be an open set of βX with $W \cap (\beta X \backslash X) = (\beta X \backslash X) \backslash V'$. Then W is CI, as is $W' \equiv W \cap V$. Since $K_p \subseteq W' \subseteq V$, W' is the desired open set. ◻

The locally compact part of X is denoted by $L(X)$; $\mathrm{cl}_{\beta X}(\beta X \backslash X) = \beta X \backslash L(X)$.

Corollary 2.3. *If X is a 0-space, then for $p \in \beta X \backslash X$, K_p is the compact connected quasicomponent of p in $\beta X \backslash X$.*

Proof. Suppose that $p \in \beta X \backslash X$ and $z \in X$. According to the previous result, it suffices to show that $x \notin K_p$. Without loss of generality, $x \notin L(X)$. Since the connected component C_p of p in $\beta X \backslash X$ is compact, there is an open set W of βX with $x \in W$ and $C_p \cap \mathrm{cl}_{\beta X} W = \phi$. Since X is a 0-space, there is a CI open subset V of βX with $C_p \subseteq V$ and $V \cap \mathrm{cl}_{\beta X} W = \phi$. Then $[\mathrm{cl}_{\beta X}[V \cap (\beta X \backslash X)]] \cap [\beta X \backslash X] = V$, and $T = \mathrm{cl}_{\beta X}(\beta X \backslash X) \backslash \mathrm{cl}_{\beta X} V$ is an open subset of $\mathrm{cl}_{\beta X}(\beta X \backslash X)$ with $T \cap (\beta X \backslash X) = (\beta X \backslash X) \backslash V$. There is a CI open subset V' of βX with $V' \cap \mathrm{cl}_{\beta X}(\beta X \backslash X) = T$. Since $x \in W \cap \mathrm{cl}_{\beta X}(\beta X \backslash X) \subseteq T$, $x \notin K_p$. $\square$

Lemma 2.4. *Suppose that $f : X \to Y$ is a monotone open map and that X is a 0-space. For $p \in \beta Y \backslash Y$, $K_p \subseteq \beta Y \backslash Y$.*

Proof. Choose $p \in \beta Y \backslash Y$ and $y \in Y$. Since f^β is monotone, $f^{\beta\leftarrow}(p)$ is a connected compact subset of $\beta X \backslash X$, hence is contained in some compact connected quasicomponent C of $\beta X \backslash X$. Choose $x \in f^{\leftarrow}(y) \cap X$; Corollary 2.3 implies that there is a CI open subset V of βX with $x \in V$ and $C \cap V = \phi$. Then $y \in f^\beta[V]$ while $p \notin f^\beta[V]$. According to Lemma 2.1, $f^\beta[V]$ is CI and open in βX, thus $K_p \subseteq \beta Y \backslash Y$. $\square$

To complete the proof of the main result, we show that $\mathbf{C}(\beta Y)$ (recall Definition 1.2) is an upper semicontinuous decomposition of βY.

The following definitions will be useful in the proof of this result.

Definitions 2.5. 1) A space X has *property* $(*)$ if whenever $x \in U \cap X$ (for U open in βX) there is an open set W of βX with $x \in W \subseteq \mathrm{cl}_{\beta X} W \subseteq U$ and such that $\mathrm{cl}_{\beta X} W \backslash \cup \{V : V \text{ is } CI \text{ open in } \beta X, V \subseteq U\}$ is a compact subset of X.

2) If U is open in βX, let $U^s = \cup\{d : d \in \mathbf{C}(\beta X), d \subseteq U\}$.

Note that $U^s \cap X = U \cap X$ and that U^s is open in βX for each open set U of βX if and only if $\mathbf{C}(\beta X)$ is an upper semicontinuous decomposition of βX.

Theorem 2.6. *Suppose that X is a 0-space and that the map $f : X \to Y$ is monotone and open. Then Y is a 0-space.*

Proof. As mentioned above, it is sufficient to show that $\mathbf{C}(\beta Y)$ is an upper semi-continuous decomposition of βY, or, equivalently, that for U open in βY, U^s is open in βY. We first show that both X and Y have property $(*)$.

Suppose that $x \in U \cap X$, where U is open in βX. Since U^s is open in βX, there is an open set W of βX with $x \in W \subseteq \mathrm{cl}_{\beta X} W \subseteq U^s$. For $p \in \mathbf{C}_{\beta X} W \backslash W$, $C_p \subseteq U$. Thus, there is a CI open set V_p of βX with $p \in V_p \subseteq U$. Then $\mathrm{cl}_{\beta X} W \backslash \cup \{V : V \text{ is } CI \text{ open in } \beta X, V \subseteq U\} \subseteq \mathrm{cl}_{\beta X} W \backslash \cup \{V_p : p \in \mathrm{cl}_{\beta X} W \backslash X\} \subseteq X$ and is a compact subset of X, thus X has property $(*)$.

To show that Y has property $(*)$, suppose that U is open in βY with $y \in U \cap Y$. Choose $x \in f^{\beta\leftarrow}(y) \cap X$; since X has property $(*)$, there is an open set W' of βX with $x \in W' \subseteq \mathrm{cl}_{\beta X} W' \subseteq f^{\beta\leftarrow}[U]$ and such that

$$\mathrm{cl}_{\beta X} W' \backslash \cup \{V : V \text{ is } CI \text{ open in } \beta X, V \subseteq f^{\beta\leftarrow}[U]\}$$

is a compact subset of X. Since $y \in f[W \cap X]$, which is open in Y, $y \in \mathrm{int}_{\beta Y} \mathrm{cl}_{\beta Y} f[W \cap X]$, while $\mathrm{cl}_{\beta Y} f[W \cap X] = f^\beta[\mathrm{cl}_{\beta X} W] \subseteq U$. Finally,

$$\begin{aligned}
\mathrm{cl}_{\beta Y} & f[W \cap X] \backslash \cup \{V' : V' \text{ is } CI \text{ open in } \beta Y, V' \subseteq U\} \\
&= f^\beta[\mathrm{cl}_{\beta X} W] \backslash \cup \{V' : V' \text{ is } CI \text{ open in } \beta Y, V' \subseteq U\} \\
&\subseteq f^\beta[\mathrm{cl}_{\beta X} W] \backslash \cup \{f^\beta[V] : V \text{ is } CI \text{ open in } \beta X, V \subseteq f^{\beta\leftarrow}[U]\} \\
&\subseteq f^\beta[\mathrm{cl}_{\beta X} W \backslash \cup \{V : V \text{ is } CI \text{ open in } \beta X, V \subseteq f^{\beta\leftarrow}[U]\}]
\end{aligned}$$

and thus is a compact subset of Y. The set $\mathrm{int}_{\beta Y} \mathrm{cl}_{\beta Y} f[W \cap X]$ is the desired neighborhood of y.

Assume that U is open in βY; we show that U^s is open in βY. Let $p \in U^s \cap (\beta Y \backslash Y)$. According to Lemmas 2.2 and 2.4, $K_p \subseteq U$. It follows from Lemma 2.2 that there is a CI open set V of βY with $p \in K_p \subseteq V \subseteq U$. To show that $p \in \mathrm{int}_{\beta Y} U^s$, it suffices to show that $V \subseteq U^s$. Since $V \cap (\beta Y \backslash Y)$ is clopen in $\beta Y \backslash Y$, if $q \in V \cap (\beta Y \backslash Y)$, then $K_q \subseteq V \cap (\beta Y \backslash Y)$, $K_q \subseteq U$ and thus $q \in U^s$. Hence, $V \cap (\beta X \backslash X) \subseteq U^s$. Since $V \cap Y \subseteq U \cap Y \subseteq U^s$, $V \subseteq U^s$.

Choose $y \in U \cap Y$. Since Y has property $(*)$, there is W open in βY with $y \in W \subseteq \mathrm{cl}_{\beta Y} W \subseteq U$ and $\mathrm{cl}_{\beta Y} W \backslash \cup \{V : V$ is CI open in $\beta Y, V \subseteq U\}$ a compact subset of Y. If $q \in \mathrm{cl}_{\beta Y} W \backslash Y$, $q \in V$ for some CI open set V of βY with $V \subseteq U$, so that $K_q \subseteq V \subseteq U$ and $q \in U^s$. Since $(\mathrm{cl}_{\beta Y} W) \cap Y \subseteq U \cap Y \subseteq U^s$, $\mathrm{cl}_{\beta Y} W \subseteq U^s$, thus U^s is open in βY. □

A result more general than Theorem 2.6 is possible when working with rimcompactness or almost rimcompactness (2.9). The next two results allow us to work with π-open sets.

Lemma 2.7. *If $f : X \to Y$ is a monotone open map and U is π-open in X, then $f[U]$ is π-open in Y and $\mathrm{cl}_Y f[U] = f[\mathrm{cl}_X U]$.*

Proof. Since f is monotone quotient, it follows from [**8**] that $\mathrm{bd}_Y f[U] \subseteq f[\mathrm{bd}_X U]$. Then $f[U]$ is an open subset of Y with compact boundary. Also,

$$\mathrm{cl}_Y f[U] = \mathrm{bd}_Y f[U] \cup f[U] \subseteq f[\mathrm{cl}_X U],$$

while $f[\mathrm{cl}_X U] \subseteq \mathrm{cl}_Y f[U]$ for any continuous function f, so that $\mathrm{cl}_Y f[U] = f[\mathrm{cl}_X U]$. □

Lemma 2.8. *Suppose that F and U are closed and open in X, respectively. The set F is nearly π-contained in U if and only if $\mathrm{cl}_{\beta X} F \backslash \cup \{\mathrm{Ex}_{\beta X} V : V$ π-open in X, $\mathrm{cl}_X V \subseteq U\}$ is a compact subset of X. In this case, $\mathrm{cl}_{\beta X} F \subseteq \mathrm{Ex}_{\beta X} U$; if $f : X \to Y$ is a monotone open map, then $\mathrm{cl}_Y f[F] \subseteq f[U]$.*

Proof. Suppose that the compact subset K of F witnesses the fact that F is nearly π-contained in U. For $p \in \mathrm{cl}_{\beta X} F \backslash F$, there is a closed subset F_p of F with $p \in \mathrm{cl}_{\beta X} F_p$ and $F_p \cap K = \phi$. Choose V_p to be π-open in X with $F_p \subseteq V_p \subseteq \mathrm{cl}_X V_p \subseteq U$. Then $\mathrm{cl}_{\beta X} F_p \subseteq \mathrm{cl}_{\beta X} V_p$; since $\mathrm{Ex}_{\beta X} V_p \cap (\beta X \backslash X) = \mathrm{cl}_{\beta X} V_p \cap (\beta X \backslash X)$, $p \in \mathrm{cl}_{\beta X} F_p \subseteq \mathrm{Ex}_{\beta x} V_p$. It follows that

$$\begin{aligned}&\mathrm{cl}_{\beta X} F \backslash \cup \{\mathrm{Ex}_{\beta X} V : V \pi\text{-open in } X, \mathrm{cl}_X V \subseteq U\} \\ &\qquad \subseteq \mathrm{cl}_{\beta X} F \backslash \cup \{\mathrm{Ex}_{\beta X} V_p : p \in \mathrm{cl}_{\beta X} F \backslash F\} \subseteq X,\end{aligned}$$

hence is a compact subset of X.

Conversely, suppose that $\mathrm{cl}_{\beta X}F\backslash \cup \{\mathrm{Ex}_{\beta X}V : V\ \pi\text{-open in } X, \mathrm{cl}_X V \subseteq U\}$ is a compact subset K of X. If F' is closed in F with $F' \cap K = \phi$, then $\mathrm{cl}_{\beta X}F' \subseteq \cup\{\mathrm{Ex}_{\beta X}V : V\ \pi\text{-open in } X, \mathrm{cl}_X V \subseteq U\}$. Compactness and the fact that a finite union of π-open sets is π-open yield a π-open set V with $F' \subseteq V \subseteq \mathrm{cl}_X V \subseteq U$. Since $V \subseteq U$ implies that $\mathrm{Ex}_{\beta X}V \subseteq \mathrm{Ex}_{\beta X}U$, $\mathrm{cl}_{\beta X}F \subseteq \mathrm{Ex}_{\beta X}U$. If V is π-open in U and $\mathrm{cl}_X V \subseteq U$, then $f^{\beta}[\mathrm{Ex}_{\beta X}V] \cap Y \subseteq f^{\beta}[\mathrm{cl}_{\beta X}V] \cap Y = (\mathrm{cl}_{\beta Y}f[V]) \cap Y = \mathrm{cl}_Y f[V] =$ (by Lemma 2.7) $f[\mathrm{cl}_X V] \subseteq f[U]$. Hence,

$$\begin{aligned}\mathrm{cl}_Y f[F] &= (\mathrm{cl}_{\beta Y}f[F]) \cap Y = f^{\beta}[\mathrm{cl}_{\beta X}F] \cap Y \\ &= (f[F] \cup f^{\beta}[\mathrm{cl}_{\beta X}F\backslash F]) \cap Y \\ &\subseteq (f[U] \cup f^{\beta}[\cup\{\mathrm{Ex}_{\beta X}V : V\pi\text{-open in } X, \mathrm{cl}_X V \subseteq U\}]) \\ &\qquad\qquad \cap Y \subseteq f[U]. \qquad \square\end{aligned}$$

Theorem 2.9. *Suppose that $f : X \to Y$ is a monotone open map and that, for each $y \in Y$, $f^{\leftarrow}(y)$ contains a point x at which X is almost rimcompact* (*rimcompact*)*. Then Y is almost rimcompact* (*rimcompact*).

Proof. Let $y \in Y$ and choose $x \in f^{\leftarrow}(y)$ at which X is almost rimcompact. If K_x witnesses the fact that X is quasi-rimcompact at x, let $K_y = f[K_x]$. The set K_y is compact. If F is closed in Y and $F \cap f[K_x] = \phi$, then $f^{\leftarrow}[F]$ is a closed subset of X with $f^{\leftarrow}[F] \cap K_x = \phi$. Choose V π-open in X with $x \in V \subseteq \mathrm{cl}_X V \subseteq X\backslash f^{\leftarrow}[F]$. Then $y \in f[V] \subseteq \mathrm{cl}_Y f[V] =$ (by Lemma 2.7) $f[\mathrm{cl}_X V] \subseteq Y\backslash F$. Since $f[V]$ is π-open in Y, K_y witnesses the fact that Y is quasi-rimcompact at y.

If X is rimcompact at x, then we can choose $K_x = \{x\}$, in which case $K_y = \{y\}$ and Y is rimcompact at y.

If $y \in U$ open in Y, and x is as above, $x \in f^{\leftarrow}[U]$. Choose W open in X with $x \in W$ and $\mathrm{cl}_X W$ nearly π-contained in $f^{\leftarrow}[U]$. The set $f[W]$ is an open neighborhood of y; according to Lemma 2.8, $\mathrm{cl}_Y f[W] \subseteq U$. If V is π-open in X, then

$$\begin{aligned}\mathrm{Ex}_{\beta Y}f[V] \cap (\beta Y\backslash Y) &= \mathrm{cl}_{\beta Y}f[V] \cap (\beta Y\backslash Y) \text{ (by 2.7)} \\ &= f^{\beta}[\mathrm{cl}_{\beta X}V] \cap (\beta Y\backslash Y) \\ &= f^{\beta}[\mathrm{Ex}_{\beta X}V] \cap (\beta Y\backslash Y).\end{aligned}$$

Since

$$\begin{aligned} &f^\beta[\mathrm{cl}_{\beta X}W]\setminus\cup\{f^\beta[\mathrm{Ex}_{\beta X}V] : V\pi\text{-open in } X, \mathrm{cl}_X V\subseteq U\}\\ &\qquad\subseteq f^\beta[\mathrm{cl}_{\beta X}W\setminus\cup\{\mathrm{Ex}_{\beta X}V : V\pi\text{-open in } X, \mathrm{cl}_X V\subseteq U\}]\\ &\qquad\subseteq f^\beta[X]\subseteq Y \end{aligned}$$

(with the second inclusion following from Lemma 2.8),

$$\begin{aligned} &\mathrm{cl}_{\beta Y}f[W]\setminus\cup\{\mathrm{Ex}_{\beta Y}V' : V'\pi\text{-open in } Y, \mathrm{cl}_X V'\subseteq U\}\\ &\quad= f^\beta[\mathrm{cl}_{\beta X}W]\setminus\cup\{\mathrm{Ex}_{\beta Y}V' : V'\pi\text{-open in } Y, \mathrm{cl}_X V'\subseteq U\}\\ &\quad\subseteq f^\beta[\mathrm{cl}_{\beta X}W]\setminus\cup\{\mathrm{Ex}_{\beta Y}f[V] : V\pi\text{-open in } X, \mathrm{cl}_X V\subseteq f^{\leftarrow}[U]\}\\ &\quad\subseteq f^\beta[X]=Y, \end{aligned}$$

and thus is a compact subset of Y. Then $\mathrm{cl}_Y f[W]$ is nearly π-contained in Y, and Y is quasi-rimcompact, thus almost rimcompact at y. □

It is reasonably easy to build a nonrimcompact space X and a monotone open map onto a rimcompact space Y satisfying the hypotheses of Theorem 2.9. (For instance, if Z is a connected space which is not rimcompact but has points of rimcompactness and Y is locally compact and zero-dimensional, then the projection map from $Z\times Y$ onto Y will have the desired properties. The space Z can be constructed by taking the square of a connected rimcompact space which is neither locally compact nor nowhere locally compact (see 2.3 of [**5**]).) Thus, Theorem 2.9 is stronger than the rimcompact version of Theorem 2.6.

Any completely regular space X can be written as the perfect image of a zero-dimensional (in fact, extremally disconnected) space [**9**], so that the perfect image of a rimcompact space need not be rimcompact. Also, in 3.4 of [**3**], a rimcompact space X, nonrimcompact space Y and monotone closed map $f : X \to Y$ are constructed. Thus, the hypotheses of monotone open in the above theorems cannot be replaced by either perfect or monotone closed.

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