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BIFURCATION OF SYNCHRONIZED PERIODIC SOLUTIONS IN SYSTEMS OF COUPLED OSCILLATORS I: PERTURBATION RESULTS FOR WEAK AND STRONG COUPLING

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ABSTRACT. This paper concerns a class of differential equations that govern the evolution of indirectly coupled oscillators. We establish the existence of synchronized periodic solutions for weak and strong coupling under certain conditions. The stability of the periodic solutions is also analyzed.

1. Introduction. It is shown in [**14**] that a number of problems in physics, chemistry and biology lead to systems of ordinary differential equations that represent oscillatory subunits coupled indirectly through a passive medium. In this paper we study the case where the oscillators, which govern the states of the uncoupled subunits, are all identical. That is, we study the following system of ordinary differential equations.

$$\begin{aligned} \frac{dx_i}{dt} &= f(x_i) + \delta P(x_0 - x_i), \qquad i = 1, \dots, N, \\ \frac{dx_0}{dt} &= \varepsilon \delta P\bigg(\frac{1}{N}\sum_{i=1}^{N} x_i - x_0\bigg). \end{aligned} \tag{1}$$

Here the variable x_0 represents the state of the coupling medium through which the subunits are coupled. P is an $n \times n$ constant matrix of permeability coefficients or conductances, and the parameters ε^{-1} and δ measure the relative capacity of the coupling medium and the coupling strength, respectively [**8, 9**]. In the absence of coupling the evolution in the i^{th} subunit is governed by the n-dimensional system $dx_i/dt = f(x_i)$ and it is assumed that this system has a nonconstant periodic solution. We show when (1) has periodic solutions and analyze the stability of these periodic solutions.

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In Section 2 we study systems that involve two parameters and prove some general results that are applicable to (1). In Section 2.1 we study a system that involves two parameters and has a family of periodic solutions when these parameters are set equal to zero. The variational system associated with each member of the family has more than one multiplier that is equal to 1 and a standard technique (cf. [**3, 5, 6, 11**]) often used to establish the persistence of periodic solutions is not applicable. We prove that, under certain conditions, a two-parameter family of periodic solutions bifurcates from a particular member of the family. We also study the stability of these periodic solutions by computing the estimates for the multipliers associated with them. In Section 2.2 we study a singularly perturbed system that also involves two parameters. A center manifold reduction leads to a system for which the standard technique is applicable and the reduced system has periodic solutions for certain ranges of parameters. The techniques used in Sections 2.1 and 2.2 to prove the existence of periodic solutions are more or less standard. However, we present a new result concerning the behavior of the multipliers associated with the periodic solutions.

In Section 3 we establish sufficient conditions for the existence and stability of periodic solutions of (1) for some extreme values of the parameters. On the subspace defined by $x_1 = x_2 = \cdots = x_N$, the problem is reduced to the case where $N = 1$. The solutions on this subspace give rise to synchronized solutions of (1) in which the evolutions of $x_1, \dots, x_N$ are all identical. In particular, periodic solutions on the subspace correspond to synchronized periodic solutions of (1). We apply the results of Section 2 to the reduced problem and show that, under certain conditions, (1) has synchronized periodic solutions when $|\varepsilon\delta|$ and $|\delta|$ are both small, $|\delta|$ is small and $|\varepsilon\delta|$ is large, or $\varepsilon \neq -1, 0$ and $|\delta|$ is large.

The analysis of Sections 2 and 3 leads to the conclusion that under certain conditions synchronized periodic solutions exist when the coupling is sufficiently weak or sufficiently strong. In [**13**] we present a result obtained in [**12**] in which a two-parameter family of global branches of synchronized periodic solutions is constructed with a particular choice of f and P in (1). We find in [**13**] that the behavior of the branches for intermediate values of δ depends on ε and another parameter β which represents the frequency of the oscillator. When β is large, the periodic solutions disappear via a Hopf bifurcation for a certain range of δ. On the other hand, when ε is small, the branch has turning points. Moreover, the coexistence of the Hopf bifurcation points and the turning points takes place in a region in the (ε, β) plane. We also present some results concerning the stability of the synchronized periodic solutions in [**13**].

Studies related to (1) are found in a number of publications including [**9, 8, 12, 14, 15, 11, 1**, and **4**]. The relationship between the results obtained in these references and this paper is discussed in Section 4. Preliminary results of this paper have appeared in [**12**].

2. Periodic solutions of systems with small parameters. As is indicated in the introduction, (1) is reduced to the case $N = 1$ on the subspace $x_1 = x_2 = \cdots = x_N$ (cf. Section 3). In this section we study two problems, which we call a degenerate problem and a singular problem, that include the reduced system as a special case when the parameters take on some extreme values. We show when these systems have periodic solutions and analyze the stability of these solutions. These problems involve functions $F : \mathbf{R}^m \to \mathbf{R}^m$, $G : \mathbf{R}^{m+n+2} \to \mathbf{R}^m$, and $H : \mathbf{R}^{m+n+2} \to \mathbf{R}^n$, on which the following assumptions are made.

Assumption 1. (a) *$F : \mathbf{R}^m \to \mathbf{R}^m$ is k-times continuously differentiable, and there is a nonconstant periodic solution $\gamma(t)$ of the m-dimensional system,*

$$\frac{du}{dt} = F(u) \tag{2}$$

with least period $T > 0$.

(b) *1 is a simple multiplier of the variational system of* (2) *with respect to $\gamma(t)$:*

$$\frac{du}{dt} = DF(\gamma(t))u. \tag{3}$$

(c) *$G : \mathbf{R}^{m+n+2} \to \mathbf{R}^m$ and $H : \mathbf{R}^{m+n+2} \to \mathbf{R}^n$ are k-times continuously differentiable.*

We assume that $k \geq 3$ for the degenerate problem and $k \geq 2$ for the singular problem.

Remark 1. (a) Under Assumption 1 (a) there is an $m\times(m-1)$-matrix $\Psi(\theta)$ such that each entry of $\Psi(\theta)$ is k-times continuously differentiable and

$$\Psi(\theta+T)=\Psi(\theta),\qquad \Psi(\theta)^T\Psi(\theta)=I_{(m-1)\times(m-1)},$$
$$F(\gamma(\theta))^T\Psi(\theta)=O_{1\times(m-1)}$$

for all $\theta\in\mathbf{R}$.

(b) One of the multipliers of (3) is 1. The remaining $m-1$ multipliers of (3) are the multipliers of

$$\frac{dy}{dt}=A(t)y, \tag{4}$$

where

$$A(t)=\Psi(t)^T[DF(\gamma(t))\Psi(t)-\Psi'(t)]. \tag{5}$$

(cf. [**11, 5**]. Let $Y(t,t_0)$ be the fundamental matrix solution of (4) with $Y(t_0,t_0)=I_{(m-1)\times(m-1)}$. Then Assumption 1 (b) implies that 1 is not an eigenvalue of $Y(t_0+jT,t_0)$ for any integer $j\neq 0$.

2.1. Bifurcation of periodic solutions in the degenerate problem. We first study the system of ordinary differential equations

$$\frac{du}{dt}=F(u)+G(u,v,\mu,\kappa),\qquad \frac{dv}{dt}=\mu H(u,v,\mu,\kappa) \tag{6}$$

under Assumption 1 with $k\geq 3$ and the following additional assumptions.

Assumption 2. (a) *$G(u,v,0,0)=0$ for all $u\in\mathbf{R}^m$ and $v\in\mathbf{R}^n$, i.e., $G(u,v,\mu,\kappa)$ can be written in the form $G(u,v,\mu,\kappa)=\mu G_1(u,v,\mu,\kappa)+\kappa G_2(u,v,\mu,\kappa)$ where $G_i:\mathbf{R}^{m+n+2}\to\mathbf{R}^m$ is $(k-1)$-times continuously differentiable for $i=1,2$.*

(b) *There is a $\xi^*\in\mathbf{R}^n$ such that $a(\xi^*)=0$ and $\det[Da(\xi^*)]\neq 0$, where a is the function defined by*

$$a(v)=\int_0^T H(\gamma(s),v,0,0)\,ds.$$

Assumption 3. (a) l_1 *multipliers of* (3) *have modulus less than* 1 *and* $m - l_1 - 1$ *multipliers have modulus greater than* 1.

(b) l_2 *eigenvalues of* $Da(\xi^*)$ *have negative real part and the remaining* $n - l_2$ *eigenvalues have positive real parts.*

Under Assumptions 1 and 2, (6) has an n-parameter family of periodic solutions

$$u = \gamma(t), \qquad v = c, \qquad c \in \mathbf{R}^n$$

when $\mu = 0$ and $\kappa = 0$. We show that (6) has periodic solutions for all small $|\mu|$ and $|\kappa|$. These periodic solutions tend to the solution

$$u = \gamma(t), \qquad v = \xi^* \tag{7}$$

as $|\mu| + |\kappa| \to 0$. That is, a two-parameter family of periodic solutions bifurcates from the n-parameter family and (7) is the bifurcation point. We also compute the estimates for the multipliers associated with the periodic solutions and determine their stability under Assumption 3.

We first convert (6) using $\Psi(\theta)$ (cf. Remark 1 (a)). There is a neighborhood of the origin in $\mathbf{R}^{m-1}$, which we call W, such that

$$u = \gamma(\theta) + \Psi(\theta)y \tag{8}$$

defines a transformation between a neighborhood of the orbit of $\gamma(t)$ and $\mathbf{R} \times W$, and we obtain the following system of ordinary differential equations for θ, y, and v:

$$\begin{aligned} \frac{d\theta}{dt} &= 1 + \Theta_0(\theta, y) + \Theta_1(\theta, y, v, \mu, \kappa), \\ \frac{dy}{dt} &= A(\theta)y + Y_0(\theta, y) + Y_1(\theta, y, v, \mu, \kappa), \\ \frac{dv}{dt} &= \mu V(\theta, y, v, \mu, \kappa), \end{aligned} \tag{9}$$

where the matrix A is as defined at (5) and

$$\Theta_0(\theta, y) = \frac{F(\gamma(\theta))^T F(\gamma(\theta) + \Psi(\theta)y)}{F(\gamma(\theta))^T [F(\gamma(\theta)) + \Psi'(\theta)y]} - 1, \tag{10}$$

$$\Theta_1(\theta, y, v, \mu, \kappa) = \frac{F(\gamma(\theta))^T G(\gamma(\theta) + \Psi(\theta)y, v, \mu, \kappa)}{F(\gamma(\theta))^T [F(\gamma(\theta)) + \Psi'(\theta)y]}, \tag{11}$$

$$Y_0(\theta, y) = \Psi(\theta)^T [F(\gamma(\theta) + \Psi(\theta)y) - DF(\gamma(\theta))\Psi(\theta)y - \Theta_0(\theta, y)\Psi'(\theta)y], \tag{12}$$

$$Y_1(\theta, y, v, \mu, \kappa) = \Psi(\theta)^T [G(\gamma(\theta) + \Psi(\theta)y, v, \mu, \kappa) - \Theta_1(\theta, y, v, \mu, \kappa)\Psi'(\theta)y], \tag{13}$$

$$V(\theta, y, v, \mu, \kappa) = H(\gamma(\theta) + \Psi(\theta)y, v, \mu, \kappa). \tag{14}$$

The functions $\Theta_0(\theta, y)$ and $Y_0(\theta, y)$ are $(k-1)$-times continuously differentiable in $\mathbf{R} \times W$ and periodic in θ with period τ. Moreover, they satisfy

$$\Theta_0(\theta, 0) = 0, \tag{15}$$

$$Y_0(\theta, 0) = 0, \tag{16}$$

$$\frac{\partial Y_0}{\partial y}(\theta, 0) = 0. \tag{17}$$

$\Theta_1(\theta, y, v, \mu, \kappa)$ and $Y_1(\theta, y, v, \mu, \kappa)$ are $(k-1)$-times continuously differentiable in $\mathbf{R} \times W \times \mathbf{R}^{n+2}$ and periodic in θ with period τ. Furthermore, Assumption 2(a) implies that

$$\Theta_1(\theta, y, v, 0, 0) = 0, \tag{18}$$

$$Y_1(\theta, y, v, 0, 0) = 0. \tag{19}$$

We show that there is a two-parameter family of solutions of (9) given by $\theta = \theta(t, \mu, \kappa)$, $y = y(t, \mu, \kappa)$, and $v = \nu(t, \mu, \kappa)$ and a function, which we call $T(\mu, \kappa)$, such that

$$\begin{aligned} \theta(t + T(\mu, \kappa), \mu, \kappa) &= \theta(t, \mu, \kappa) + T, \\ y(t + T(\mu, \kappa), \mu, \kappa) &= y(t, \mu, \kappa), \\ \nu(t + T(\mu, \kappa), \mu, \kappa) &= \nu(t, \mu, \kappa). \end{aligned} \tag{20}$$

Then we define

$$\gamma(t, \mu, \kappa) = \gamma(\theta(t, \mu, \kappa)) + \Psi(\theta(t, \mu, \kappa))y(t, \mu, \kappa). \tag{21}$$

It follows from (8) that

$$u = \gamma(t, \mu, \kappa), \qquad v = \nu(t, \mu, \kappa) \tag{22}$$

is a periodic solution of period $T(\mu, \kappa)$. We state and prove this result in Theorem 1 (a). We also study the stability of these periodic solutions. The stability is determined by the variational system:

$$\begin{aligned} \frac{du}{dt} &= [DF(\gamma(t, \mu, \kappa)) + A_{11}(t, \mu, \kappa)]u + A_{12}(t, \mu, \kappa)v, \\ \frac{dv}{dt} &= \mu[A_{21}(t, \mu, \kappa)u + A_{22}(t, \mu, \kappa)v], \end{aligned} \tag{23}$$

where

$$\begin{aligned} A_{11}(t, \mu, \kappa) &= \frac{\partial G}{\partial u}(\gamma(t, \mu, \kappa), \nu(t, \mu, \kappa), \mu, \kappa), \\ A_{12}(t, \mu, \kappa) &= \frac{\partial G}{\partial v}(\gamma(t, \mu, \kappa), \nu(t, \mu, \kappa), \mu, \kappa), \\ A_{21}(t, \mu, \kappa) &= \frac{\partial H}{\partial u}(\gamma(t, \mu, \kappa), \nu(t, \mu, \kappa), \mu, \kappa), \\ A_{22}(t, \mu, \kappa) &= \frac{\partial H}{\partial v}(\gamma(t, \mu, \kappa), \nu(t, \mu, \kappa), \mu, \kappa). \end{aligned} \tag{24}$$

One of the multipliers of (23) equals 1. The remaining $m + n - 1$ multipliers give us the information concerning the orbital stability of (22). If j multipliers of (23) have modulus less than 1 and the remaining $m + n - 1 - j$ multipliers have modulus greater than 1, then there are a $(j + 1)$-dimensional stable manifold and an $(m + n - j)$-dimensional unstable manifold of the orbit of (22). We state a result concerning the behavior of the multipliers in Theorem 1(b). In particular, we determine how many multipliers lie inside the unit disk and how many lie outside under Assumption 3. This result is summarized in Theorem 1(c).

Theorem 1. *Suppose that the conditions stated in Assumptions* 1 *and* 2 *are satisfied with* $k \geq 3$.

(a) *There are open intervals* I *and* J *about* 0, *and functions* $\gamma : \mathbf{R} \times I \times J \to \mathbf{R}^m$, $\nu : \mathbf{R} \times I \times J \to \mathbf{R}^n$, *and* $T : I \times J \to \mathbf{R}$ *such that*

$$u = \gamma(t, \mu, \kappa), \qquad v = \nu(t, \mu, \kappa) \tag{25}$$

is a periodic solution of (6) *with period* $T(\mu,\kappa)$. *The functions* $\gamma(t,\mu,\kappa)$, $\nu(t,\mu,\kappa)$, *and* $T(\mu,\kappa)$ *are* $(k-2)$*-times continuously differentiable and satisfy*

$$\gamma(t,0,0)=\gamma(t), \qquad \nu(t,0,0)=\xi^*, \qquad T(0,0)=T. \tag{26}$$

Moreover, for a fixed positive integer j, *the orbit of* $u=\gamma(t)$, $v=\xi^*$ *has a neighborhood* W_j, *in which the orbit of any periodic solution whose period is close to* jT *must coincide with that of* (25), *i.e.,* (25) *is the only periodic solution in* W_j *whose period is close to* jT ((25) *is also periodic with period* $jT(\mu,\kappa)$).

(b) *The* $m+n-1$ *multipliers of* (23) *have the forms*

$$\begin{gathered}\lambda_j+\tilde{\lambda}_j(\mu,\kappa), \qquad j=2,\dots,m,\\ 1+\mu[\lambda_j+\tilde{\lambda}_j(\mu,\kappa)], \qquad j=m+1,\dots,m+n,\end{gathered}$$

where $\lambda_2,\dots,\lambda_m$ *are the multipliers of* (4), $\lambda_{m+1},\dots,\lambda_{m+n}$ *are the eigenvalues of* $Da(\xi^*)$, *and* $\tilde{\lambda}_j(\mu,\kappa)\to 0$, $j=2,\dots,m+n$, *as* $|\mu|+|\kappa|\to 0$.

(c) *Under Assumption* 3, *for* $\mu>0$, l_1+l_2 *multipliers of* (23) *have modulus less than* 1, *and* $m+n-l_1-l_2-1$ *multipliers have modulus greater than* 1. *For* $\mu<0$, l_1+n-l_2 *multipliers of* (23) *have modulus less than* 1, *and* $m-l_1+l_2-1$ *multipliers have modulus greater than* 1.

Proof. Let

$$\theta=\theta(t,\rho,\xi,\mu,\kappa), \qquad y=y(t,\rho,\xi,\mu,\kappa), \qquad v=v(t,\rho,\xi,\mu,\kappa) \tag{27}$$

be the solution of (9) with the initial value

$$\theta(0,\rho,\xi,\mu,\kappa)=0, \qquad y(0,\rho,\xi,\mu,\kappa)=\rho, \qquad v(0,\rho,\xi,\mu,\kappa)=\xi. \tag{28}$$

Define a function $u(t,\rho,\xi,\mu,\kappa)$ by

$$u(t,\rho,\xi,\mu,\kappa)=\gamma(\theta(t,\rho,\xi,\mu,\kappa))+\Psi(\theta(t,\rho,\xi,\mu,\kappa))y(t,\rho,\xi,\mu,\kappa). \tag{29}$$

Then

$$u=u(t,\rho,\xi,\mu,\kappa), \qquad v=v(t,\rho,\xi,\mu,\kappa) \tag{30}$$

is a solution of (6) which satisfies

$$u(0,\rho,\xi,\mu,\kappa) = \gamma(0) + \Psi(0)\rho, \qquad v(0,\rho,\xi,\mu,\kappa) = \xi. \tag{31}$$

Let j be a positive integer and consider the following system of equations for τ, ρ, ξ, μ, and κ.

$$p_j(\tau,\rho,\xi,\mu,\kappa) = 0, \tag{32}$$

$$q(\tau,\rho,\xi,\mu,\kappa) = 0, \tag{33}$$

$$r(\tau,\rho,\xi,\mu,\kappa) = 0, \tag{34}$$

where

$$p_j(\tau,\rho,\xi,\mu,\kappa) = \theta(\tau,\rho,\xi,\mu,\kappa) - jT, \tag{35}$$

$$q(\tau,\rho,\xi,\mu,\kappa) = y(\tau,\rho,\xi,\mu,\kappa) - \rho, \tag{36}$$

$$r(\tau,\rho,\xi,\mu,\kappa) = v(\tau,\rho,\xi,\mu,\kappa) - \xi. \tag{37}$$

If τ, ρ, ξ, μ, and κ satisfy (32), (33), and (34), then (30) is a periodic solution of (6) with period τ. Note that there are neighborhoods of the origin B_1 and B_2 in $\mathbf{R}^{(m-1)}$ and $\mathbf{R}^n$, respectively, and open intervals I_1 and I_2 containing 0 such that the functions p_j, q, and r are defined on $\mathbf{R} \times B_1 \times B_2 \times I_1 \times I_2$ and $(k-1)$-times continuously differentiable in this region.

In view of (15), (16), (18), and (19), we find that

$$\theta(t,0,\xi,0,0) = t, \qquad y(t,0,\xi,0,0) = 0. \tag{38}$$

On the other hand,

$$v(t,\rho,\xi,0,\kappa) = \xi. \tag{39}$$

It follows that, for any $\xi \in \mathbf{R}^n$, $\tau = jT$, $\rho = 0$, $\mu = 0$, $\kappa = 0$ satisfy (32), (33), and (34). The properties of the functions $p_j(\tau,\rho,\xi,\mu,\kappa)$, $q(\tau,\rho,\xi,\mu,\kappa)$, and $r(\tau,\rho,\xi,\mu,\kappa)$ are summarized in Lemma 1. The proof of this lemma is given in the appendix.

Lemma 1.

$$\frac{\partial p_j}{\partial \tau}(\tau, 0, \xi, 0, 0) = 1,$$

$$\frac{\partial p_j}{\partial \rho}(jT, 0, \xi, 0, 0) = \int_0^{jT} \frac{\partial \Theta_0}{\partial y}(s, 0) Y(s, 0)\, ds, \tag{40}$$

$$\frac{\partial q}{\partial \tau}(jT, 0, \xi, 0, 0) = 0,$$

$$\frac{\partial q}{\partial \rho}(jT, 0, \xi, 0, 0) = Y(jT, 0) - I_{(m-1)\times(m-1)},$$

$$\frac{\partial r}{\partial \mu}(jT, 0, \xi, 0, 0) = ja(\xi). \tag{41}$$

Recall that $Y(t, t_0)$ is the fundamental matrix solution of (4) with $Y(t_0, t_0) = I_{(m-1)\times(m-1)}$. Because of (40) and Remark 1 (b), the Implicit Function theorem guarantees the existence of functions $\tau_j(\xi, \mu, \kappa)$ and $\rho_j(\xi, \mu, \kappa)$ that satisfies

$$\tau_j(\xi, 0, 0) = jT, \qquad \rho_j(\xi, 0, 0) = 0, \tag{42}$$

and

$$p_j(\tau_j(\xi, \mu, \kappa), \rho_j(\xi, \mu, \kappa), \xi, \mu, \kappa) = 0,$$
$$q(\tau_j(\xi, \mu, \kappa), \rho_j(\xi, \mu, \kappa), \xi, \mu, \kappa) = 0.$$

Now we substitute $\tau = \tau_j(\xi, \mu, \kappa)$ and $\rho = \rho_j(\xi, \mu, \kappa)$ in (34) and look for solutions of

$$r(\tau_j(\xi, \mu, \kappa), \rho_j(\xi, \mu, \kappa), \xi, \mu, \kappa) = 0.$$

Note that, in view of (37) and (39), there is a function $s(\tau, \rho, \xi, \mu, \kappa)$ such that $r(\tau, \rho, \xi, \mu, \kappa) = \mu s(\tau, \rho, \xi, \mu, \kappa)$. It follows that $r(\tau_j(\xi, \mu, \kappa),$ $\rho_j(\xi, \mu, \kappa), \xi, \mu, k) = \mu s_j(\xi, \mu, \kappa)$ where $s_j(\xi, \mu, \kappa) = s(\tau_j(\xi, \mu, \kappa),$ $\rho_j(\xi, \mu, \kappa), \xi, \mu, \kappa)$. Now Assumption 2 (b) and (41) guarantee the existence of a function $\xi_j(\mu, \kappa)$ that satisfies

$$\xi_j(0, 0) = \xi^* \tag{43}$$

and $s_j(\xi_j(\mu, \kappa), \mu, \kappa) = 0$.

We have shown that $\tau = \tau_j(\xi_j(\mu,\kappa),\mu,\kappa)$, $\rho = \rho_j(\xi_j(\mu,\kappa),\mu,\kappa)$, and $\xi = \xi_j(\mu,\kappa)$ satisfy (32), (33), and (34). However, it is easily seen that

$$\begin{aligned} \tau_j(\xi_j(\mu,\kappa),\mu,\kappa) &= j\tau_1(\xi_1(\mu,\kappa),\mu,\kappa), \\ \rho_j(\xi_j(\mu,\kappa),\mu,\kappa) &= \rho_1(\xi_1(\mu,\kappa),\mu,\kappa), \\ \xi_j(\mu,\kappa) &= \xi_1(\mu,\kappa) \end{aligned} \tag{44}$$

for sufficiently small $|\mu|$ and $|\kappa|$. Define

$$\begin{aligned} T(\mu,\kappa) &= \tau_1(\xi_1(\mu,\kappa),\mu,\kappa), \\ \rho(\mu,\kappa) &= \rho_1(\xi_1(\mu,\kappa),\mu,\kappa), \\ \xi(\mu,\kappa) &= \xi_1(\mu,\kappa), \end{aligned} \tag{45}$$

and

$$\begin{aligned} \theta(t,\mu,\kappa) &= \theta(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ y(t,\mu,\kappa) &= y(t,\rho(u,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ \nu(t,\mu,\kappa) &= v(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa). \end{aligned}$$

Then the functions $\theta(t,\mu,\kappa)$, $y(t,\mu,\kappa)$, and $\nu(t,\mu,\kappa)$ satisfy (20). It follows that (25) is a periodic solution of (6) with period $T(\mu,\kappa)$ provided $\gamma(t,\mu,\kappa)$ is defined by (21). Furthermore, it follows from (38), (39), (42), and (43) that $\theta(t,0,0) = t$, $\rho(t,0,0) = 0$, $\nu(t,0,0) = \xi^*$, and $T(0,0) = T$. Now (26) follows from (21).

Recall that $u = u(t,\rho,\xi,\mu,\kappa)$, $v = v(t,\rho,\xi,\mu,\kappa)$ is a solution of (6) which satisfies (31). It is easily seen that

$$\begin{aligned} \gamma(t,\mu,\kappa) &= u(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ \nu(t,\mu,\kappa) &= v(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa). \end{aligned} \tag{46}$$

Now any solution of (6) can be written in the form given by (29) and (30) provided its u-component stays sufficiently close to the orbit of $\gamma(t)$. Suppose that (30) is a periodic solution of (6) with period τ. Then τ,ρ,ξ,μ, and κ satisfy (32), (33), and (34). However, when $|\tau - jT|$, $||\rho||$, $||\xi-\xi^*||$, $|\mu|$, and $|\kappa|$ are sufficiently small, τ, ρ, and ξ must be given by $\tau = \tau_j(\xi_j(\mu,\kappa),\mu,\kappa)$, $\rho = \rho_j(\xi_j(\mu,\kappa),\mu,\kappa)$, and $\xi = \xi_j(\mu,\kappa)$. Now (44) leads to the conclusion concerning the uniqueness of the periodic solution in a neighborhood of the orbit of $u = \gamma(t)$, $\xi = \xi^*$. This completes the proof of (a). The proof of (b) is given in the Appendix. (c) follows from (b). □

2.2. Persistence of periodic solutions in the singular problem. In this section we study the system of ordinary differential equations

$$
\begin{aligned}
\frac{du}{dt} &= F(u) + G(u, v, \mu, \kappa), \\
\kappa \frac{dv}{dt} &= Cv + H(u, v, \mu, \kappa)
\end{aligned} \tag{47}
$$

under Assumption 1 and the following assumption.

Assumption 4. (a) $G(u, 0, 0, 0) = 0$ *for all* $\mu \in \mathbf{R}^m$.

(b) $H(u, 0, 0, 0) = 0$, $(\partial H/\partial v)(u, 0, 0, 0) = 0$ *for all* $u \in \mathbf{R}^m$.

(c) *The eigenvalues of* C *have negative real parts.*

We show that, for sufficiently small $|\mu|$ and $|\kappa|$, (47) has a periodic solution $u = \gamma(t, \mu, \kappa)$, $v = \nu(t, \mu, \kappa)$ of period $T(\mu, \kappa)$ and that the functions $\gamma(t, \mu, \kappa)$, $\nu(t, \mu, \kappa)$, and $T(\mu, \kappa)$ are $(k-1)$-times continuously differentiable and satisfy $\gamma(t, 0, 0) = \gamma(t)$, $\nu(t, 0, 0) = 0$, and $T(0, 0) = T$. We also study the stability of these periodic solutions.

Remark 2. (a) Choose a bounded open sphere $\tilde{B}$ in $\mathbf{R}^m$ that contians the orbit of $\gamma(t)$. Then there are open intervals $\tilde{I}$ and $\tilde{J}$ about 0, and a k-times continuously diffferentiable function $c : \tilde{B} \times \tilde{I} \times \tilde{J} \to \mathbf{R}^n$ (a center manifold in the fast time t/κ, cf. [**4, 7, 14**]) such that

$$
c(u, 0, 0) = 0 \tag{48}
$$

and the manifold defined by $v = c(u, \mu, \kappa)$ is an invariant manifold of (47). A standard technique (cf. [**7**]) can be used to construct a center manifold of (47) whose domain contains a certain bounded set in $\mathbf{R}^m$.

(b) Assumption 4(c) guarantees the existence of positive numbers α and K such that

$$
||e^{C_t}|| \le K e^{-\alpha t} \qquad \text{for } t \ge 0.
$$

As is done in Section 2.1, (8) leads to the system of ordinary differential equations for θ, y, and v:

$$\begin{aligned}\frac{d\theta}{dt} &= 1 + \Theta_0(\theta, y) + \Theta_1(\theta, y, v, \mu, \kappa),\\ \frac{dy}{dt} &= A(\theta)y + Y_0(\theta, y) + Y_1(\theta, y, v, \mu, \kappa),\\ \kappa\frac{dv}{dt} &= Cv + V(\theta, y, v, \mu, \kappa),\end{aligned} \tag{49}$$

where the $(m-1)\times(m-1)$-matrix A and the functions Θ, Y, and V are exactly as defined by the formulas (5) and (10)–(14). $\Theta_0(\theta, y)$ and $Y_0(\theta, y)$ satisfy (15), (16) and (17). Because of (11), (13) and Assumption 4(a),

$$\Theta_1(\theta, y, 0, 0, 0) = 0, \qquad Y_1(\theta, y, 0, 0, 0) = 0. \tag{50}$$

We show that (49) has solutions $\theta = \theta(t, \mu, \kappa)$, $y = y(t, \mu, \kappa)$, and $v = \nu(t, \mu, \kappa)$ that satisfy (20). Then (47) has a two-parameter family of periodic soutions $u = \gamma(t, \mu, \kappa)$, $v = \nu(t, \mu, \kappa)$ provided $\gamma(t, \mu, \kappa) = \gamma(\theta(t, \mu, \kappa)) + \Psi(\theta(t, \mu, \kappa))y(t, \mu, \kappa)$. The variational system of (47) with respect to this periodic soution is

$$\begin{aligned}\frac{du}{dt} &= [DF(\gamma(t, \mu, \kappa)) + A_{11}(t, \mu, \kappa)]u + A_{12}(t, \mu, \kappa)v,\\ \kappa\frac{dv}{dt} &= A_{21}(t, \mu, \kappa)u + [C + A_{22}(t, \mu, \kappa)]v,\end{aligned} \tag{51}$$

where $A_{ij}(t, \mu, \kappa)$, $i = 1, 2$, $j = 1, 2$, are defined by the formulas given at (24). We prove results concerning the multipliers of this variational system. We summarize these results in the following theorem.

Theorem 2. *Suppose that the conditions stated in Assumptions* 1 *and* 4 *are satisfied with* $k \geq 2$.

(a) *There are open intervals* I *and* J *about* 0, *and functions* $\gamma : \mathbf{R} \times I \times J \to \mathbf{R}^m$, $\nu : \mathbf{R} \times I \times J \to \mathbf{R}^n$, *and* $T : I \times J \to \mathbf{R}$ *such that*

$$u = \gamma(t, \mu, \kappa), \qquad v = \nu(t, \mu, \kappa) \tag{52}$$

is a periodic solution of (47) *with period* $T(\mu,\kappa)$. *The functions* $\gamma(t,\mu,\kappa)$, $\nu(t,\mu,\kappa)$, *and* $T(\mu,\kappa)$ *are* $(k-1)$*-times continuously differentiable and satisfy*

$$\gamma(t,0,0)=\gamma(t), \qquad \nu(t,0,0)=0, \qquad T(0,0)=T. \tag{53}$$

Moreover, for a fixed positive integer j, *the orbit of* $u=\gamma(t)$, $v=0$ *has a neighborhood* W_j, *in which the orbit of any periodic solution whose period is close to* jT *must coincide with that of* (52), *i.e.,* (52) *is the only periodic solution in* W_j *whose period is close to* jT.

(b) *For* $\kappa>0$, *the* $m+n-1$ *multipliers of* (51) *have the forms*:

$$\begin{gathered}\lambda_j+\tilde{\lambda}_j^+(\mu,\kappa), \qquad j=2,\dots,m,\\ \tilde{\lambda}_j^+(\mu,\kappa), \qquad j=m+1,\dots,m+n,\end{gathered}$$

and for $\kappa<0$, *their inverses have the forms*:

$$\begin{gathered}\lambda_j^{-1}+\tilde{\lambda}_j^-(\mu,\kappa), \qquad j=2,\dots,m,\\ \tilde{\lambda}_j^-(\mu,\kappa), \qquad j=m+1,\dots,m+n,\end{gathered}$$

where $\lambda_2,\dots,\lambda_m$ *are the multipliers of* (4) *and* $\tilde{\lambda}_j^\pm(\mu,\kappa)\to 0$, $j=2,\dots,m+n$, *as* $|\mu|+|\kappa|\to 0$.

(c) *Under Assumption* 3(a), *for* $\kappa>0$, l_1+n *multipliers of* (51) *have modulus less than* 1 *and* $m-l_1-1$ *multipliers have modulus greater than* 1. *For* $\kappa<0$, l_1 *multipliers of* (51) *have modulus less than* 1 *and* $m+n-l_1-1$ *multipliers have modulus greater than* 1.

Proof. According to Remark 2(a), (49) has an invariant manifold defined by $v=d(\theta,y,\mu,\kappa)$ where

$$d(\theta,y,\mu,\kappa)=c(\gamma(\theta)+\Psi(\theta)y,\mu,\kappa). \tag{54}$$

(48) and (54) imply that

$$d(\theta,y,0,0)=0. \tag{55}$$

The reduced form of (49) on this manifold is

$$\begin{aligned}\frac{d\theta}{dt}&=1+\Theta_0(\theta,y)+\tilde{\Theta}_1(\theta,y,\mu,\kappa),\\ \frac{dy}{dt}&=A(\theta)y+Y_0(\theta,y)+\tilde{Y}_1(\theta,y,\mu,\kappa),\end{aligned} \tag{56}$$

where

$$\begin{aligned}\tilde{\Theta}_1(\theta, y, \mu, \kappa) &= \Theta_1(\theta, y, d(\theta, y, \mu, \kappa), \mu, \kappa), \\ \tilde{Y}_1(\theta, y, \mu, \kappa) &= Y_1(\theta, y, d(\theta, y, \mu, \kappa), \mu, \kappa).\end{aligned} \tag{57}$$

There is a neighborhood of 0 in $\mathbf{R}^{m-1}$, which we call $\tilde{W}$, such that $\Theta_0 : \mathbf{R} \times \tilde{W} \to \mathbf{R}$, $Y_0 : \mathbf{R} \times \tilde{W} \to \mathbf{R}^{m-1}$, $\tilde{\Theta}_1 : \mathbf{R} \times \tilde{W} \times \tilde{I} \times \tilde{J} \to \mathbf{R}$, and $\tilde{Y}_1 : \mathbf{R} \times \tilde{W} \times \tilde{I} \times \tilde{J} \to \mathbf{R}^{m-1}$ are $(k-1)$-times continuously differentiable.

Let

$$\theta = \tilde{\theta}(t, \rho, \mu, \kappa), \qquad y = \tilde{y}(t, \rho, \mu, \kappa)$$

be the solution of (56) with the initial value

$$\tilde{\theta}(0, \rho, \mu, \kappa) = 0, \qquad \tilde{y}(0, \rho, \mu, \kappa) = \rho.$$

In view of (15), (16), (50), (55), and (57),

$$\tilde{\theta}(t, 0, 0, 0) = t, \tag{58}$$

$$\tilde{y}(t, 0, 0, 0) = 0. \tag{59}$$

We look for solutions of the system of the equations

$$p_j(\tau, \rho, \mu, \kappa) = 0, \tag{60}$$

$$q(\tau, \rho, \mu, \kappa) = 0, \tag{61}$$

where j is a positive integer and

$$p_j(\tau, \rho, \mu, \kappa) = \tilde{\theta}(\tau, \rho, \mu, \kappa) - jT, \tag{62}$$

$$q(\tau, \rho, \mu, \kappa) = \tilde{y}(\tau, \rho, \mu, \kappa) - \rho. \tag{63}$$

As we saw in Section 2.1, such a solution corresponds to a periodic solution of (47).

It follows from (58), (59), (62), and (63) that $\tau = jT$, $\rho = 0$, $\mu = 0$, and $\kappa = 0$ satisfy (60) and (61). On the other hand, as is seen in the previous section, (15), (16), (17), (50), (55), and (57) lead to the properties of p_j and q summarized in the following lemma. The proof is left to the reader.

Lemma 2.

$$\begin{aligned}
\frac{\partial p_j}{\partial \tau}(\tau, 0, 0, 0) &= 1, \\
\frac{\partial p_j}{\partial \rho}(jT, 0, 0, 0) &= \int_0^{jT} \frac{\partial \Theta_0}{\partial y}(s, 0) Y(s, 0)\, ds, \\
\frac{\partial q}{\partial \tau}(jT, 0, 0, 0) &= 0, \\
\frac{\partial q}{\partial \rho}(jT, 0, 0, 0) &= Y(jT, 0) - I_{(m-1)\times(m-1)}.
\end{aligned}$$

According to the Implicit Function theorem, Lemma 2 and Assumption 1(b) lead to the existence of functions, which we call $\tau_j(\mu, \kappa)$ and $\rho_j(\mu, \kappa)$ such that $\tau_j(0, 0) = jT$ and $\rho_j(0, 0) = 0$, and

$$\begin{aligned}
p_j(\tau_j(\mu, \kappa), \rho_j(\mu, \kappa), \mu, \kappa) &= 0, \\
q(\tau_j(\mu, \kappa), \rho_j(\mu, \kappa), \mu, \kappa) &= 0.
\end{aligned}$$

As is seen in the previous section, for all sufficiently small $|\mu|$ and $|\kappa|$,

$$\tau_j(\mu, \kappa) = j\tau_1(\mu, \kappa), \qquad \rho_j(\mu, \kappa) = \rho_1(\mu, \kappa). \tag{64}$$

Define

$$\begin{aligned}
T(\mu, \kappa) &= \tau_1(\mu, \kappa), \qquad \rho(\mu, \kappa) = \rho_1(\mu, \kappa), \\
\theta(t, \mu, \kappa) &= \tilde{\theta}(t, \rho(\mu, \kappa), \mu, \kappa), \qquad y(t, \mu, \kappa) = \tilde{y}(t, \rho(\mu, \kappa), \mu, \kappa),
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
\gamma(t, \mu, \kappa) &= \gamma(\theta(t, \mu, \kappa)) + \Psi(\theta(t, \mu, \kappa)) y(t, \mu, \kappa), \\
\nu(t, \mu, \kappa) &= d(\theta(t, \mu, \kappa), y(t, \mu, \kappa), \mu, \kappa).
\end{aligned}$$

Then (52) is a periodic solution of (47) with period $T(\mu, \kappa)$. As is seen in Section 2.1, (64) leads to a conclusion concerning the uniqueness of a periodic solution in a neighborhood of the closed curve $u = \gamma(t)$, $v = 0$. This completes the proof of (a). The proof of (b) is given in the appendix and (c) follows immediately from (b).

Remark 3. Note that

$$\theta = \theta(t, \mu, \kappa), \qquad y = y(t, \mu, \kappa), \qquad v = \nu(t, \mu, \kappa) \tag{66}$$

is a solution of (49) that satisfies

$$\theta(t,0,0) = t, \qquad y(t,0,0) = 0, \qquad \nu(t,0,0) = 0.$$

In fact, if $\xi(\mu,\kappa)$ is the function defined by

$$\xi(\mu,\kappa) = d(0,\rho(\mu,\kappa),\mu,\kappa) = c(\gamma(0) + \Psi(0)\rho(\mu,\kappa),\mu,\kappa) \tag{67}$$

and

$$\theta = \theta(t,\rho,\xi,\mu,\kappa), \qquad y = y(t,\rho,\xi,\mu,\kappa), \qquad v = v(t,\rho,\xi,\mu,\kappa)$$

is the solution of (49) which satisfies (28), then

$$\begin{aligned} \theta(t,\mu,\kappa) &= \theta(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ y(t,\mu,\kappa) &= y(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa). \end{aligned}$$

Note also that

$$\begin{aligned} \tilde{\theta}(t,\rho,\mu,\kappa) &= \theta(t,\rho,d(0,\rho,\mu,\kappa),\mu,\kappa), \\ \tilde{y}(t,\rho,\mu,\kappa) &= y(t,\rho,d(0,\rho,\mu,\kappa),\mu,\kappa). \end{aligned} \tag{68}$$

Furthermore, if $u(t,\rho,\xi,\mu,\kappa)$ is the function defined by (29), then the pair $u = u(t,\rho,\xi,\mu,\kappa)$, $v = v(t,\rho,\xi,\mu,\kappa)$ is the solution of (47) which satisfies (31) and

$$\begin{aligned} \gamma(t,\mu,\kappa) &= u(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ \nu(t,\mu,\kappa) &= v(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa). \end{aligned}$$

Remark 4. Suppose that the conditions stated in Assumption 1 and Assumption 4(a) and (b) are satisfied with $k \geq 2$. Instead of Assumption 4(c), assume that the eigenvalues of C have positive real parts. To establish the existence of periodic solutions in this case, we set $\kappa = -\tilde{\kappa}$ and obtain the following system from (47).

$$\begin{aligned} \frac{du}{dt} &= F(u) + G(u,v,\mu,-\tilde{\kappa}), \\ \tilde{\kappa}\frac{dv}{dt} &= -Cv - H(u,v,\mu,-\tilde{\kappa}). \end{aligned}$$

According to Theorem 2(a), this system has periodic solutions whenever $|\mu|$ and $|\tilde{\kappa}|$ are sufficiently small. In fact, the statement in Theorem 2 is still valid. However, under the additional Assumption 3(a), we obtain the following result concerning the multipliers of (51). For $\kappa > 0$, l_1 multipliers of (51) have modulus less than 1 and $m+n-l_1-1$ multipliers have modulus greater than 1. For $\kappa < 0$, l_1+n multipliers of (51) have modulus less than 1 and $m-l_1-1$ multipliers have modulus greater than 1.

3. Bifurcaton of synchronized periodic solutions in coupled oscillators. In this section we apply the results of Section 2 to (1) to show when this system has periodic solutions. We show, in particular, that periodic solutions exist in three cases where $|\varepsilon\delta|$ and $|\delta|$ are both small, $|\delta|$ is small and $|\varepsilon\delta|$ is large, and $\varepsilon \neq -1, 0$ and $|\delta|$ is large. We also analyze the stability of these periodic soutions.

Let

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad w_i = x_i - \bar{x}, \qquad i = 1, \dots, N-1.$$

Note that

$$\begin{aligned} x_i &= w_i + \bar{x}, \qquad i = 1, \dots, N-1, \\ x_N &= \bar{x} - \sum_{i=1}^{N-1} w_i, \end{aligned}$$

and (1) becomes

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \frac{1}{N}\left[\sum_{j=1}^{N-1} f(w_j + \bar{x}) + f\left(\bar{x} - \sum_{j=1}^{N-1} w_j\right)\right] - \delta P(\bar{x} - x_0), \\ \frac{dx_0}{dt} &= \varepsilon\delta P(\bar{x} - x_0), \\ \frac{dw_i}{dt} &= f(w_i + \bar{x}) - \frac{1}{N}\left[\sum_{j=1}^{N-1} f(w_j + \bar{x}) + f\left(\bar{x} - \sum_{j=1}^{N-1} w_j\right)\right] \\ &\quad - \delta P w_i, \qquad i = 1, \dots, N-1. \end{aligned} \tag{69}$$

This transformation is introduced in [**9**] to analyze the steady state solutions of (1).

The subspace of $\mathbf{R}^{(N+1)n}$ defined by $w_i = 0$, $i = 1, \dots, N-1$, is an invariant subspace of (69), and in this subspace (69) becomes

$$\frac{d\bar{x}}{dt} = f(\bar{x}) + \delta P(x_0 - \bar{x}), \qquad \frac{dx_0}{dt} = \varepsilon\delta P(\bar{x} - x_0). \tag{70}$$

That is, if

$$\bar{x} = \bar{\phi}(t, \varepsilon, \delta), \quad x_0 = \phi_0(t, \varepsilon, \delta) \tag{71}$$

is a solution of (70), then

$$\begin{aligned} \bar{x} &= \bar{\phi}(t, \varepsilon, \delta), \qquad x_0 = \phi_0(t, \varepsilon, \delta), \\ w_i &= 0, \qquad i = 1, \dots, N-1 \end{aligned} \tag{72}$$

is a solution of (69). In the original coordinates, the invariant subspace is given by $x_1 = x_2 = \cdots = x_N$. It follows that

$$x_0 = \phi_0(t, \varepsilon, \delta), \qquad x_i = \bar{\phi}(t, \varepsilon, \delta), \qquad i = 1, \dots, N \tag{73}$$

is a solution of (1). Thus solutions of (70) generate synchronized solutions of (1) in which the evolutions of $x_1, \dots, x_N$ are all identical. In particular, periodic solutions of (70) give rise to synchronized periodic solutions of (1).

The variational equations associated with (71) and (72) are also related. The variational equation of (70) with respet to (71) is

$$\frac{d}{dt}\begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix} = \begin{bmatrix} Df(\bar{\phi}(t, \varepsilon, \delta)) - \delta P & \delta P \\ \varepsilon\delta P & -\varepsilon\delta P \end{bmatrix} \begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix}, \tag{74}$$

whereas the variational equation of (69) with respect to (72) consists of (74) and the additional $N-1$ linear systems

$$\frac{dw_i}{dt} = [Df(\bar{\phi}(t, \varepsilon, \delta)) - \delta P]w_i, \qquad i = 1, \dots, N-1. \tag{75}$$

Therefore, (74) determines the stability of (73) with respect to the solutions in the subspace $x_1 = x_2 = \cdots = x_N$ and (75) determines its stability in the complement.

When $|\varepsilon\delta|$ and $|\delta|$ are sufficiently small, Theorem 1 guarantees the existence of periodic solutions of (70) under the following Assumption 5. Under Assumption 6, the stability of these periodic solutions in the invariant subspace can also be determined. We summarize these results in Theorem 3.

Assumption 5. (a) *$f : \mathbf{R}^n \to \mathbf{R}^n$ is k-times continuously differentiable and the n-dimensional system*

$$\frac{dx}{dt} = f(x)$$

has a nonconstant periodic solution $\eta(t)$ with least period $T > 0$.

(b) *1 is a simple multiplier of the variational system*

$$\frac{dx}{dt} = Df(\eta(t))x. \tag{76}$$

(c) *The matrix P is nonsingular.*

Assumption 6. (a) *l_1 multipliers of (76) have modulus less than 1 and $n - l_1 - 1$ multipliers have modulus greater than 1.*

(b) *$n-l_2$ eigenvalues of P have negative real parts and the remaining l_2 eigenvalues have positive real parts.*

Theorem 3. *Suppose that the conditions stated in Assumption 5 are satisfied with $k \geq 3$. Then there are open intervals I_1 and J_1 about 0, and functions $\gamma_1 : \mathbf{R} \times I_1 \times J_1 \to \mathbf{R}^n$, $\nu_1 : \mathbf{R} \times I_1 \times J_1 \to \mathbf{R}^n$, and $T_1 : I_1 \times J_1 \to \mathbf{R}$ such that*

$$\bar{x} = \gamma_1(t, \varepsilon\delta, \delta), \qquad x_0 = \nu_1(t, \varepsilon\delta, \delta)$$

is a periodic solution of (70) with period $T_1(\varepsilon\delta, \delta)$. $\gamma_1(t, \mu, \kappa)$, $\nu_1(t, \mu, \kappa)$, and $T_1(\mu, \kappa)$ are $(k-2)$-times continuously differentiable and satisfy

$$\gamma_1(t, 0, 0) = \eta(t), \qquad \nu_1(t, 0, 0) = \frac{1}{T}\int_0^T \eta(s)\, ds, \qquad T_1(0, 0) = T.$$

In addition, suppose that Assumption 6 *is satisfied. Then, for* $\varepsilon\delta > 0$, *the variational system* (74) *with* $\bar{\phi}(t, \varepsilon, \delta) = \gamma_1(t, \varepsilon\delta, \delta)$ *has* $l_1 + l_2$ *multipliers with modulus less than* 1 *and* $2n - l_1 - l_2 - 1$ *multipliers with modulus greater than* 1. *For* $\varepsilon\delta < 0$, (74) *has* $l_1 + n - l_2$ *multipliers with modulus less than* 1 *and* $n - l_1 + l_2 - 1$ *multipliers with modulus greater than* 1.

Proof. Set $\mu = \varepsilon\delta$, $\kappa = \delta$, $u = \bar{x}$, $v = x_0$, and

$$a(v) = -TP\left(v - \frac{1}{T}\int_0^T \eta(s)\, ds\right).$$

Then Theorem 3 follows immediately from Theorem 1. □

Next we consider the case where $|\delta|$ is small and $|\varepsilon\delta|$ is large under the following additional assumption.

Assumption 7. *The eigenvalues of* P *have positive real parts.*

We summarize the existence and stability of periodic solutions in Theorem 4.

Theorem 4. *Supopse that the conditions stated in Assumption* 5(a) *and* (b) *and Assumption* 7 *are satisfied with* $k \geq 2$. *Then there are open intervals* I_2 *and* J_2 *about 0, and functions* $\gamma_2 : \mathbf{R} \times I_2 \times J_2 \to \mathbf{R}^n$, $\nu_2 : \mathbf{R} \times I_2 \times J_2 \to \mathbf{R}^n$, *and* $T_2 : I_2 \times J_2 \to \mathbf{R}$ *such that*

$$\bar{x} = \gamma_2(t, \delta, (\varepsilon\delta)^{-1}), \quad x_0 = \gamma_2(t, \delta, (\varepsilon\delta)^{-1}) + \nu_2(t, \delta, (\varepsilon\delta)^{-1})$$

is a periodic solution of (70) *with period* $T_2(\delta, (\varepsilon\delta)^{-1})$. $\gamma_2(t, \mu, \kappa)$, $\nu_2(t, \mu, \kappa)$, *and* $T_2(\mu, \kappa)$ *are* $(k-1)$*-times continuously differentiable and satisfy*

$$\gamma_2(t, 0, 0) = \eta(t), \qquad \nu_2(t, 0, 0) = 0, \qquad T_2(0, 0) = T.$$

In addition, suppose that Assumption 6(a) *is satisfied. Then, for* $\varepsilon\delta > 0$, *the variational system* (74) *with* $\bar{\phi}(t, \varepsilon, \delta) = \gamma_2(t, \delta, (\varepsilon\delta)^{-1})$ *has* $l_1 + n$ *multipliers with modulus less than* 1 *and* $n - l_1 - 1$ *multipliers with modulus greater than* 1. *For* $\varepsilon\delta < 0$, (74) *has* l_1 *multipliers with* *modulus less than* 1 *and* $2n - l_1 - 1$ *multipliers with modulus greater than* 1.

Proof. Set $\mu = \delta$, $\kappa = (\varepsilon\delta)^{-1}$, $u = \bar{x}$, and $v = x_0 - \bar{x}$, and use the result of Theorem 2. □

Finally, we consider the case where $|\delta|$ is large and prove the results concerning the existence and stability of periodic solutions in Theorem 5.

Theorem 5. *Suppose that the conditions stated in Assumption* 5(a) *and* (b) *and Assumption* 7 *are satisfied with* $k \geq 2$. *Given* $\varepsilon \neq -1, 0$, *there are an open interval* J_3 *about* 0, *and functions* $\gamma_3 : \mathbf{R} \times J_3 \to \mathbf{R}^n$, $\nu_3 : \mathbf{R} \times J_3 \to \mathbf{R}^n$, *and* $T_3 : J_3 \to \mathbf{R}$ *such that*

$$\begin{aligned} \bar{x} &= (1+\varepsilon)^{-1}[\gamma_3(t,\delta^{-1}) + \nu_3(t,\delta^{-1})], \\ x_0 &= (1+\varepsilon)^{-1}[\gamma_3(t,\delta^{-1}) - \varepsilon\nu_3(t,\delta^{-1})] \end{aligned} \tag{77}$$

is a periodic solution of (70) *with period* $T_3(\delta^{-1})$. $\gamma_3(t,\kappa), \nu_3(t,\kappa)$, *and* $T_3(\kappa)$ *are* $(k-1)$*-times continuously differentiable and satisfy*

$$\begin{aligned} \gamma_3(t,0) &= (1+\varepsilon)\eta(\varepsilon(1+\varepsilon)^{-1}t), \\ \nu_3(t,0) &= 0, \\ T_3(0) &= \varepsilon^{-1}(1+\varepsilon)T. \end{aligned} \tag{78}$$

In addition suppose that Assumption 6(a) *is satisfied. Set*

$$\bar{\phi}(t,\varepsilon,\delta) = (1+\varepsilon)^{-1}[\gamma_3(t,\delta^{-1}) + \nu_3(t,\delta^{-1})]$$

in the variational system (74). *First, suppose that* $\varepsilon > 0$. *Then, for* $\delta > 0$, (74) *has* $l_1 + n$ *multipliers with modulus less than* 1 *and* $n - l_1 - 1$ *multipliers with modulus greater than* 1. *For* $\delta < 0$, (74) *has* l_1 *multipliers with modulus less than* 1 *and* $2n - l_1 - 1$ *multipliers with modulus greater than* 1. *Next, suppose that* $-1 < \varepsilon < 0$. *Then, for* $\delta > 0$, (74) *has* $2n - l_1 - 1$ *multipliers with modulus less than* 1 *and* l_1 *multipliers with modulus greater than* 1. *For* $\delta < 0$, (74) *has* $n - l_1 - 1$ *multipliers with modulus less than* 1 *and* $l_1 + n$ *multipliers with modulus greater than* 1. *Finally, suppose that* $\varepsilon < -1$. *Then for* $\delta > 0$, (74) *has*

$n - l_1 - 1$ multipliers with modulus less than 1 and $l_1 + n$ multipliers with modulus greater than 1. For $\delta < 0$, (74) has $2n - l_1 - 1$ multipliers with modulus less than 1 and l_1 multipliers with modulus greater than 1.

Proof. Let

$$u = \varepsilon\bar{x} + x_0, \qquad v = \bar{x} - x_0.$$

Then (70) becomes

$$\begin{aligned} \frac{du}{dt} &= \varepsilon f((1+\varepsilon)^{-1}(u+v)), \\ \frac{dv}{dt} &= f((1+\varepsilon)^{-1}(u+v)) - (1+\varepsilon)\delta P v. \end{aligned} \tag{79}$$

This system can be written in the form given at (47) with $\kappa = \delta^{-1}$, $F(u) = \varepsilon f((1+\varepsilon)^{-1}u))$, $G(u, v, \mu, \kappa) = \varepsilon[f((1+\varepsilon)^{-1}(u+v)) - f((1+\varepsilon)^{-1}u))]$, $C = -(1+\varepsilon)P$, and $H(u, v, \mu, \kappa) = \kappa f((1+\varepsilon)^{-1}(u+v))$. Under Assumption 5(a),

$$\frac{du}{dt} = F(u) = \varepsilon f((1+\varepsilon)^{-1}u) \tag{80}$$

has a nonconstant periodic solution

$$u = (1+\varepsilon)\eta(\varepsilon(1+\varepsilon)^{-1}t) \tag{81}$$

whose period is $|\varepsilon^{-1}(1+\varepsilon)|T$.

The variational system of (80) with respect to (81) is

$$\frac{du}{dt} = \varepsilon(1+\varepsilon)^{-1} Df(\eta(\varepsilon(1+\varepsilon)^{-1}t)). \tag{82}$$

Let $\Phi(t)$ be a fundamental matrix solution of (76). Thus, if $1, \lambda_2, \dots, \lambda_n$ are the multipliers of (76), they are the eigenvalues of the matrix $\Phi^{-1}(0)\Phi(T)$. Now note that $\Phi(\varepsilon(1+\varepsilon)^{-1}t)$ is a fundamental matrix solution of (82). Therefore, the multipliers of (82) are the eigenvalues of the matrix

$$\Phi^{-1}(0)\Phi(\varepsilon(1+\varepsilon)^{-1}|\varepsilon^{-1}(1+\varepsilon)|T) = \begin{cases} \Phi^{-1}(0)\Phi(T) & \text{for } \varepsilon < -1 \text{ or } \varepsilon > 0, \\ \Phi^{-1}(0)\Phi(-T) & \text{for } -1 < \varepsilon < 0. \end{cases}$$

It follows that $1, \lambda_2, \ldots, \lambda_n$ are the multipliers of (82) for $\varepsilon < -1$ or $\varepsilon > 0$. On the other hand, when $-1 < \varepsilon < 0, 1, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$ are the multipliers of (82). In particular, if 1 is a simple multiplier of (76), it is also a simple multiplier of (82). Now Theorem 2(a) and Remark 4 lead to the following conclusion. Under Assumption 5(a) and (b), and Assumption 7, there are open intervals I and J about 0, and functions $\gamma : \mathbf{R} \times I \times J \to \mathbf{R}^m$, $\nu : \mathbf{R} \times I \times J \to \mathbf{R}^n$, and $T : I \times J \to \mathbf{R}$ such that $u = \gamma(t, \mu, \kappa)$, $v = \nu(t, \mu, \kappa)$ is a periodic solution of (79) with period $T(\mu, \kappa)$. Moreover, the functions $\gamma(t, \mu, \kappa)$, $\nu(t, \mu, \kappa)$ and $T(\mu, \kappa)$ are $(k-1)$-times continuously differentiable and satisfy $\gamma(t, 0, 0) = (1+\varepsilon)\eta(\varepsilon(1+\varepsilon)^{-1}t)$, $\nu(t, 0, 0) = 0$, and $T(0, 0) = |\varepsilon^{-1}(1+\varepsilon)|T$. Now define $J_3 = J$, $\gamma_3(t, \kappa) = \gamma(t, 0, \kappa)$, $\nu_3(t, \kappa) = \nu(t, 0, \kappa)$, and $T_3(\kappa) = T(0, \kappa)$. Then (77) is a periodic solution of (70) and the functions $\gamma_3 : \mathbf{R} \times J_3 \to \mathbf{R}^n$, $\nu_3 : \mathbf{R} \times J_3 \to \mathbf{R}^n$, and $T_3 : J_3 \to \mathbf{R}$ satisfy (78). To complete the proof, recall that the multipliers of (76) equal the multipliers of (82) when $\varepsilon < -1$ or $\varepsilon > 0$. On the other hand, the multipliers of (76) equal the inverses of the multipliers of (82) when $-1 < \varepsilon < 0$. Now the results concerning the multipliers of (74) follow from Theorem 2(c) and Remark 4. □

To simplify the notation we denote by $\bar{x} = \bar{\phi}(t, \varepsilon, \delta)$ and $x_0 = \phi_0(t, \varepsilon, \delta)$ the periodic solutions of (70) whose existence for the extreme values of $\varepsilon\delta$ and δ are guaranteed by Theorems 3, 4, and 5. Let the corresponding periods be denoted by $T(\varepsilon, \delta)$. These solutions exist when $|\varepsilon\delta|$ and $|\delta|$ are both small, $|\delta|$ is small and $|\varepsilon\delta|$ is large, or $\varepsilon \neq -1, 0$ and $|\delta|$ is large.

Remark 5. According to Theorems 3, 4, and 5, $\bar{\phi}(t, \varepsilon, \delta)$, $\phi_0(t, \varepsilon, \delta)$, and $T(\varepsilon, \delta)$ have the following limits.

(a) As $|\delta| \to 0$ and $|\varepsilon\delta| \to 0$,

$$\bar{\phi}(t, \varepsilon, \delta) \to \eta(t), \qquad \phi_0(t, \varepsilon, \delta) \to \frac{1}{T}\int_0^T \eta(s)\, ds,$$
$$\text{and } T(\varepsilon, \delta) \to T.$$

(b) As $|\delta| \to 0$ and $|\varepsilon\delta| \to \infty$,

$$\bar{\phi}(t, \varepsilon, \delta) \to \eta(t), \phi_0(t, \varepsilon, \delta) \to 0, \quad \text{and} \quad T(\varepsilon, \delta) \to T.$$

(c) As $|\delta| \to \infty$,

$$\bar{\phi}(t, \varepsilon, \delta) \to \eta(\varepsilon(1+\varepsilon)^{-1}t), \qquad \phi_0(t, \varepsilon, \delta) \to \eta(\varepsilon(1+\varepsilon)^{-1}t),$$
$$\text{and} \quad T(\varepsilon, \delta) \to |\varepsilon^{-1}(1+\varepsilon)|T,$$

for $\varepsilon \neq -1, 0$.

We now consider an example where f is a truncated normal form of a planar oscillator near a Hopf bifurcation [**2**]. Specifically, we choose $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$f(y, z) = \begin{pmatrix} \alpha y + \beta z - y(y^2 + z^2) \\ -\beta y + \alpha z - z(y^2 + z^2) \end{pmatrix}, \qquad \alpha > 0, \beta > 0. \tag{83}$$

Then (70) has a nonconstant periodic solution $x = \eta(t)$ with

$$\eta(t) = \sqrt{\alpha} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}$$

and its period is $2\pi/\beta$.

$$X(t) = \begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} e^{-2\alpha t} & 0 \\ 0 & 1 \end{bmatrix}$$

is a fundamental matrix solution of the variational equation of (70) with respect to $\eta(t)$ and therefore its multipliers are 1 and $e^{-4\alpha\pi/\beta}$. For simplicity, we assume that the eigenvalues of P have positive real parts. Then, for some ranges of ε and δ, (70) has a periodic solution $\bar{x} = \bar{\phi}(t, \varepsilon, \delta)$, $x_0 = \phi_0(t, \varepsilon, \delta)$ whose period equals $T(\varepsilon, \delta)$. As $|\varepsilon\delta| \to 0$ and $|\delta| \to 0$, or as $|\varepsilon\delta| \to \infty$ and $|\delta| \to 0$,

$$\begin{aligned} &\bar{\phi}(t, \varepsilon, \delta) \to \sqrt{\alpha} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}, \\ &\phi_0(t, \varepsilon, \delta) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad T(\varepsilon, \delta) \to \frac{2\pi}{\beta}. \end{aligned} \tag{84}$$

As $|\delta| \to \infty$,

$$\begin{aligned} &\bar{\phi}(t, \varepsilon, \delta) \to \sqrt{\alpha} \begin{pmatrix} \cos(\varepsilon\beta t/(1+\varepsilon)) \\ -\sin(\varepsilon\beta t/(1+\varepsilon)) \end{pmatrix}, \\ &\phi_0(t, \varepsilon, \delta) \to \sqrt{\alpha} \begin{pmatrix} \cos(\varepsilon\beta t/(1+\varepsilon)) \\ -\sin(\varepsilon\beta t/(1+\varepsilon)) \end{pmatrix}, \\ &T(\varepsilon, \delta) \to \frac{2(1+\varepsilon)\pi}{\varepsilon\beta}. \end{aligned} \tag{85}$$

We will again consider this example with $\alpha = 1$ in [**13**]. Given $\varepsilon > 0$ and $\beta > 0$, we will construct periodic solutions for all admissible values of δ and study their global behavior. Note that this shows how to generate long-period oscillation via coupling.

4. Discussion. Invariant manifolds of systems that include (1) as a special case are studied in [**8, 12, 14** and **15**]. Suppose that $\varepsilon = \varepsilon_0 \delta^{-p}$ for $\varepsilon_0 > 0$ and $p > 0$. The results obtained in [**12, 14** and **15**] are applicable to (1) under Assumptions 5(a), 6(a) with $l_1 = n - 1$, and 7.

First assume that $p = 1$. Then the results of [**12**] and [**14**] assert that the unperturbed system of (1) ($\delta = 0$) has an N-dimensional invariant torus whose parametric representation is given by

$$x_i = \eta(\theta_i), \qquad i = 1, \dots, N,$$
$$x_0 = -\varepsilon_0 \sum_{i=1}^{N} [I - e^{-\varepsilon_0 PT}]^{-1} \int_0^T e^{\varepsilon_0 P(s-T)} P\eta(\theta_i + s)\, ds,$$

and that this invariant torus persists for weak coupling.

When $p > 1$, $\varepsilon\delta \to \infty$ as $\delta \to 0$. It is shown in [**14**] that, for all small $\delta > 0$, (1) has invariant tori that lie in a center manifold in the fast time near the N-dimensional torus defined by

$$x_0 = -\sum_{i=1}^{N} \eta(\theta_i), \qquad x_i = \eta(\theta_i), \qquad i = 1, \dots, N.$$

When $0 < p < 1$, the unperturbed system of (1) has an n-parameter family of invariant tori defined by

$$x_0 = c, \quad c \in \mathbf{R}^n, \qquad x_i = \eta(\theta_i), \quad i = 1, \dots, N.$$

A result of [**12**] and [**15**] guarantees that

$$x_i = \eta_i(\theta), \quad i = 1, \dots, N, \qquad x_0 = \frac{1}{T} \int_0^T \eta(s)\, ds \tag{86}$$

persists for all small $\delta > 0$, i.e., a one-parameter family of the invariant tori bifurcates from (86). A similar result is obtained in [**8**] for (1) in which $f : \mathbf{R}^2 \to \mathbf{R}^2$. It is easily seen that the synchronized periodic solutions which exist for weak coupling lie in these invariant tori.

The technique used in Section 2.1 to prove the existence of periodic solutions of (6) is similar to the one introduced in [**11**] to prove a bifurcation of periodic solutions in a one-parameter family of systems, which are called "partially oscillatory systems." The stability of the periodic solutions is also studied there. Here we have studied a two-parameter family of problems and established the existence of periodic solutions. We have also obtained a new result concerning the estimates for the multipliers associated with the periodic solutions. A one-parameter family of singularly-perturbed systems similar to (47) are treated in [**1**] and [**4**], and the existence and stability of periodic solutions are studied. Here we have studied a two-parameter family of singularly-perturbed systems. We have established the existence of periodic solutions and also obtained a result concerning the behavior of the multipliers associated with them.

Significant results concerning planar oscillators of directly coupled type

$$\frac{dx_1}{dt} = f(x_1) + \delta D(x_2 - x_1),$$
$$\frac{dx_2}{dt} = f(x_2) + \delta D(x_1 - x_2)$$

are obtained in [**2**]. Here $f : \mathbf{R}^2 \to \mathbf{R}^2$ is as defined by (83) and D is a constant matrix. The persistence of the invariant torus for the unperturbed system ($\delta = 0$) is established. The behavior of two types of periodic solutions, which are called the in-phase orbits and the out-of-phase orbits, is also studied. The stability of these solutions is also analyzed. These solutions are also studied in [**10**] in case D is a multiple of the identity matrix.

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APPENDIX

The proofs of Lemma 1, Theorem 1(b), and Theorem 2(b). In this section we prove Lemma 1, Theorem 1(b), and Theorem 2(b). Recall that $\theta = \theta(t,\rho,\xi,\mu,\kappa)$, $y = y(t,\rho,\xi,\mu,\kappa)$, $v = v(t,\rho,\xi,\mu,\kappa)$ is the solution of (9) which satisfies (28) and that the functions $p_j(\tau,\rho,\xi,\mu,\kappa)$, $q(\tau,\rho,\xi,\mu,\kappa)$, and $r(\tau,\rho,\xi,\mu,\kappa)$ are defined by (35), (36) and (37), respectively.

Proof of Lemma 1. In view of (35), (36), and (38),

$$\frac{\partial p_j}{\partial \tau}(\tau,0,\xi,0,0) = 1, \qquad \frac{\partial q}{\partial \tau}(jT,0,\xi,0,0) = 0. \tag{87}$$

$\theta(t,\rho,\xi,\mu,\kappa)$ satisfies

$$\frac{\partial \theta}{\partial t}(t,\rho,\xi,\mu,\kappa) = 1 + \Theta(\theta(t,\rho,\xi,\mu,\kappa), y(t,\rho,\xi,\mu,\kappa), v(t,\rho,\xi,\mu,\kappa),\mu,\kappa).$$

Differentiating this equation with respect to ρ, and using (15), (18), (38) and (39), one finds that

$$\frac{\partial}{\partial t}\frac{\partial \theta}{\partial \rho}(t,0,\xi,0,0) = \frac{\partial \Theta_0}{\partial y}(t,0)\frac{\partial y}{\partial \rho}(t,0,\xi,0,0).$$

It follows from (35) that

$$\frac{\partial p_j}{\partial \rho}(jT,0,\xi,0,0) = \int_0^{jT} \frac{\partial \Theta_0}{\partial y}(s,0)\frac{\partial y}{\partial \rho}(s,0,\xi,0,0)\,ds. \tag{88}$$

In view of (36),

$$\frac{\partial q}{\partial \rho}(jT,0,\xi,0,0) = \frac{\partial y}{\partial \rho}(jT,0,\xi,0,0) - I_{(m-1)\times(m-1)}. \tag{89}$$

On the other hand, $y(t,\rho,\xi,\mu,\kappa)$ satisfies

$$\frac{\partial y}{\partial t}(t,\rho,\xi,\mu,\kappa) = A(\theta(t,\rho,\xi,\mu,\kappa))y(t,\rho,\xi,\mu,\kappa) + Y(\theta(t,\rho,\xi,\mu,\kappa), y(t,\rho,\xi,\mu,\kappa), v(t,\rho,\xi,\mu,\kappa),\mu,\kappa).$$

Differentiating this equation with respect to ρ, and using (16), (17), (19), (38), and (39), we obtain

$$\frac{\partial}{\partial t}\frac{\partial y}{\partial \rho}(t,0,\xi,0,0) = A(t)\frac{\partial y}{\partial \rho}(t,0,\xi,0,0).$$

We also find that

$$\frac{\partial y}{\partial \rho}(0,\rho,\xi,\mu,\kappa) = I_{(m-1)\times(m-1)}.$$

This shows that

$$\frac{\partial y}{\partial \rho}(t,0,\xi,0,0) = Y(t,0). \tag{90}$$

(40) now follows from (87), (88), (89), and (90).

In view of (37),

$$\frac{\partial r}{\partial \mu}(jT,0,\xi,0,0) = \frac{\partial v}{\partial \mu}(jT,0,\xi,0,0). \tag{91}$$

Now $v(t,\rho,\xi,\mu,\delta)$ satisfies

$$\frac{\partial v}{\partial t}(t,\rho,\xi,\mu,\kappa) = \mu V(\theta(t,\rho,\xi,\mu,\kappa), y(t,\rho,\xi,\mu,\kappa), v(t,\rho,\xi,\mu,\kappa),\mu,\kappa).$$

Differentiating this equation with respect to μ, and using (14), (38), and (39), we obtain

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial \mu}(t,0,\xi,0,0) = H(\eta(t),\xi,0,0). \tag{92}$$

It follows from (91) and (92) that

$$\begin{aligned}\frac{\partial r}{\partial \mu}(jT,0,\xi,0,0) &= \int_0^{jT} H(\eta(s),\xi,0,0)\,ds \\ &= j\int_0^{T} H(\eta(s),\xi,0,0)\,ds = ja(\xi).\end{aligned}$$

The proof of Lemma 1 is now complete. □

Before we prove Theorems 1(b) and 2(b), we derive some useful properties associated with solutions of (23) and (51). Recall that (27) is the solution of (9) which satisfies (28) and that (30) is the solution of (6) which satisfies (31) provided the function $u(t,\rho,\xi,\mu,\kappa)$ is defined by (29). Furthermore, (25) is a periodic solution of (6) provided the functions $\gamma(t,\mu,\kappa)$ and $\nu(t,\mu,\kappa)$ are defined by (45) and (46). Therefore, the $(m+n)\times(m+n)$-matrix $Z(t,\mu,\kappa)$ defined by

$$(93)\quad Z(t,\mu,\kappa) = \begin{pmatrix} \frac{\partial\gamma}{\partial t}(t,\mu,\kappa) & \frac{\partial u}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) & \frac{\partial u}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \\ \frac{\partial\nu}{\partial t}(t,\mu,\kappa) & \frac{\partial v}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) & \frac{\partial v}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \end{pmatrix}$$

is a matrix solution of (23). $Z(t,\mu,\kappa)$ is also a matrix solution of the variational system (51) when (27) is the solution of (49), $u(t,\rho,\xi,\mu,\kappa)$ is defined by (29), and $\rho(\mu,\kappa)$ and $\xi(\mu,\kappa)$ are defined by (65) and (67), respectively.

Note that $Z(t,\mu,\kappa)$ satisfies

$$Z(0,\mu,\kappa) = \begin{bmatrix} \frac{\partial\gamma}{\partial t}(0,\mu,\kappa) & \Psi(0) & O_{m\times n} \\ \frac{\partial\nu}{\partial t}(0,\mu,\kappa) & O_{n\times(m-1)} & I_{n\times n} \end{bmatrix}.$$

Denote by $\Lambda(\mu,\kappa)$ the $m\times m$-matrix

$$(94)\qquad \Lambda(\mu,\kappa) = [\,\tfrac{\partial\gamma}{\partial t}(0,\mu,\kappa)\quad \Psi(0)\,].$$

Note that (26) or (53) gives

$$(95)\qquad \Lambda(0,0) = [\,F(\gamma(0))\quad \Psi(0)\,].$$

Therefore, for all small $|\mu|$ and $|\kappa|$, (94) is nonsingular. It follows that, for small $|\mu|$ and $|\kappa|$, $Z(t,\mu,\kappa)$ is a fundamental matrix solution and the multipliers of (23) or (51) are the eigenvalues of the matrix $C(\mu,\kappa)$ defined by $C(\mu,\kappa) = Z^{-1}(0,\mu,\kappa)Z(T(\mu,\kappa),\mu,\kappa)$. Furthermore, its inverse $C^{-1}(\mu,\kappa)$ is given by $C^{-1}(\mu,\kappa) = Z^{-1}(0,\mu,\kappa)Z(-T(\mu,\kappa),\mu,\kappa)$. Thus, we obtain

$$(96)\qquad C^{\pm 1}(\mu,\kappa) = Z^{-1}(0,\mu,\kappa)Z(\pm T(\mu,\kappa),\mu,\kappa).$$

$C^{\pm 1}(\mu,\kappa)$ can be written in terms of the derivatives in (93). We summarize these results in the following lemma.

Lemma 3. *Denote by* (27) *the solution of* (9) *or* (49) *which satisfies* (28). *Let* $u(t,\rho,\xi,\mu,\kappa)$ *be the function defined by* (29). *If* $\rho(\mu,\kappa)$, $\xi(\mu,\kappa)$, $\gamma(t,\mu,\kappa)$, *and* $\nu(t,\mu,\kappa)$ *are defined by* (45) *and* (46), *then the eigenvalues of* $C(\mu,\kappa)$ *are the multipliers of* (23). *If* $\rho(\mu,\kappa)$, $\xi(\mu,\kappa)$, $\gamma(t,\mu,\kappa)$ *and* $\nu(t,\mu,\kappa)$ *are defined by* (65) *and* (67), *then the eigenvalues of* $C(\mu,\kappa)$ *are the multipliers of* (51). *Furthermore,* $C^{\pm 1}(\mu,\kappa)$ *can be written in the form*

$$C^{\pm 1}(\mu,\kappa) = \begin{bmatrix} 1 & C^{\pm}_{12}(\mu,\kappa) & C^{\pm}_{13}(\mu,\kappa) \\ O_{(m-1)\times 1} & C^{\pm}_{22}(\mu,\kappa) & C^{\pm}_{23}(\mu,\kappa) \\ O_{n\times 1} & C^{\pm}_{32}(\mu,\kappa) & C^{\pm}_{33}(\mu,\kappa) \end{bmatrix}, \tag{97}$$

where

$$\begin{aligned} C^{\pm}_{12}(\mu,\kappa) &= [\,1 \quad O_{1\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa) \\ &\quad \frac{\partial u}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \\ C^{\pm}_{22}(\mu,\kappa) &= [\,O_{(m-1)\times 1} \quad I_{(m-1)\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa) \\ &\quad \frac{\partial u}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \end{aligned} \tag{98}$$

$$\begin{aligned} C^{\pm}_{32}(\mu,\kappa) &= [\,-\tfrac{\partial \nu}{\partial t}(0,\mu,\kappa) \quad O_{n\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa) \\ &\quad \frac{\partial u}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \\ &\quad + \frac{\partial v}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \end{aligned} \tag{99}$$

$$\begin{aligned} C^{\pm}_{13}(\mu,\kappa) &= [\,1 \quad O_{1\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa) \\ &\quad \frac{\partial u}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \end{aligned}$$

$$\begin{aligned} C^{\pm}_{23}(\mu,\kappa) &= [\,O_{(m-1)\times 1} \quad I_{(m-1)\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa) \\ &\quad \frac{\partial u}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa), \end{aligned} \tag{100}$$

$$C_{33}^{\pm}(\mu,\kappa) = [\,-\tfrac{\partial\nu}{\partial t}(0,\mu,\kappa) \quad O_{n\times(m-1)}\,]\,\Lambda^{-1}(\mu,\kappa)$$
$$\frac{\partial u}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \tag{101}$$
$$+\frac{\partial v}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa).$$

Proof. One finds that

(102)

$$Z^{-1}(0,\mu,\kappa) = \begin{bmatrix} 1 & O_{1\times(m-1)} & O_{1\times n} \\ O_{(m-1)\times 1} & I_{(m-1)\times(m-1)} & O_{(m-1)\times n} \\ -\frac{\partial\nu}{\partial t}(0,\mu,\kappa) & O_{n\times(m-1)} & I_{n\times n} \end{bmatrix} \begin{bmatrix} \Lambda^{-1}(\mu,\kappa) & O_{m\times n} \\ O_{n\times m} & I_{n\times n} \end{bmatrix}.$$

(97) follows from (93), (94), (96), and (102). □

For the degenerate problem we need the formulas for $C(\mu,\kappa)$ only. The formulas for $C^{\pm 1}(\mu,\kappa)$ are used when we study the stability of periodic solutions in the singular problem.

Now we prove Theorem 1(b).

Proof of Theorem 1(b). Differentiating both sides of (29) with respect to ρ, and using (38), we obtain

$$\frac{\partial u}{\partial \rho}(t,0,\xi,0,0) = F(\gamma(t))\frac{\partial \theta}{\partial \rho}(t,0,\xi,0,0) + \Psi(t)\frac{\partial y}{\partial \rho}(t,0,\xi,0,0). \tag{103}$$

In view of (29) and (38), we find that

$$\frac{\partial u}{\partial \xi}(t,0,\xi,0,0) = O_{m\times n}. \tag{104}$$

Because of (39),

$$\frac{\partial v}{\partial \rho}(t,\rho,\xi,0,\kappa) = O_{n\times(m-1)}. \tag{105}$$

From (39) and (92), we obtain

$$\frac{\partial v}{\partial \xi}(t,\rho,\xi,0,\kappa) = I_{n\times n}, \qquad \frac{\partial^2 v}{\partial\mu\partial\xi}(T,0,\xi,0,0) = Da(\xi). \tag{106}$$

It follows from (95), (98) and (103) that there is an $(m-1)\times(m-1)$-matrix, which we call $\tilde{C}^+_{22}(\mu,\kappa)$, such that

$$\begin{aligned} C^+_{22}(\mu,\kappa) &= Y(T,0) + \tilde{C}^+_{22}(\mu,\kappa), \\ \tilde{C}^+_{22}(0,0) &= O_{(m-1)\times(m-1)}. \end{aligned} \tag{107}$$

(39), (99), and (105) show that

$$C^+_{32}(0,\kappa) = O_{n\times(m-1)}. \tag{108}$$

From (100) and (104) we obtain

$$C^+_{23}(0,0) = O_{(m-1)\times n}. \tag{109}$$

Finally, (39), (101), (104), and (106) show that there is an $n\times n$-matrix $\tilde{C}^+_{33}(\mu,\kappa)$ such that

$$\begin{aligned} C^+_{33}(\mu,\kappa) &= I_{n\times n} + \mu[Da(\xi^*) + \tilde{C}^+_{33}(\mu,\kappa)], \\ \tilde{C}^+_{33}(0,0) &= O_{n\times n}. \end{aligned} \tag{110}$$

Now Theorem 1(b) follows from Assumption 2(b), Lemma 3, and (107)–(110). □

Next we prove Theorem 2(b). We first summarize a result concerning a differential inequality in Property 1. The statement is identical to what is proven in Lemma 4 in [**15**] in case $i = 0$. A similar proof leads to the case $i = 1$ and is left to the reader.

Property 1. *Suppose that h is a continuous nonnegative function on the closed interval $[t_0, t_1]$ such that*

$$h(t) \le e^{A_1|t-t_i|}\left\{B_1 + C_1\left|\int_{t_i}^{t} e^{(A_2-A_1)|s-t_i|}\left[B_2 + C_2\left|\int_{t_i}^{s} e^{-A_2|\tau-t_i|}h(\tau)\,d\tau\right|\right]ds\right|\right\}$$

for all $t \in [t_0, t_1]$, *where* $i = 0$ *or* 1, A_j, B_j, *and* C_j *are real numbers, and* $C_j \geq 0$ *for* $j = 1$ *and* 2. *Then*

$$h(t) \leq e^{\mathcal{R}_0|t-t_i|}\left[B_1 + (B_2C_1 - \operatorname{sgn}(t-t_i)B_1r_0)\frac{1-e^{-\mathcal{R}|t-t_i|}}{\mathcal{R}}\right]$$

for all $t \in [t_0, t_1]$, *where*

$$\begin{aligned}
\mathcal{R} &= \sqrt{(A_2 - A_1)^2 + 4C_1C_2} = |A_2 - A_1| + 2\mathcal{I},\\
\mathcal{R}_0 &= \frac{A_1 + A_2 + \mathcal{R}}{2} = \max\{A_1, A_2\} + \mathcal{I},\\
r_0 &= \frac{A_1 - A_2 + \mathcal{R}}{2} = \frac{A_2 - A_1 + |A_2 - A_1|}{2} + \mathcal{I},
\end{aligned}$$

$$\mathcal{I} = \int_0^1 \frac{C_1C_2\,ds}{\sqrt{(A_2 - A_1)^2 + 4C_1C_2s}}.$$

Using the results of Property 1, we next state and prove some properties concerning solutions of a linear system in the following lemma.

Lemma 4. *Let the pair* $x_1 = x_1(t)$, $x_2 = x_2(t)$ *be a solution of the following linear system*:

$$\begin{aligned}
\frac{dx_1}{dt} &= C_{11}(t)x_1 + C_{12}(t)x_2,\\
\frac{dx_2}{dt} &= C_{21}(t)x_1 + C_{22}(t)x_2,
\end{aligned}$$

where $x_i \in \mathbf{R}^{n_i}$ *for* $i = 1$ *and* 2, *and each entry of the* $n_i \times n_j$*-matrix* $C_{ij}(t)$, $i = 1, 2$, $j = 1, 2$, *is defined for all* $t \in \mathbf{R}$ *and is a continuous bounded function of* t. *For* $i = 1$ *and* 2, *let* $X_i(t, s)$, $X_i(s, s) = I_{n_i \times n_i}$, *be the fundamental matrix solution of the linear system*:

$$\frac{dx_i}{dt} = C_{ii}(t)x_i.$$

Let a_i and K_i be constants with $K_i > 0$. Define

$$R = \sqrt{(a_1 - a_2)^2 + 4\lambda_{12}\lambda_{21}K_1K_2} = |a_1 - a_2| + 2I,$$
$$R_0 = \frac{a_1 + a_2 + R}{2} = \max\{a_1, a_2\} + I,$$
$$r_i = \frac{a_j - a_i + R}{2} = \frac{a_j - a_i + |a_j - a_i|}{2} + I,$$

$i = 1, 2,\ j = 1, 2,\ i \neq j,$

$$I = \int_0^1 \frac{\lambda_{12}\lambda_{21}K_1K_2\, ds}{\sqrt{(a_1 - a_2)^2 + 4\lambda_{12}\lambda_{21}K_1K_2 s}},$$
$$\lambda_{ij} = \sup_{t\in\mathbf{R}}\{||C_{ij}(t)||\}, \quad i = 1, 2,\ j = 1, 2.$$

(a) *If*

$$||X_i(t, s)|| \leq K_i e^{a_i(t-s)} \qquad \textit{for } t \geq s, \tag{111}$$

then, for $t \geq t_0$,

$$||x_i(t)|| \leq K_i e^{R_0(t-t_0)}\bigg[||x_i(t_0)|| + (\lambda_{ij}K_j||x_j(t_0)|| - r_i||x_i(t_0)||)\frac{1 - e^{-R(t-t_0)}}{R}\bigg]. \tag{112}$$

(b) *If*

$$||X_i(t, s)|| \leq K_i e^{a_i(s-t)} \qquad \textit{for } t \leq s, \tag{113}$$

then, for $t \leq t_0$,

$$||x_i(t)|| \leq K_i e^{R_0(t_0-t)}\bigg[||x_i(t_0)|| + (\lambda_{ij}K_j||x_j(t_0)|| + r_j||x_i(t_0)||)\frac{1 - e^{-R(t_0-t)}}{R}\bigg]. \tag{114}$$

Proof. For $i = 1$ and 2, $x_i(t)$ is given by

$$x_i(t) = X_i(t, t_0)x_i(t_0) + \int_{t_0}^t X_i(t, s)C_{ij}(s)x_j(s)\, ds,$$
$$j = 1, 2,\ i \neq j.$$

Substituting one equation in the other, we obtain

$$(115)\quad x_i(t) = X_i(t,t_0)x_i(t_0) + \int_{t_0}^{t} X_i(t,s)C_{ij}(s)\left[X_j(s,t_0)x_j(t_0) + \int_{t_0}^{s} X_j(s,\tau)C_{ji}(\tau)x_i(\tau)\,d\tau\right] ds.$$

Using (111), we find that, for $t \geq t_0$,

$$||x_i(t)|| \leq K_i e^{a_i(t-t_0)}\left\{||x_i(t_0)|| + \int_{t_0}^{t} \lambda_{ij}K_j e^{(a_j-a_i)(s-t_0)}\left[||x_j(t_0)|| + \int_{t_0}^{s} \lambda_{ji} e^{-a_j(\tau-t_0)}||x_i(\tau)||\,d\tau\right] ds\right\}.$$

(112) now follows from Property 1. This completes the proof of (a).

The proof of (b) is similar. From (113) and (115), we obtain

$$||x_i(t)|| \leq K_i e^{a_i(t_0-t)}\left\{||x_i(t_0)|| + \int_{t}^{t_0} \lambda_{ij}K_j e^{(a_j-a_i)(t_0-s)}\left[||x_j(t_0)|| + \int_{s}^{t_0} \lambda_{ji} e^{-a_j(t_0-\tau)}||x_i(\tau)||\,d\tau\right] ds\right\}.$$

Now Property 1 leads to (114). □

Remark 6. Note that $X_i(t,s)$ can be written in the form

$$X_i(t,s) = I_{n_i\times n_i} + \int_{s}^{t} C_{ii}(\tau)X_i(\tau,s)\,d\tau.$$

From this equation, we obtain

$$||X_i(t,s)|| \leq 1 + \int_{s}^{t} \lambda_{ii}||X_i(\tau,s)||\,d\tau \quad \text{for } t \geq s.$$

It follows from Gronwall's inequality that $||X_i(t,s)|| \leq e^{\lambda_{ii}(t-s)}$ for $t \geq s$. Similarly, one shows that $||X_i(t,s)|| \leq e^{\lambda_{ii}(s-t)}$ for $t \leq s$. Therefore, we may take $a_i = \lambda_{ii}$ and $K_i = 1$ in (111) and (113).

To prove Theorem 2(b) we proceed as follows. The variational system (51) is closely related to the variational system of (49) with respect to (66):

$$
\begin{aligned}
d\theta/dt &= B_{11}(t,\mu,\kappa)\theta + B_{12}(t,\mu,\kappa)y + B_{13}(t,\mu,\kappa)v,\\
dy/dt &= B_{21}(t,\mu,\kappa)\theta + B_{22}(t,\mu,\kappa)y + B_{23}(t,\mu,\kappa)v,\\
\kappa(dv/dt) &= B_{31}(t,\mu,\kappa)\theta + B_{32}(t,\mu,\kappa)y + [C + B_{33}(t,\mu,\kappa)]v,
\end{aligned}
\tag{116}
$$

where

$$
B_{i1}(t,\mu,\kappa) = \frac{\partial R_i}{\partial \theta}(\theta(t,\mu,\kappa), y(t,\mu,\kappa), \nu(t,\mu,\kappa), \mu, \kappa),
\tag{117}
$$

$$
B_{i2}(t,\mu,\kappa) = \frac{\partial R_i}{\partial y}(\theta(t,\mu,\kappa), y(t,\mu,\kappa), \nu(t,\mu,\kappa), \mu, \kappa),
\tag{118}
$$

$$
B_{i3}(t,\mu,\kappa) = \frac{\partial R_i}{\partial v}(\theta(t,\mu,\kappa), y(t,\mu,\kappa), \nu(t,\mu,\kappa), \mu, \kappa)
\tag{119}
$$

for $i = 1, 2$ and 3, and

$$
\begin{aligned}
R_1(\theta,y,v,\mu,\kappa) &= \Theta_0(\theta,y) + \Theta_1(\theta,y,v,\mu,\kappa),\\
R_2(\theta,y,v,\mu,\kappa) &= A(\theta)y + Y_0(\theta,y) + Y_1(\theta,y,v,\mu,\kappa),\\
R_3(\theta,y,v,\mu,\kappa) &= V(\theta,y,v,\mu,\kappa).
\end{aligned}
\tag{120}
$$

Write

$$
w = \begin{pmatrix}\theta\\ y\end{pmatrix}.
$$

Let the pairs $w = w_1(t)$, $v = v_1(t)$ and $w = w_2(t)$, $v = v_2(t)$ be the solutions of (116) with the initial values

$$
\begin{aligned}
w_1(0) &= I_{m\times m}, \quad w_2(0) = O_{m\times n},\\
v_1(0) &= O_{n\times m}, \quad v_2(0) = I_{n\times n}.
\end{aligned}
$$

It is easily seen that

$$
\begin{aligned}
\begin{bmatrix} \frac{\partial\theta}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ \frac{\partial y}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\end{bmatrix} &= w_1(t)\begin{bmatrix} O_{1\times(m-1)}\\ I_{(m-1)\times(m-1)}\end{bmatrix},\\
\frac{\partial v}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) &= v_1(t)\begin{bmatrix} O_{1\times(m-1)}\\ I_{(m-1)\times(m-1)}\end{bmatrix},\\
\begin{bmatrix} \frac{\partial\theta}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ \frac{\partial y}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\end{bmatrix} &= w_2(t),\\
\frac{\partial v}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) &= v_2(t).
\end{aligned}
\tag{121}
$$

Let

$$\beta_{ij}(\mu,\kappa) = \sup_{t\in\mathbf{R}}\{||B_{ij}(t,\mu,\kappa)||\}, \qquad i=1,2,3,\ j=1,2,3.$$

In view of (119), (120), and Assumption 4(b),

$$\beta_{33}(\mu,\kappa) \sim \mathcal{O}(|\mu|+|\kappa|). \tag{122}$$

Define

(123)

$$\begin{aligned}
\zeta_{11}(\mu,\kappa) &= \sup_{t\in\mathbf{R}}\left\{\left\| \begin{bmatrix} B_{11}(t,\mu,\kappa) & B_{12}(t,\mu,\kappa) \\ B_{21}(t,\mu,\kappa) & B_{22}(t,\mu,\kappa)\end{bmatrix}\right\|\right\},\\
\zeta_{12}(\mu,\kappa) &= \sup_{t\in\mathbf{R}}\left\{\left\| \begin{bmatrix} B_{13}(t,\mu,\kappa) \\ B_{23}(t,\mu,\kappa)\end{bmatrix}\right\|\right\},\\
\zeta_{21}(\mu,\kappa) &= \sup_{t\in\mathbf{R}}\{||\,[\,B_{31}(t,\mu,\kappa) \quad B_{32}(t,\mu,\kappa)\,]\,||\}.
\end{aligned}$$

Then

$$\begin{aligned}
\zeta_{11}(\mu,\kappa) &\le \max_{1\le j\le 2}\left\{\sum_{i=1}^{2}\beta_{ij}(\mu,\kappa)\right\},\\
\zeta_{12}(\mu,\kappa) &\le \sum_{i=1}^{2}\beta_{i3}(\mu,\kappa),\\
\zeta_{21}(\mu,\kappa) &\le \max_{1\le j\le 2}\{\beta_{3j}(\mu,\kappa)\}.
\end{aligned}$$

Let $W(t,s,\mu,\kappa)$, $W(s,s,\mu,\kappa) = I_{m\times m}$, be the fundamental matrix solution of the linear system:

$$\frac{dw}{dt} = \mathcal{B}(t,\mu,\kappa)w,$$

where

$$\mathcal{B}(t,\mu,\kappa) = \begin{bmatrix} B_{11}(t,\mu,\kappa) & B_{12}(t,\mu,\kappa) \\ B_{21}(t,\mu,\kappa) & B_{22}(t,\mu,\kappa)\end{bmatrix}. \tag{124}$$

Let $V(t,s,\mu,\kappa)$, $V(s,s,\mu,\kappa) = I_{n\times n}$, be the fundamental matrix solution of the linear system:

$$\kappa\frac{dv}{dt} = [C + B_{33}(t,\mu,\kappa)]v.$$

Lemma 5.

$$(125)\qquad \begin{gathered} \|W(t,s,\mu,\kappa)\| \le e^{\zeta_{11}(\mu,\kappa)|t-s|}, \\ \|V(t,s,\mu,\kappa)\| \le Ke^{-\kappa^{-1}[\alpha - K\beta_{33}(\mu,\kappa)](t-s)} \\ \text{for } \kappa^{-1}(t-s) \ge 0, \end{gathered}$$

where K and α are the constants defined in Remark 2.

Proof. (125) follows from (123), (124), and Remark 6. $V(t,s,\mu,\kappa)$ is given by

$$V(t,s,\mu,\kappa) = e^{\kappa^{-1}C(t-s)} + \kappa^{-1}\int_s^t e^{\kappa^{-1}C(t-\tau)} B_{33}(\tau,\mu,\kappa)V(\tau,s,\mu,\kappa)\,d\tau.$$

It follows from Remark 2(b) that for $\kappa > 0$ and $t \ge s$,

$$\begin{aligned} \|V(t,s,\mu,\kappa)\| \le {} & Ke^{-\kappa^{-1}\alpha(t-s)} \\ & + \kappa^{-1}\int_s^t Ke^{-\kappa^{-1}\alpha(t-\tau)}\beta_{33}(\mu,\kappa)\|V(\tau,s,\mu,\kappa)\|\,d\tau \end{aligned}$$

and

$$\begin{aligned} e^{\kappa^{-1}\alpha t}\|V(t,s,\mu,\kappa)\| \le {} & Ke^{\kappa^{-1}\alpha s} \\ & + \kappa^{-1}\int_s^t Ke^{\kappa^{-1}\alpha\tau}\beta_{33}(\mu,\kappa)\|V(\tau,s,\mu,\kappa)\|\,d\tau. \end{aligned}$$

It follows from Gronwall's inequality that

$$e^{\kappa^{-1}\alpha t}\|V(t,s,\mu,\kappa)\| \le Ke^{\kappa^{-1}\alpha s}e^{\kappa^{-1}K\beta_{33}(\mu,\kappa)(t-s)}.$$

Therefore

$$\|V(t,s,\mu,\kappa)\| \le Ke^{-\kappa^{-1}[\alpha - K\beta_{33}(\mu,\kappa)](t-s)} \qquad \text{for } \kappa > 0 \text{ and } t \ge s.$$

The proof for the case $\kappa < 0$ and $t \le s$ is similar. □

Now we complete the proof of Theorem 2(b).

Proof of Theorem 2(b). In view of (122), $\alpha - K\beta_{33}(\mu,\kappa) > 0$ for sufficiently small $|\mu|+|\kappa|$. We set

$$\begin{aligned} a_1 &= \zeta_{11}(\mu,\kappa), \qquad K_1 = 1,\\ a_2 &= -|\kappa|^{-1}[\alpha - K\beta_{33}(\mu,\kappa)], \qquad K_2 = K,\\ \lambda_{11} &= \zeta_{11}(\mu,\kappa), \qquad \lambda_{12} = \zeta_{12}(\mu,\kappa),\\ \lambda_{21} &= |\kappa|^{-1}\zeta_{21}(\mu,\kappa). \end{aligned}$$

Then, according to Lemma 4, for $\kappa^{-1}t \geq 0$,

$$\begin{aligned} ||v_1(t)|| &\leq K_2 e^{R_0|t|}\lambda_{21}K_1||w_1(0)||\frac{1-e^{-R|t|}}{R}\\ &\leq |\kappa|^{-1}\zeta_{21}(\mu,\kappa)Ke^{R_0|t|}\frac{1-e^{-R|t|}}{R},\\ ||w_2(t)|| &\leq K_1 e^{R_0|t|}\lambda_{12}K_2 - 2||v_2(0)||\frac{1-e^{-R|t|}}{R}\\ &\leq \zeta_{12}(\mu,\kappa)Ke^{R_0|t|}\frac{1-e^{-R|t|}}{R}, \end{aligned} \tag{126}$$

where

$$\begin{aligned} R &= |a_1 - a_2| + 2I = \zeta_{11}(\mu,\kappa) + |\kappa|^{-1}[\alpha - K\beta_{33}(\mu,\kappa)] + 2I,\\ R_0 &= \max\{a_1, a_2\} + I = \zeta_{11}(\mu,\kappa) + I, \end{aligned}$$

(127)
$$\begin{aligned} I &= \int_0^1 \frac{\lambda_{12}\lambda_{21}K_1K_2\,ds}{\sqrt{(a_1-a_2)^2 + 4\lambda_{12}\lambda_{21}K_1K_2 s}},\\ &= \int_0^1 \frac{K\zeta_{12}(\mu,\kappa)\zeta_{21}(\mu,\kappa)\,ds}{\sqrt{[\alpha - K\beta_{33}(\mu,\kappa) + |\kappa|\zeta_{11}(\mu,\kappa)]^2 + 4|\kappa|\zeta_{12}(\mu,\kappa)\zeta_{21}(\mu,\kappa)s}}. \end{aligned}$$

In view of (117), (118), (120), and Assumption 4(b),

$$\zeta_{21}(\mu,\kappa) \sim \mathcal{O}(|\mu| + |\kappa|). \tag{128}$$

It is easily seen that the integral I in (127) is bounded for all μ and κ in a neighborhood of 0. In fact, (128) leads to the estimate $I \sim \mathcal{O}(|\mu|+|\kappa|)$. It is also seen that

$$R^{-1} \sim \mathcal{O}(|\kappa|). \tag{129}$$

It follows from (126), (128), and (129) that

$$||v_1(t)|| \sim \mathcal{O}(|\mu| + |\kappa|), \qquad ||w_2(t)|| \sim \mathcal{O}(|\kappa|). \tag{130}$$

Let the pair $w = w(t)$ and $v = v(t)$ be the solution of (116). Then

$$v(t)=V(t,0,\mu,\kappa)v(0)+\kappa^{-1}\int_0^t V(t,s,\mu,\kappa)[B_{31}(s,\mu,\kappa), B_{32}(s,\mu,\kappa)]w(s)ds.$$

It follows from Lemma 5 that for $\kappa^{-1}t \geq 0$,

(131)

$$\begin{aligned} ||v(t)|| &\leq Ke^{-\kappa^{-1}[\alpha - K\beta_{33}(\mu,\kappa)]t}||v(0)|| \\ &\quad + \sup_{s\in[0,t]}\{||w(s)||\}\frac{K\zeta_{21}(\mu,\kappa)}{\alpha - K\beta_{33}(\mu,\kappa)}\{1 - e^{-\kappa^{-1}[\alpha - K\beta_{33}(\mu,\kappa)]t}\}. \end{aligned}$$

We find from the second inequality of (126), (128), and (131) that there is a positive number α_1 such that

$$||v_2(t)|| \sim \mathcal{O}(e^{-\kappa^{-1}\alpha_1 t} + |\mu| + |\kappa|) \quad \text{for } \kappa^{-1}t \geq 0. \tag{132}$$

According to (68),

$$\begin{aligned} \frac{\partial\tilde{\theta}}{\partial\rho}(t,\rho(\mu,\kappa),\mu,\kappa) &= \frac{\partial\theta}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \\ &\quad + \frac{\partial\theta}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\frac{\partial d}{\partial y}(0,\rho(\mu,\kappa),\mu,\kappa), \\ \frac{\partial\tilde{y}}{\partial\rho}(t,\rho(\mu,\kappa),\mu,\kappa) &= \frac{\partial y}{\partial\rho}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \\ &\quad + \frac{\partial y}{\partial\xi}(t,\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\frac{\partial d}{\partial y}(0,\rho(\mu,\kappa),\mu,\kappa). \end{aligned} \tag{133}$$

Furthermore, (29) gives

$$\begin{aligned}(134)\quad &\frac{\partial u}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &= \Psi(0)\frac{\partial y}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &+ [F(\gamma(0)) + \Psi'(0)\rho(\mu,\kappa)]\frac{\partial \theta}{\partial \rho}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa),\\ &\frac{\partial u}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &= \Psi(0)\frac{\partial y}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &+ [F(\gamma(0)) + \Psi'(0)\rho(\mu,\kappa)]\frac{\partial \theta}{\partial \xi}(\pm T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa).\end{aligned}$$

It follows from (121), (130), (132), (133), and (134) that

$$\begin{aligned}(135)\quad &\frac{\partial u}{\partial \rho}(\mathrm{sgn}\,(\kappa)T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &- F(\gamma(0))\frac{\partial \tilde\theta}{\partial \rho}(\mathrm{sgn}\,(\kappa)T,0,0,0)\\ &- \Psi(0)\frac{\partial \tilde y}{\partial \rho}(\mathrm{sgn}\,(\kappa)T,0,0,0) \sim \mathcal{O}(|\mu|+|\kappa|),\\ &\frac{\partial v}{\partial \rho}(\mathrm{sgn}\,(\kappa)T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \sim \mathcal{O}(|\mu|+|\kappa|),\\ &\frac{\partial u}{\partial \xi}(\mathrm{sgn}\,(\kappa)T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa) \sim \mathcal{O}(|\mu|+|\kappa|),\\ &\frac{\partial v}{\partial \xi}(\mathrm{sgn}\,(\kappa)T(\mu,\kappa),\rho(\mu,\kappa),\xi(\mu,\kappa),\mu,\kappa)\\ &\sim \mathcal{O}(e^{-|\kappa|^{-1}\alpha_2 T} + |\mu| + |\kappa|) \qquad \text{for some } \alpha_2 > 0,\end{aligned}$$

where

$$\mathrm{sgn}\,(\kappa) = \begin{cases} -1 & \text{for } \kappa < 1,\\ 1 & \text{for } \kappa > 1.\end{cases}$$

On the other hand, an argument similar to the one used in the proof of Lemma 1 leads to the conclusion that

$$(136)\qquad \frac{\partial \tilde y}{\partial \rho}(t,0,0,0) = Y(t,0)$$

Theorem 2(b) now follows from Lemma 3, (135), and (136). □

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