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EXISTENCE THEORY FOR A STRONGLY DEGENERATE PARABOLIC SYSTEM

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ABSTRACT. An existence theory is established for the system $a\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi)$, $bu_t = \operatorname{div}(k(u)\nabla u) + \sigma(u)|\nabla\varphi|^2$ in a bounded domain of $\mathbf{R}^N$ coupled with initial-boundary conditions. We only assume that σ, k are positive, and thus the system may become degenerate as u goes to infinity. As a result, solutions of the problem display new phenomena that cannot be incorporated into the classical weak formulation. A generalized notion of a solution developed in [**9, 10**] is employed to handle the problem.

1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and T a positive number. Set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$. Consider the following initial-boundary-value problem:

$$a\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi) \qquad \text{in } Q_T \tag{1.1a}$$

$$bu_t = \operatorname{div}(k(u)\nabla u) + \sigma(u)|\nabla\varphi|^2 \qquad \text{in } Q_T \tag{1.1b}$$

$$\varphi = \bar{\varphi} \qquad \text{on } S_T \tag{1.1c}$$

$$u = 0 \qquad \text{on } S_T \tag{1.1d}$$

$$u = u_0, \qquad \varphi = \varphi_0 \qquad \text{on } \Omega \times \{0\}. \tag{1.1e}$$

Here a and b are given positive constants and $\sigma(u), k(u)$ are known functions of u.

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We may view (1.1) as a model for an incompressible, unidirectional flow with temperature-dependent viscosity; see [**8**] and its references. In this situation, φ is the nonzero component of the velocity field of the flow and u the temperature. The Navier-Stokes equations and the energy equation reduce to (1.1a) and (1.1b), respectively. Problem (1.1) also arises in the study of heating in a massive conductor caused by the eddy currents; see [**2**] where a long, conducting, and homogeneous cylinder is considered. An external variable magnetic field induces the Foucault currents in the cylinder which, in turn, generate the so-called Joule heating. The question of finding the magnetic field and the temperature in the cylinder as a consequence of Joule heating leads to the consideration of problems of type (1.1).

Problem (1.1) is considered to be well-understood in the case $N = 1$; see, e.g., [**5, 1, 3, 4**], where the existence of a classical solution and the large time behavior of the solution are investigated under various assumptions on the data. An existence assertion is established for a problem, which is a slight variation of (1.1), in [**8**] under the assumption that σ and k are continuous and satisfy

$$0 < m \le \sigma(s), \qquad k(s) \le M \qquad \text{on } \mathbf{R},$$

where M and m are two constants. In particular, no restriction on N, the space dimension, is imposed in [**8**].

As pointed out in [**8**], the main mathematical difficulty in this situation is due to the quadratic gradient growth in the nonlinearity. In general, nonlinearities of this nature make it impossible to obtain the usual energy estimates which are such celebrated and fundamental properties of equations of parabolic types. Owing to the lack of a priori estimates, the classical compactness and regularity results are no longer applicable. This suggests that one has to be able to extract some extra information from the explicit nonlinear structure of the problem in order to tackle it successfully. It is observed in [**8**] that the relation

$$\varphi \operatorname{div}(\sigma(u)\nabla\varphi) + \sigma(u)|\nabla\varphi|^2 = \operatorname{div}(\sigma(u)\varphi\nabla\varphi)$$

can be employed in a certain way to eliminate the effect of the quadratic term, and, as a consequence, an existence theorem is established there.

In this paper we shall consider the case where σ and k are only assumed to be positive. This leaves open the possibility that $\sigma(s)$ and

$k(s)$ may tend to 0 as $|s| \to \infty$. A new mathematical difficulty arises from this. In order to view the system in the sense of distributions, one should know that φ belongs to $L^2(0,T;W^{1,2}(\Omega))$. This information would be implied by the boundedness of the temperature, u, which in turn depends upon the regularity of φ. Existing results on the regularity of weak solutions to degenerate parabolic equations of type (1.1b) indicate that there is a gap between the regularity of φ obtained from assuming u is bounded and that needed to yield the boundedness of u. This gap does not seem to be of a technical nature. Thus, if u is unbounded on Q_T, the system will degenerate on the set $\{|u| = \infty\}$ where $\nabla\varphi$ is not well-defined. This implies that no a priori estimate for $\nabla\varphi$ will be possible. This new phenomenon cannot be incorporated into the classical weak formulation which we followed in [**8**]. As a result, the classical notion of a weak solution is not appropriate in analyzing (1.1).

The above discussion suggests that we will have to look for a solution in some L^p-space instead of the usual space $L^2(0,T;W^{1,2}(\Omega)) \times L^2(0,T;W_0^{1,2}(\Omega))$. This gives rise to the possibility that $\nabla\varphi$ is only a distribution. Consequently, our system may involve the multiplication of distributions. Thus the sense in which the system is satisfied is an issue. To overcome this difficulty, we first establish the following lemma.

Lemma 1.1. *Assume*

(H1) *σ and k are continuous, positive, and bounded above*;

(H2) $\bar{\varphi} \in L^\infty(Q_T) \cap W^{1,2}(Q_T)$, $u_0 \in L^2(\Omega)$, $\varphi_0 \in L^\infty(\Omega)$.

Then (φ, u) is a classical weak solution of (1.1) *which can be defined in an obvious way* (*see also* [**8**]) *if and only if*

$$\varphi - \bar{\varphi}, \qquad u \in L^2(0,T;W_0^{1,2}(\Omega)), \tag{1.2a}$$

$$\varphi \in L^\infty(Q_T), \tag{1.2b}$$

and

$$-a\int_{Q_T} \varphi\xi_t\,dx\,dt + \int_{Q_T} \sigma(u)\nabla\varphi\nabla\xi\,dx\,dt = a\int_\Omega \varphi_0(x)\xi(x,0)\,dx, \tag{1.2c}$$

(1.2d)
$$-\int_{Q_T}\left(\frac{a}{2}\varphi^2+bu\right)\xi_t\,dx\,dt+\int_{Q_T}(\sigma(u)\varphi\nabla\varphi+k(u)\nabla u)\nabla\xi\,dx\,dt = \int_\Omega\left(bu_0(x)+\frac{a}{2}\varphi_0^2\right)\xi(x,0)\,dx$$

for all $\xi\in H^1(0,T;W_0^{1,2}(\Omega))$ *such that* $\xi(x,T)\equiv 0$.

We may view (1.2) as the classical weak formulation of the following problem:

(1.3a)
$$a\varphi_t=\operatorname{div}(\sigma(u)\nabla\varphi)\qquad\text{in } Q_T$$

(1.3b)
$$\left(\frac{a}{2}\varphi^2+bu\right)_t=\operatorname{div}(k(u)\nabla u)+\operatorname{div}(\sigma(u)\varphi\nabla\varphi)\qquad\text{in } Q_T,$$

(1.3c)
$$\varphi=\bar{\varphi},\qquad u=0\qquad\text{on } S_T,$$

(1.3d)
$$\varphi=\varphi_0,\qquad u=u_0\qquad\text{on } \Omega\times\{0\}.$$

Now we are in a position to employ the notion of a capacity solution developed in [**9, 10**] to study (1.1). For this purpose, let

$$\beta(s)=\int_0^s k(\tau)\,d\tau,\qquad v=\beta(u),\qquad \mathcal{A}=\{\rho\in C_0^1(\mathbf{R}):\rho(0)=1\}.$$

Definition. By a capacity solution to (1.1), we mean a triplet (u,φ,g) such that

(i) $u\in L^2(Q_T)$, $\varphi\in L^\infty(Q_T)$, $g\in[L^2(Q_T)]^N$, $v\in L^2(0,T;W_0^{1,2}(\Omega))$;

(ii) (1.2c) and (1.2d) hold with $\sigma(u)\nabla\varphi$ replaced by g and $k(u)\nabla u$ replaced by ∇v;

(iii) for each $\rho\in\mathcal{A}$, $\rho(v)\varphi-\bar{\varphi}\in L^2(0,T;W_0^{1,2}(\Omega))$ and $\rho(v)g=\sigma(u)(\nabla(\rho(v)\varphi)-\varphi\nabla\rho(v))$ in $L^2(Q_T)$.

Let us analyze (iii) a little bit further. If u is indeed bounded, then we can choose $\rho \in \mathcal{A}$ so that $\rho = 1$ on the range of v. Then it immediately follows from (iii) that $\varphi - \bar{\varphi} \in L^2(0,T;W_0^{1,2}(\Omega))$ and $g = \sigma(u)\nabla\varphi$. Thus, in this case, (u,φ) is a classical weak solution. Even if u is unbounded, we can still recover $\nabla\varphi$ in the almost everywhere sense. To see this, set $E_m = \{(x,t) \in Q_T : |v(x,t)| \leq m\}$ for each $m > 0$. Select a ρ_m in $\mathcal{A}$ so that $\rho_m = 1$ on $[-m,m]$. Then $\rho_m(v)\varphi = \varphi$ on E_m. But $\rho_m(v)\varphi \in L^2(0,T;W^{1,2}(\Omega))$, and we can evaluate $\nabla(\rho_m(v)\varphi)$. Define $\nabla\varphi = \nabla(\rho_m(v)\varphi)$ for $(x,t) \in E_m$. Then, by (iii), $g = \sigma(u)\nabla\varphi$ on E_m. Since $Q_T \setminus \cup_{m=1}^{\infty} E_m \equiv \{(x,t) : |v(x,t)| = \infty\}$ is of measure 0, $\nabla\varphi$ can be defined for almost all (x,t) in Q_T. Then $g = \sigma(u)\nabla\varphi$ in Q_T. Note that $\nabla\varphi$ obtained here may only be a measurable function, and the product $g = \sigma(u)\nabla\varphi$ is taken as a product of two measurable functions. A remarkable possibility is that in this case $\nabla\varphi$ in the sense of distributions may not be a pointwise function, i.e., a pure distribution. An example situation is that φ has a jump discontinuity across a manifold in Q_T and $v(x,t)$ goes to infinity as (x,t) approaches the manifold. Another point here is that φ may not be regular enough to allow the definition of a trace. That is why the boundary condition for φ is characterized by $\rho(v)\varphi - \bar{\varphi} \in L^2(0,T;W_0^{1,2}(\Omega))$ for each $\rho \in \mathcal{A}$. Note that $\rho(v) = 1$ on S_T. This enables us to say $\varphi = \bar{\varphi}$ on S_T in some sense. We shall show that this notion of a capacity solution is indeed general enough to encompass the new phenomena caused by the relaxation of assumptions on σ and k.

Theorem 1.2. *Let the assumptions of Lemma* 1.1 *be satisfied. Assume that $Rg\beta$, the range of $\beta, = (-\infty,\infty)$ and that there exist two constants m and M such that $0 < m \leq \sigma(s)/k(s) \leq M$ for all $s \in \mathbf{R}$. Then* (1.1) *has a capacity solution.*

This paper is organized as follows. The proofs of Lemma 1.1 and Theorem 1.2 are presented in Section 2. In Section 3 we give a proof of the existence theorem for (1.1) in the case where σ and k stay away from zero, which seems to be simpler than that presented in [**8**].

Finally, we remark that the situation considered here is also very interesting from the point of view of applications. In the electrical heating of a conductor such as a thermistor, $\sigma(u)$ represents the temperature-dependent electric conductivity. As the temperature increases, the conductivity decreases. A very high temperature essentially leads to the shut-down of the electric current, i.e., $\sigma(u) = 0$ if u is "large." Our model may be considered as an approximation to this situation.

2. The degenerate case. In this section we first recall some information on Banach-space-valued functions. Then we proceed to prove Lemma 1.1 and Theorem 1.2.

Proposition 2.1. *Let the Banach space V be dense and continuously embedded in the Hilbert space H; identify $H = H'$ so that $V \hookrightarrow H \hookrightarrow V'$. Then the Banach space $W_p(0,T) \equiv \{u \in L^p(0,T;V) : u_t \in L^{p'}(0,T;V')\}$ is contained in $C([0,T],H)$. Moreover, if $u, v \in W_p(0,T)$, then $(u(\cdot), v(\cdot))_H$ is absolutely continuous on $[0,T]$, and*

$$\frac{d}{dt}(u(t), v(t))_H = (u'(t), v(t)) + (v'(t), u(t)).$$

Here and in what follows we use the convention that p' is such that $1/p + 1/p' = 1$, and $(\cdot,\cdot)$ denotes the duality pairing between a topological vector space V and its dual V'. Next, we cite Lions-Aubin's theorem for the reader's convenience.

Proposition 2.2 (Lions, Aubin). *Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$, and assume that $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, B_0 and B_1 be reflexive, and define*

$$W = \{u \in L^p(0,T;B_0) : u_t \in L^q(0,T;B_1)\}.$$

Then the inclusion $W \hookrightarrow L^p(0,T;B)$ is compact.

Now we are ready to prove Lemma 1.1.

Proof of Lemma 1.1. Let (φ, u) be a classical weak solution of (1.1). Then we only need to prove (1.2d). It is sufficient to prove that (1.2d) holds for all $\xi \in C_0^\infty(\Omega \times (-\infty, T))$. We infer from (1.2c) that

$$a\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi) \qquad \text{in } L^2(0,T;W^{-1,2}(\Omega)).$$

Thus, for any ξ in $C_0^\infty(\Omega\times(-\infty,T))$, we have

$$(a\varphi_t,\varphi\xi) = -\int_{Q_T}\sigma(u)\nabla\varphi\nabla(\varphi\xi)\,dx\,dt. \tag{2.1}$$

Keeping this in mind, we compute

$$\begin{aligned}\int_{Q_T}\sigma(u)\varphi\nabla\varphi\nabla\xi\,dx\,dt &= \int_{Q_T}\sigma(u)\nabla\varphi(\nabla(\varphi\xi)-\xi\nabla\varphi)\,dx\,dt\\ &= -(a\varphi_t,\varphi\xi) - \int_{Q_T}\xi\sigma(u)|\nabla\varphi|^2\,dx\,dt.\end{aligned} \tag{2.2}$$

The weak formulation of (1.1b) gives

$$\begin{aligned}-\int_{Q_T} bu\xi_t dx\,dt = &-\int_{Q_T}(k(u)\nabla u\nabla\xi+\sigma(u)|\nabla\varphi|^2\xi)\,dx\,dt\\ &+ b\int_\Omega u_0(x)\xi(x,0)\,dx.\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\begin{aligned}\int_{Q_T}(\sigma(u)\varphi\nabla\varphi+k(u)\nabla u)\nabla\xi\,dx\,dt = -(a\varphi_t,\varphi\xi) &+ \int_{Q_T} bu\xi_t\,dx\,dt\\ &+ b\int_\Omega u_0(x)\xi(x,0)\,dx.\end{aligned} \tag{2.4}$$

Thus, to establish (1.2d) it is enough to show

$$\int_{Q_T}\frac{a}{2}\varphi^2\xi_t\,dx\,dt + \frac{a}{2}\int_\Omega \varphi_0^2(x)\xi(x,0)\,dx = -(a\varphi_t,\varphi\xi). \tag{2.5}$$

By Proposition 2.1, we have

$$\begin{aligned}&\frac{d}{dt}(a\varphi(\cdot,t)-a\bar{\varphi}(\cdot,t),\varphi(\cdot,t)\xi(\cdot,t))_{L^2(\Omega)}\\ &\qquad = (a\varphi_t(\cdot,t)-a\bar{\varphi}_t(\cdot,t),\varphi(\cdot,t)\xi(\cdot,t))\\ &\qquad\quad + ((\varphi(\cdot,t)\xi(\cdot,t))_t, a\varphi(\cdot,t)-a\bar{\varphi}(\cdot,t)).\end{aligned} \tag{2.6}$$

Recall that $\xi\in C_0^\infty(\Omega\times(-\infty,T))$. Hence, $\varphi_t\xi$ makes sense in $L^2(0,T;W^{-1,2}(\Omega))$, i.e., $(\varphi_t\xi,\eta)=(\varphi_t,\xi\eta)$ for any η in $L^2(0,T;W_0^{1,2}(\Omega))$.

Then we have that $(\varphi\xi)_t = \xi\varphi_t + \varphi\xi_t$ in $L^2(0,T;W^{-1,2}(\Omega))$. Use this in (2.6) and integrate the resulting equation over $(0,T)$ to get

$$-\int_\Omega a\varphi_0^2\xi(x,0)\,dx = 2(a\varphi_t,\varphi\xi) + \int_{Q_T} a\xi_t\varphi^2\,dx\,dt.$$

This is equivalent to (2.5).

Since (2.5) holds for any φ such that $\varphi - \bar\varphi \in L^2(0,T;W_0^{1,2}(\Omega))$ and $\varphi_t \in L^2(0,T;W^{-1,2}(\Omega))$ and $\xi \in C_0^\infty(\Omega\times(-\infty,T))$, it is easy to see that (1.2c) and (1.2d) also imply (2.3). This completes our proof. □

Now we turn our attention to Theorem 1.2. A capacity solution will be constructed as the limit of a sequence of classical weak solutions of the approximation problems.

Proof of Theorem 1.2. For each positive integer n set

$$\sigma_n(s) = \sigma(s) + 1/n, \qquad k_n(s) = k(s) + 1/n,$$

and then consider the following problem:

$$a\varphi_t = \operatorname{div}(\sigma_n(u)\nabla\varphi) \quad \text{in } Q_T \tag{2.7a}$$

$$bu_t = \operatorname{div}(k_n(u)\nabla u) + \sigma_n(u)|\nabla\varphi|^2 \quad \text{in } Q_T \tag{2.7b}$$

$$\varphi = \bar\varphi \quad \text{on } S_T \tag{2.7c}$$

$$u = 0 \quad \text{on } S_T \tag{2.7d}$$

$$u = u_0, \qquad \varphi = \varphi_0 \quad \text{on } \Omega\times\{0\}. \tag{2.7e}$$

By Theorem 3.1 in the subsequent section, for each n there exists a classical weak solution (φ_n, u_n) to (2.7). Let

$$\beta_n(s) = \int_0^s k_n(\tau)\,d\tau, \qquad \alpha_n(s) = \beta_n^{-1}(s),$$

and

$$v_n = \beta_n(u_n).$$

Then we have

$$a\varphi_{nt} = \operatorname{div}(\sigma_n(u_n)\nabla\varphi_n) \qquad \text{in } L^2(0,T;W^{-1,2}(\Omega)), \tag{2.8}$$

$$\begin{gathered} b\alpha_n(v_n)_t = \Delta v_n + \sigma_n(u_n)|\nabla\varphi_n|^2 \\ \text{in } (L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(Q_T))'. \qquad \square \end{gathered} \tag{2.9}$$

Lemma 2.1. *Let the assumptions of Lemma* 1.1 *be satisfied. Assume that* $Rg\beta = (-\infty,\infty)$. *Then* $\{\varphi_n\}$ *is precompact in* $L^p(Q_T)$ *for each* $p \geq 1$.

Proof. We infer from (H2) and the weak maximum principle that

$$\operatorname*{ess\,sup}_{(x,t)\in Q_T} |\varphi_n(x,t)| \le \max\{||\bar\varphi||_{L^\infty(S_T)}, ||\varphi_0||_{L^\infty(\Omega)}\} \qquad \text{for all } n. \tag{2.10}$$

Use $\varphi_n - \bar\varphi$ as a test function in (2.8) to derive

$$\begin{aligned} \frac{a}{2}\frac{d}{dt}\int_\Omega (\varphi_n-\bar\varphi)^2\,dx + a\int_\Omega \bar\varphi_t(\varphi_n-\bar\varphi)\,dx \\ = -\int_\Omega \sigma_n(u_n)\nabla\varphi_n(\nabla\varphi_n - \nabla\bar\varphi)\,dx. \end{aligned}$$

Integration with respect to t, application of Hölder's inequality, and the inequality that

$$ab \le \frac{a^2}{2\varepsilon} + \frac{b^2\varepsilon}{2} \qquad \text{for all } a,b\in\mathbf{R},\ \varepsilon>0 \tag{2.11}$$

yield

$$\begin{aligned} \frac{a}{2}\int_\Omega (\varphi_n-\bar\varphi)^2\,dx + \frac{1}{2}\int_0^t\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\,dx\,d\tau \\ \le \frac{a}{2}\int_0^t\int_\Omega (\varphi_n-\bar\varphi)^2\,dx\,d\tau + c. \end{aligned}$$

Here c depends on the $W^{1,2}(Q_T)$-norm of $\bar{\varphi}$ and the upper bound M for σ. By Gronwall's inequality and our assumptions on $\bar{\varphi}$,

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_\Omega \varphi_n^2(x,t)\,dx+\int_{Q_T}\sigma_n(u_n)|\nabla\varphi_n|^2\,dx\,dt\le c_1 \tag{2.12}$$

for all n. Here, and in what follows, c, c_i, $i=0,1,\dots$, represent positive constants which are independent of n.

Now we proceed to obtain a priori estimates for $\{v_n\}$. For this purpose, let

$$l_\delta(s)=\begin{cases}\delta & \text{if } s\ge\delta,\\ s & \text{if } |s|<\delta,\\ -\delta & \text{if } s\le-\delta,\end{cases}$$

where $\delta>0$. Use $l_\delta(v_n)$ as a test function in (2.9) to get

$$\begin{aligned}\int_0^T(bu_{nt},l_\delta(v_n))\,dt+\int_{\{|v_n|\le\delta\}}|\nabla v_n|^2\,dx\,dt&\\ &=\int_0^T\int_\Omega\sigma_n(u_n)|\nabla\varphi_n|^2 l_\delta(v_n)\,dx\,dt\\ &\le c\delta.\end{aligned} \tag{2.13}$$

Here and in what follows c, c_i, $i=0,1,\dots$, are also independent of δ unless explicitly indicated otherwise. By the chain rule,

$$(bu_{nt},l_\delta(v_n))=\frac{d}{dt}\int_\Omega\left(\int_0^{u_n}bl_\delta(\beta_n(s))\,ds\right)dx.$$

Furthermore,

$$\int_0^{u_n}l_\delta(\beta_n(s))\,ds\ge0\quad\text{almost everywhere on } Q_T$$

because $l_\delta(\beta_n(s))$ is an increasing function and $l_\delta(\beta_n(0))=0$. In view of these, we deduce from (2.13) that

$$\int_{\{|v_n|\le\delta\}}|\nabla v_n|^2\,dx\,dt\le c_1\delta+c_2\quad\text{for each } n \text{ and each } \delta>0. \tag{2.14}$$

Claim 1. *For each $\delta > 0$, there exists a positive number $c(\delta)$ such that*

$$\int_{\{|v_n|\le\delta\}} |\nabla\varphi_n|^2\,dx\,dt \le c(\delta) \qquad \textit{for all } n. \tag{2.15}$$

Proof of Claim 1. Since

$$\int_0^\infty k(\tau)\,d\tau = \infty, \qquad \int_{-\infty}^0 k(\tau)\,d\tau = -\infty$$

by our assumptions,

$$\sup_{n\ge1}\{\alpha_n(\delta), -\alpha_n(-\delta)\} \equiv A(\delta)$$

is finite for each $\infty > \delta > 0$. Recall that $\sigma(s)$ is strictly positive on $\mathbf{R}$. For $A(\delta) > 0$ there exists a $B(\delta) > 0$ such that

$$\sigma_n(s) \ge B(\delta) \qquad \text{on } [-A(\delta), +A(\delta)]$$

for all n. We are ready to calculate, using (2.12), that

$$\begin{aligned}\int_{\{|v_n|\le\delta\}} |\nabla\varphi_n|^2\,dx\,dt &= \int_{\{\alpha_n(-\delta)\le u_n\le\alpha_n(\delta)\}} |\nabla\varphi_n|^2\,dx\,dt \\ &\le \int_{\{|u_n|\le A(\delta)\}} |\nabla\varphi_n|^2\,dx\,dt \\ &\le \frac{1}{B(\delta)}\int_{\{|u_n|\le A(\delta)\}} \sigma_n(u_n)|\nabla\varphi_n|^2\,dx\,dt \\ &\le \frac{1}{B(\delta)}\int_{Q_T} \sigma_n(u_n)|\nabla\varphi_n|^2\,dx\,dt \\ &\le \frac{c_1}{B(\delta)} \equiv c(\delta). \qquad \square\end{aligned}$$

Claim 2. *For each $\delta > 0$ and each $\theta \in \mathcal{C} = \{\theta \in C^1(\mathbf{R}) : \theta = 0$ on $(-\infty, 0]$, $\theta' \ge 0$, and $\theta = 1$ on $[1,\infty)\}$, $\{\psi_n \equiv \varphi_n(1-\theta(|v_n|/\delta))\}$ is bounded in $L^2(0,T;W^{1,2}(\Omega))$.*

Proof of Claim 2. Clearly, $\{\psi_n\}$ is bounded in $L^\infty(Q_T)$. We compute

$$\nabla\psi_n = \left(1-\theta\left(\frac{|v_n|}{\delta}\right)\right)\nabla\varphi_n - \varphi_n\theta'\left(\frac{|v_n|}{\delta}\right)\frac{1}{\delta}\mathrm{sign}\,(v_n)\nabla v_n.$$

Note that

$$\begin{aligned} 1-\theta(|v_n|/\delta) &= 0 \quad \text{almost everywhere on } \{|v_n|>\delta\},\\ \theta'(|v_n|/\delta) &= 0 \quad \text{almost everywhere on } \{|v_n|\geq\delta\}. \end{aligned}$$

Consequently,

$$\begin{aligned} ||\nabla\psi_n||_{L^2} &\leq \left(\int_{\{|v_n|\leq\delta\}}|\nabla\varphi_n|^2\,dx\,dt\right)^{1/2} \\ &\quad + \frac{c}{\delta}\left(\int_{\{|v_n|\leq\delta\}}|\nabla v_n|^2 dx\,dt\right)^{1/2} \\ &\leq c_1(\delta) \end{aligned} \tag{2.16}$$

due to (2.15) and (2.14). Here we also use the fact that $\max_{s\in\mathbf{R}}|\theta'(s)|\leq c_0$.

Now we are ready to show that

$$\{\varphi_n\} \text{ is precompact in } L^p(Q_T) \text{ for each } p\geq 1.$$

By (2.12) and (2.8), $\{\varphi_{nt}\}$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. Take $B_0=L^2(\Omega)$, $B=B_1=W^{-1,2}(\Omega)$. Since the inclusion $L^2(\Omega)\hookrightarrow W^{-1,2}(\Omega)$ is compact, we are in a position to use Lions-Aubin's theorem to conclude that $\{\varphi_n\}$ is precompact in $L^2(0,T;W^{-1,2}(\Omega))$. Without loss of generality, assume

$$\varphi_n\to\varphi \text{ weakly in } L^2(Q_T) \text{ and strongly in } L^2(0,T;W^{-1,2}(\Omega)). \tag{2.17}$$

For $\theta\in\mathcal{C}$, $\delta>0$, we calculate

$$\begin{aligned} \int_{Q_T}(\varphi_n-\varphi)\varphi_n\,dx\,dt &= \int_{Q_T}(\varphi_n-\varphi)\varphi_n\left(1-\theta\left(\frac{|v_n|}{\delta}\right)\right)dx\,dt \\ &\quad + \int_{Q_T}(\varphi_n-\varphi)\varphi_n\theta\left(\frac{|v_n|}{\delta}\right)dx\,dt \\ &\equiv I_1+I_2. \end{aligned}$$

We have, using (2.17) and (2.16), that

$$\begin{aligned}
|I_1| &\le \left|\left(\varphi_n-\varphi, \varphi_n\left(1-\theta\left(\frac{|v_n|}{\delta}\right)\right)-\bar{\varphi}\right)\right| \\
&\quad + \left|\int_{Q_T}(\varphi_n-\varphi)\bar{\varphi}\,dx\,dt\right| \\
&\le ||\varphi_n-\varphi||_{L^2(0,T;W^{-1,2}(\Omega)}\left\|\varphi_n\left(1-\theta\left(\frac{|v_n|}{\delta}\right)\right)-\bar{\varphi}\right\|_{L^2(0,T;W_0^{1,2}(\Omega)} \\
&\quad + \left|\int_{Q_T}(\varphi_n-\varphi)\bar{\varphi}\,dx\,dt\right| \\
&\le c(\delta)||\varphi_n-\varphi||_{L^2(0,T;W^{-1,2}(\Omega))} + \left|\int_{Q_T}(\varphi_n-\varphi)\bar{\varphi}\,dx\,dt\right| \to 0 \\
&\qquad \text{as } n\to\infty.
\end{aligned}$$

We use Poincaré's inequality to estimate I_2:

$$\begin{aligned}
|I_2| &\le c\left(\int_0^T\int_\Omega \theta^2\left(\frac{|v_n|}{\delta}\right)dx\,dt\right)^{1/2} \\
&\le c_1\left(\int_0^T\int_\Omega \left|\theta'\left(\frac{|v_n|}{\delta}\right)\frac{1}{\delta}\operatorname{sign} v_n\nabla v_n\right|^2 dx\,dt\right)^{1/2} \\
&\le c_3\frac{1}{\delta}\left(\int_{\{|v_n|\le\delta\}}|\nabla v_n|^2\,dx\,dt\right)^{1/2} \\
&\le c\frac{1}{\delta}(c_1\delta+c_2)^{1/2}.
\end{aligned}$$

The last step is due to (2.14). Also, note that here c, c_1, c_2 are independent of δ. Hence,

$$\limsup_{n\to\infty}\left|\int_{Q_T}(\varphi_n-\varphi)\varphi_n\,dx\,dt\right| \le c\frac{1}{\delta}(c_1\delta+c_2)^{1/2}.$$

Here $\delta > 0$ is arbitrary. In particular, taking $\delta\to\infty$ yields

$$\lim_{n\to\infty}\int_{Q_T}(\varphi_n-\varphi)\varphi_n\,dx\,dt = 0.$$

We have

$$\int_{Q_T} (\varphi_n - \varphi)^2 \, dx \, dt = \int_{Q_T} (\varphi_n - \varphi)\varphi_n \, dx \, dt - \int_{Q_T} (\varphi_n - \varphi) \, \varphi \, dx \, dt$$
$$\to 0 \qquad \text{as } n \to \infty.$$

Since $\{\varphi_n\}$ is also bounded in $L^\infty(Q_T)$, this completes the proof. □

Lemma 2.2. *If there exists* $0 < m_1 \le m_2$ *such that*

$$m_1 \le \frac{\sigma(s)}{k(s)} \le m_2 \qquad \text{on } \mathbf{R},$$

then

$$\sup_{0 \le t \le T} \int_\Omega b u_n^2(x,t) \, dx + \int_{Q_T} k_n(u_n) |\nabla u_n|^2 \, dx \, dt \le c.$$

Proof. By Lemma 1.1, we have

$$\left(\frac{a}{2} \varphi_n^2 + b u_n \right)_t = \operatorname{div}\left(k_n(u_n) \nabla u_n\right) + \operatorname{div}\left(\sigma_n(u_n) \varphi_n \nabla \varphi_n\right) \tag{2.18}$$

in

$$L^2(0,T; W^{-1,2}(\Omega)).$$

Use $(a/2)\varphi_n^2 + b u_n - (1/2)\bar\varphi^2$ as a test function to obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_\Omega \left(\frac{a}{2} \varphi_n^2 + b u_n - \frac{a}{2} \bar\varphi^2 \right)^2 dx + \int_\Omega a \overline{\varphi \varphi_t} \left(\frac{a}{2} \varphi_n^2 + b u_n - \frac{a}{2} \bar\varphi^2 \right) dx \\
&= - \int_\Omega (k_n(u_n) \nabla u_n + \sigma_n(u_n) \varphi_n \nabla \varphi_n)(a \varphi_n \nabla \varphi_n + b \nabla u_n - a \bar\varphi \nabla \bar\varphi) \, dx \\
&= - \int_\Omega \sqrt{\frac{k_n(u_n)}{\sigma_n(u_n)}} \sqrt{k_n(u_n)} \nabla u_n a \varphi_n \sqrt{\sigma_n(u_n)} \nabla \varphi_n \, dx \\
&\quad - b \int_\Omega k_n(u_n) |\nabla u_n|^2 \, dx - a \int_\Omega \sigma_n(u_n) \varphi_n^2 |\nabla \varphi_n|^2 \, dx \\
&\quad - \int_\Omega \sqrt{\frac{\sigma_n(u_n)}{k_n(u_n)}} \varphi_n \sqrt{\sigma_n(u_n)} \nabla \varphi_n b \sqrt{k_n(u_n)} \nabla u_n \, dx \\
&\quad + \int_\Omega (k_n(u_n) \nabla u_n + \sigma_n(u_n) \varphi_n \nabla \varphi_n) a \bar\varphi \nabla \bar\varphi \, dx.
\end{aligned}$$

Consider the function $g(s) = (c_1 + s)/(c_2 + s)$. Compute

$$g'(s) = \frac{c_2 + s - (c_1 + s)}{(c_2 + s)^2} = \frac{c_2 - c_1}{(c_2 + s)^2}.$$

Thus $g(s)$ is increasing when $c_2 \geq c_1$ and decreasing when $c_1 \leq c_2$. Taking a note of this, we estimate

$$\begin{aligned}\frac{k_n(s)}{\sigma_n(s)} = \frac{k(s) + 1/n}{\sigma(s) + 1/n} &= \frac{k(s) + 1/n}{k(s)\frac{\alpha(s)}{k(s)} + 1/n} \\ &\leq \frac{k(s) + 1/n}{m_1 k(s) + 1/n} \\ &\leq \begin{cases} \dfrac{k(s) + 1}{m_1 k(s) + 1} \leq M + 1 & \text{if } m_1 \geq 1 \\ \dfrac{k(s)}{m_1 k(s)} = \dfrac{1}{m_1} & \text{if } m_1 < 1. \end{cases}\end{aligned}$$

Similarly,

$$\frac{k_n(s)}{\sigma_n(s)} \geq \frac{k(s) + 1/n}{k(s)m_2 + 1/n} \geq \begin{cases} \dfrac{1}{m_2}, & m_2 \geq 1 \\ \dfrac{k(s) + 1}{k(s)m_2 + 1} > \dfrac{1}{Mm_2 + 1}, & m_2 < 1. \end{cases}$$

With this in mind, apply Hölder's inequality and (2.11) to (2.19) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\bar{\varphi}^2\right)^2 dx + c_0 \int_\Omega k_n(u_n)|\nabla u_n|^2\, dx \\ \leq c_1 \int_\Omega \left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\bar{\varphi}^2\right)^2 dx + c_2.\end{aligned}$$

Integration with respect to t and applications of Gronwall's inequality and the fact that $\{\varphi_n\}$ is bounded in $L^\infty(Q_T)$ yield the desired result.

It follows from Lemma 2.2 that

$$\begin{aligned}\int_{Q_T} |\nabla v_n|^2\, dx\, dt &= \int_{Q_T} k_n^2(u_n)|\nabla u_n|^2\, dx\, dt \\ &\leq (M+1)\int_{Q_T} k_n(u_n)|\nabla u_n|^2\, dx\, dt \leq c.\end{aligned}$$

Poincaré's inequality implies that $||v_n||_{L^2(Q_T)} \leq c$. We may assume that

$$u_n \to u \qquad \text{weakly in } L^2(Q_T), \tag{2.20}$$

$$v_n \to v \qquad \text{weakly in } L^2(0,T;W_0^{1,2}(\Omega)) \tag{2.21}$$

(passing to subsequences if necessary). □

Claim 3. *$v_n \to v$ strongly in $L^2(Q_T)$.*

Proof of Claim 3. We see from (2.18) and Lemma 2.2 that $\{((a/2)\varphi_n^2 + bu_n)_t\}$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. Once again, we are in a position to employ Lions-Aubin's theorem to conclude that $\{(a/2)\varphi_n^2 + bu_n\}$ is precompact in $L^2(0,T;W^{-1,2}(\Omega))$. In view of Lemma 2.1, (2.17), and (2.20), we have

$$\frac{a}{2}\varphi_n^2 + bu_n \to \frac{1}{2}\varphi^2 + bu \qquad \text{strongly in } L^2(0,T;W^{-1,2}(\Omega)).$$

Consequently,

$$\begin{aligned}\int_{Q_T} &\left[\frac{a}{2}\varphi_n^2 + bu_n - \left(\frac{a}{2}\varphi^2 + bu\right)\right](v_n - v)\,dx\,dt \\ &= \left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\varphi^2 - bu, v_n - v\right) \to 0 \qquad \text{as } n \to \infty,\end{aligned}$$

from whence follows

$$\begin{aligned}&b\int_{Q_T}(u_n - u)(v_n - v)\,dx\,dt \\ &= \int_{Q_T}\left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\varphi^2 - bu\right)(v_n - v)\,dx\,dt - \frac{a}{2}\int_{Q_T}(\varphi_n^2 - \varphi^2)(v_n - v)\,dx\,dt \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&b\int_{Q_T}(u_n - u)(\beta_n(u_n) - \beta_n(u))\,dx\,dt \\ &= b\int_{Q_T}(u_n - u)(v_n - v)\,dx\,dt + b\int_{Q_T}(u_n - u)(v - \beta_n(u))\,dx\,dt \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned} \tag{2.22}$$

The second term tends to 0 as $n \to \infty$ because $|\beta_n(s)| \le (M+1)|s|$ and $\lim_{n\to\infty} \beta_n(s) = \int_0^s k(\tau)\,d\tau \equiv \beta(s)$ for all s, and thus Lebesgue's dominated convergence theorem implies that $\beta_n(u) \to \beta(u)$ strongly in $L^2(Q_T)$. We deduce from (2.22) that

$$0 \le (u_n - u)(\beta_n(u_n) - \beta_n(u)) \to 0 \quad \text{almost everywhere on } Q_T \tag{2.23}$$

(passing to a further subsequence if need be). Recall that $\beta(s)$ is strictly increasing. Then (2.23) implies

$$u_n \to u \quad \text{almost everywhere on } Q_T. \tag{2.24}$$

Subsequently, we also have

$$v_n \to v \quad \text{almost everywhere on } Q_T \tag{2.25}$$

since $\beta_n \to \beta$ uniformly on bounded subsets of $\mathbf{R}$. Note that

$$\begin{aligned} \int_{Q_T} u_n v_n\,dx\,dt &= \int_{Q_T} ((u_n - u)(v_n - v) + u_n v + u v_n - uv)\,dx\,dt \\ &= \int_{Q_T} (u_n - u)(v_n - v)\,dx\,dt + \int_{Q_T} u_n v\,dx\,dt \\ &\quad + \int_{Q_T} u_n v\,dx\,dt - \int_{Q_T} uv\,dx\,dt \\ &\to \int_{Q_T} uv\,dx\,dt \qquad \text{as } n \to \infty. \end{aligned} \tag{2.26}$$

Now for any subset $E \subset Q_T$ we have from Fatou's lemma that

$$\liminf_{n\to\infty} \int_E u_n v_n\,dx\,dt \ge \int_E uv\,dx\,dt. \tag{2.27}$$

On the other hand, by (2.26)

$$\begin{aligned} \limsup_{n\to\infty} \int_E u_n v_n\,dx\,dt & \\ &\le \limsup_{n\to\infty} \int_{Q_T} u_n v_n\,dx\,dt - \liminf_{n\to\infty} \int_{Q_T\setminus E} u_n v_n\,dx\,dt \\ &\le \int_{Q_T} uv\,dx\,dt - \int_{Q_T\setminus E} uv\,dx\,dt \\ &= \int_E uv\,dx\,dt. \end{aligned} \tag{2.28}$$

Thus, combining (2.27) and (2.28) yields

$$\lim_{n\to\infty}\int_E u_n v_n\,dx\,dt = \int_E uv\,dx\,dt$$

for each subset E of Q_T. Now we can appeal to a result in [**6**, p. 144] to conclude that $\{u_n v_n\}$ is uniformly integrable. Since $|v_n| \le (M+1)|u_n|$, $\{v_n^2\}$ is also uniformly integrable. In view of this and (2.25), we may apply Vitali's theorem to obtain our desired result.

By virtue of (2.12), we may also assume that

$$g_n \equiv \sigma_n(u_n)\nabla\varphi_n \to g \quad \text{weakly in } [L^2(Q_T)]^N. \qquad \square$$

Claim 4. *φ, g satisfy* (iii)*, where φ is given as in* (2.17)*.*

Proof of Claim 4. For each $\rho \in \mathcal{A}$ there exists a $\delta > 0$ such that the support of ρ is contained in $(-\delta, \delta)$. We compute

$$\nabla(\rho(v_n)\varphi_n) = \rho(v_n)\nabla\varphi_n + \rho'(v_n)\nabla v_n\varphi_n.$$

In view of (2.15) and the fact that $\rho(v_n) = 0$ on $\{|v_n| > \delta\}$, we obtain that $\{\rho(v_n)\varphi_n\}$ is bounded in $L^2(0,T;W^{1,2}(\Omega))$ for each ρ in $\mathcal{A}$. Thus, we may assume, using our claim that $\{v_n\}$ is precompact in $L^2(Q_T)$ and (2.17), that $\rho(v_n)\varphi_n \to \rho(v)\varphi$ weakly in $L^2(0,T;W^{1,2}(\Omega))$. That is, $\rho(v)\varphi \in L^2(0,T;W^{1,2}(\Omega))$. Note that $\rho(0) = 1$. Thus, $\rho(v_n) = 1$ on S_T. Consequently, $\rho(v)\varphi - \bar{\varphi} \in L^2(0,T;W_0^{1,2}(\Omega))$. Let us calculate

$$\begin{aligned}\rho(v_n)g_n &= \rho(v_n)\sigma_n(u_n)\nabla\varphi_n \\ &= \sigma_n(u_n)(\nabla(\rho(v_n)\varphi_n) - \varphi_n\nabla\rho(v_n)).\end{aligned}$$

By virtue of Lemma 2.1 and (2.24), we can take $n \to \infty$ in this identity to get the desired result. $\square$

In view of (2.12), lemmas 2.1 and 2.2, we can pass to the limits in (2.8) and (2.18) to get (i) and (ii). By Claim 3, $v = \beta(u)$. This, together with Claim 4, completes the proof of Theorem 1.2. $\square$

3. The nondegenerate case. The main result of this section is:

Theorem 3.1. *Assume that* σ *and* k *are continuous and satisfy*

$$0 < m \leq \sigma, k \leq M \quad \text{on } \mathbf{R}$$

for some constants m *and* M*, and assume* (H2). *Then there exists a classical weak solution to* (1.1).

If the boundary conditions are replaced by $\varphi = 0$, $\partial u/\partial v = 0$ on S_T, then this theorem is proved in [**8**] via the method of implicit discretization in time. The proof presented here entails weaker assumptions on the initial-boundary data. Set $p(x_1, \dots, x_N) = x_1^2 + x_2^2 + \cdots + x_N^2$ for $x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$. For each n define

$$p_n(x) = \begin{cases} n, & \text{if } p(x) \geq n \\ p(x), & \text{if } p(x) < n. \end{cases}$$

Lemma 3.2. *For each* n *there is a weak solution to the following problem:*

$$\begin{aligned} &a\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi) \qquad \text{in } Q_T \\ &bu_t = \operatorname{div}(k(u)\nabla u) + \sigma(u)p_n(\nabla\varphi) \qquad \text{in } Q_T \\ &\varphi = \bar{\varphi}, \qquad u = 0 \qquad \text{on } S_T, \\ &\varphi = \varphi_0, \qquad u = u_0 \qquad \text{on } \Omega \times \{0\}. \end{aligned}$$

Proof. Define a nonlinear operator F from $L^2(Q_T)$ to $L^2(Q_T)$ by: $F(u) = v$ if v is the solution of the following problem:

$$bv_t = \operatorname{div}(k(v)\nabla v) + \sigma(u)p_n(\nabla\varphi) \qquad \text{in } Q_T, \tag{3.1a}$$

$$v = 0 \qquad \text{on } S_T, \tag{3.1b}$$

$$v = u_0 \qquad \text{on } \Omega \times \{0\}, \tag{3.1c}$$

where φ solves the problem:

$$a\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi) \qquad \text{in } Q_T, \tag{3.2a}$$

$$\varphi = \bar{\varphi} \qquad \text{on } S_T, \tag{3.2b}$$

$$\varphi = \varphi_0 \qquad \text{on } \Omega \times \{0\}. \tag{3.2c}$$

Standard theory asserts that for any $u \in L^2(Q_T)$, (3.2) has a classical weak solution φ in $L^2(0,T;W^{1,2}(\Omega))$. For such a φ, we may conclude the existence of a classical weak solution v in $L^2(0,T;W_0^{1,2}(\Omega))$ to (3.1). Thus F is well-defined.

Let v_* be the solution of the problem:

$$bv_t = \operatorname{div}(k(v)\nabla v) \qquad \text{in } Q_T,$$
$$v = 0 \qquad \text{on } S_T,$$
$$v = u_0 \qquad \text{on } \Omega \times \{0\};$$

and v^* the solution of the problem

$$bv_t = \operatorname{div}(k(v)\nabla v) + H \qquad \text{in } Q_T,$$
$$v = 0 \qquad \text{on } S_T,$$
$$v = u_0 \qquad \text{on } \Omega \times \{0\},$$

where $H \equiv \operatorname{ess\,sup}_{Q_T} \sigma(u)p_n(\nabla\varphi) \leq Mn$. Then, for each $u \in L^2(Q_T)$ and each $\varphi \in L^2(0,T;W^{1,2}(\Omega))$, the solution v of (3.1) satisfies:

$$v_* \leq v \leq v^* \quad \text{almost everywhere on } Q_T$$

according to the comparison principle. Thus the range of F is contained in the set D defined by:

$$D = \{f \in L^2(Q_T) : v_* \leq f \leq v^* \text{ almost everywhere on } Q_T\}.$$

It is easy to see that D is a closed convex bounded subset of $L^2(Q_T)$.

Claim 5. *F is continuous.*

Proof of Claim 5. Suppose that $\{u_k\}$ is a sequence in $L^2(Q_T)$ such that

$$u_k \to u \qquad \text{in } L^2(Q_T) \qquad \text{as } n \to \infty.$$

For each k, let φ_k be the solution of (3.2) corresponding to u_k. Then we have

$$a\varphi_{kt} = \operatorname{div}(\sigma(u_k)\nabla\varphi_k) \qquad \text{in } Q_T, \tag{3.3a}$$
$$\varphi_k = \bar{\varphi} \qquad \text{on } S_T, \tag{3.3b}$$
$$\varphi_k = \varphi_0 \qquad \text{on } \Omega \times \{0\}. \tag{3.3c}$$

Using $\varphi_k - \bar{\varphi}$ as a test function in (3.3a) yields the usual energy estimate:

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_\Omega \varphi_k^2(x,t)\,dx + \int_{Q_T}|\nabla\varphi_k|^2\,dx\,dt \le c, \qquad k = 1,2,\dots .$$

Thus, there exists a subsequence $\{\varphi_{k_j}\} \subset \{\varphi_k\}$ which converges to an element φ weakly in $L^2(0,T;W^{1,2}(\Omega))$. Passing to the limit in (3.3) along the subsequence, we obtain that φ is a classical weak solution of (3.2) corresponding to u. However, for each u there is only one solution to (3.2). Hence, the whole sequence $\{\varphi_k\}$ converges to φ weakly in $L^2(0,T;W^{1,2}(\Omega))$. Use $\varphi_k - \varphi$ as a test function in (3.3a) to get

$$\int_0^T (a\varphi_{kt}, \varphi_k - \varphi)\,dt + \int_{Q_T}\sigma(u_k)\nabla\varphi_k\nabla(\varphi_k-\varphi)\,dx\,dt = 0.$$

Note that

$$\int_0^T (a\varphi_{kt}, \varphi_k - \varphi)\,dt = \frac{a}{2}\int_\Omega(\varphi_k(x,T)-\varphi(x,T))^2\,dx + \int_0^T(a\varphi_t,\varphi_k-\varphi)\,dt.$$

Thus, we have

$$\int_{Q_T}\sigma(u_k)|\nabla(\varphi_k-\varphi)|^2\,dx\,dt = -\int_{Q_T}\sigma(u_k)\nabla\varphi\nabla(\varphi_k-\varphi)\,dx\,dt - \int_0^T(a\varphi_t,\varphi_k-\varphi)\,dt - \frac{a}{2}\int_\Omega(\varphi_k(x,T)-\varphi(x,T))^2\,dx.$$

This implies

$$\lim_{k\to\infty}\int_{Q_T}|\nabla(\varphi_k-\varphi)|^2\,dx\,dt = 0.$$

Here we use the fact that

$$(\sigma(u_k)-\sigma(u))\nabla(\varphi_k-\varphi)\to 0 \quad \text{weakly in } [L^2(Q_T)]^N.$$

Consequently,

$$p_n(\nabla\varphi_k) \to p_n(\nabla\varphi) \quad \text{strongly in } L^1(Q_T).$$

For each k let $v_k = F(u_k)$. We have

$$bv_{kt} = \operatorname{div}(k(v_k)\nabla v_k) + \sigma(u_k)p_n(\nabla\varphi_k), \tag{3.4a}$$

$$v_k = 0 \qquad \text{on } S_T, \tag{3.4b}$$

$$v_k = u_0 \qquad \text{on } \Omega\times\{0\}. \tag{3.4c}$$

Similarly, we can show that $\{v_k\}$ is bounded in

$$W_2(0,T) \equiv \{v \in L^2(0,T;W_0^{1,2}(\Omega)) : v_t \in L^2(0,T;W^{-1,2}(\Omega))\}.$$

Hence, by Lions-Aubin's theorem, there exists a subsequence $\{v_{k_j}\} \subset \{v_k\}$ which converges to v weakly in $W_2(0,T)$ and strongly in $L^2(Q_T)$. Passing to the limit along the subsequence in (3.4), we see that v is a solution of (3.1) corresponding to u and φ. But for each $u \in L^2(Q_T)$ and each $\varphi \in L^2(0,T;W^{1,2}(\Omega))$, there is only one solution to (3.1). Consequently, the entire sequence converges to v. That is, $v_k = F(u_k) \to v = F(u)$ strongly in $L^2(Q_T)$. This completes the proof of Claim 5. □

Since $0 \le \sigma(u)p_n(\nabla\varphi) \le Mn$, the solution of (3.1) is bounded in $W_2(0,T)$ uniformly in u and φ. It suffices to recall the compact imbedding $W_2(0,T)\hookrightarrow L^2(Q_T)$ to see that F sends bounded subsets into relatively compact ones.

By the Schauder fixed point theorem, there is a u in D such that $Fu = u$. By the definition of F, this implies the conclusion of our lemma.

Proof of Theorem 3.1. For each n there exists a weak solution (φ_n, u_n) to the following problem:

$$a\varphi_{nt} = \operatorname{div}(\sigma(u_n)\nabla\varphi_n) \qquad \text{in } Q_T \tag{3.5a}$$

$$bu_{nt} = \operatorname{div}(k(u_n)\nabla u_n) + \sigma(u_n)p_n(\nabla\varphi_n) \qquad \text{in } Q_T \tag{3.5b}$$

$$\varphi_n = \bar\varphi, \qquad u_n = 0 \qquad \text{on } S_T, \tag{3.5c}$$

$$\varphi_n = \varphi_0, \qquad u_n = u_0 \qquad \text{on } \Omega\times\{0\}. \tag{3.5d}$$

We can easily show that

$$\{\varphi_n - \bar{\varphi}\} \text{ is bounded in } W_2(0,T) \text{ and } L^\infty(Q_T). \tag{3.6}$$

Since $\sigma(u_n)p_n(\nabla\varphi_n) \le \sigma(u_n)|\nabla\varphi_n|^2$, we use the proof of Lemma 1.2 to obtain

$$\begin{aligned}\left(\frac{a}{2}\varphi_n^2 + bu_n\right)_t &\le \operatorname{div}(k(u_n)\nabla u_n)\\ &\quad + \operatorname{div}(\varphi_n\sigma(u_n)\nabla\varphi_n) \qquad \text{in } \mathcal{D}'(Q_T).\end{aligned} \tag{3.7}$$

Set

$$D_n(t) = \left\{(x,t) \in Q_T : \frac{a}{2}\varphi_n^2(x,t) + bu_n(x,t) - \frac{a}{2}\bar{\varphi}^2(x,t) \ge 0\right\}.$$

Use $((a/2)\varphi_n^2 + bu_n - (a/2)\bar{\varphi}^2)^+$ as a test function in (3.7) to get

$$\begin{aligned}&\frac{a}{2}\frac{d}{dt}\int_\Omega \left[\left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\bar{\varphi}^2\right)^+\right]^2 dx + \int_\Omega a\overline{\varphi}\overline{\varphi}_t\left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{1}{2}\bar{\varphi}^2\right)^+ dx\\ &\quad\le \int_{D_n(t)}(k(u_n)\nabla u_n + \varphi_n\sigma(u_n)\nabla\varphi_n)(a\varphi_n\nabla\varphi_n + b\nabla u_n - a\bar{\varphi}\nabla\bar{\varphi})\,dx.\end{aligned}$$

This implies

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_\Omega \left[\left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\bar{\varphi}^2\right)^+\right]^2 dx + \int \cup_{0\le t\le T} D_n(t)|\nabla u_n|^2\,dx\,dt \le c. \tag{3.8}$$

Set

$$M_1 = \operatorname*{ess\,sup}_{\substack{n\ge 1\\ (x,t)\in Q_T}}\left\{\frac{a}{2}\bar{\varphi}^2(x,t) - \frac{a}{2}\varphi_n^2(x,t)\right\}.$$

Since $\{\varphi_n\}$ is bounded in $L^\infty(Q_T)$, M_1 is finite. We infer from (3.8) that

$$\int_{\{u_n\ge M_1\}} |\nabla u_n|^2\,dx\,dt \le c. \tag{3.9}$$

We have from (3.5b) that

$$bu_{nt} \ge \operatorname{div}(k(u_n)\nabla u_n) \qquad \text{in } \mathcal{D}'(Q_T).$$

Use $(u_n + M_1)^-$ as a test function to get

$$0 \geq b \int_\Omega \int_0^{u_n(x,T)} (s + M_1)^- \, ds \, dx$$
$$\geq \int_{\{u_n \leq -M_1\}} k(u_n)|u_n|^2 \, dx \, dt + b \int_\Omega \int_0^{u_0} (s + M_1)^- \, ds \, dx,$$

from whence follows

$$\int_{\{u_n \leq -M_1\}} k(u_n)|\nabla u_n|^2 \, dx \, dt \leq c. \tag{3.10}$$

Note that $\{\sigma(u_n)p_n(\nabla\varphi_n)\}$ is bounded in $L^1(Q_T)$. By a calculation similar to (2.13), we also get

$$\int_{\{|u_n| \leq M_1\}} u_n^2 \, dx \, dt + \int_{\{|u_n| \leq M_1\}} |\nabla u_n|^2 \leq c_1 M_1 + c_2. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11) yields

$$\int_{Q_T} |\nabla u_n|^2 \, dx \, dt \leq c. \tag{3.12}$$

For $0 < h < T$ define $u_h(x,t) = u(x, t+h)$ for $u \in L^2(Q_T)$. Integrate (3.5b) with respect to t over $(t, t+h)$ to get

$$b(u_{nh} - u_n) \geq \operatorname{div} \int_t^{t+h} k(u_n)\nabla u_n \, d\tau. \tag{3.13}$$

Using $(u_{nh} - u_n)^-$ as a test function in (3.13), by a calculation similar to that in [**11**, p. 129] we get

$$\int_0^{T-h} \int_\Omega [(u_{nh} - u_n)^-]^2 \, dx \, dt \leq ch^{1/2}. \tag{3.14}$$

Here, and in what follows, c is also independent of h. It follows from (3.7) that

$$\frac{a}{2}\varphi_{nh}^2 + bu_{nh} - \frac{a}{2}\varphi_n^2 - bu_n$$
$$\leq \operatorname{div} \int_t^{t+h} (k(u_n)\nabla u_n + \varphi_n \sigma(u_n)\nabla\varphi_n) \, d\tau. \tag{3.15}$$

Using

$$\left[\left(\frac{a}{2}\varphi_n^2 + bu_n - \frac{a}{2}\bar{\varphi}^2\right)_h - \frac{a}{2}\varphi_n^2 - bu_n + \frac{a}{2}\bar{\varphi}^2\right]^+$$

as a test function in (3.15), after a calculation in which we use the fact that

$$(s_1 + s_2)^+ \le s_1^+ + s_2^+ \qquad \text{for all } s_1, s_2 \in \mathbf{R},$$

$$\int_0^{T-h}\int_\Omega (\varphi_{nh} - \varphi_n)^2 \, dx \, dt \le ch^{1/2}, \tag{3.16}$$

and

$$\int_0^{T-h}\int_\Omega (\bar{\varphi}_h - \bar{\varphi})^2 \, dx \, dt \le ch^{1/2},$$

we arrive at

$$\int_0^{T-h}\int_\Omega [(u_{nh} - u_n)^+]^2 \, dx \, dt \le ch^{1/2}. \tag{3.17}$$

Here (3.16) is due to (3.6). In view of (3.14), (3.17) and (3.12), we may invoke a result in [**7**] to conclude that $\{u_n\}$ is precompact in $L^2(Q_T)$. Once we have this result, we can proceed as in the proof of Claim 5 to show that $\{\varphi_n\}$ is precompact in $L^2(0, T; W^{1,2}(\Omega))$. We are ready to pass to the limits in (3.5) to conclude our proof. □

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