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ON PSEUDOCONTINUOUS MAPPINGS

ELIZA WAJCH

ABSTRACT. This paper is devoted to an investigation of some classes of pseudocontinuous mappings of topological spaces and to those topological properties which are invariant under pseudocontinuous images.

In [**5**], R.A. Johnson and W. Wilczyński introduced various concepts of pseudocontinuous mappings which are near to continuous mappings in the sense that the inverse image of an open set becomes an open set after removing from it or adding to it a set from a fixed σ-ideal. Some types of pseudocontinuity are strictly related to the Baire property of functions, as well as to the well-known and important classes of quasicontinuous, somewhat continuous and nearly continuous mappings. The present paper, being a continuation of [**5**], investigates notions of pseudocontinuity and points out some topological properties which are preserved under pseudocontinuous images.

We shall use the standard notation. For cardinal functions not defined here, see [**1**] and [**6**]. All cardinals are assumed to be infinite. The symbol R will always stand for the space of reals with the usual topology.

In what follows, X and Y denote topological spaces. Let $\varkappa$ be an infinite cardinal number, and let $\mathcal{J}$ be a $\varkappa$-ideal of subsets of X, i.e., $\mathcal{J}$ is a collection of subsets of X such that if $A \subseteq B \in \mathcal{J}$, then $A \in \mathcal{J}$, and if $A_s \in \mathcal{J}$ for $s \in S$ with $|S| \leq \varkappa$, then $\cup_{s\in S} A_s \in \mathcal{J}$.

Definitions. A mapping $f : X \to Y$ is called:

(1) *weakly $\mathcal{J}$-pseudocontinuous* if, for any open subset V of Y, there exists an open subset W of X with $f^{-1}(V)\Delta W \in \mathcal{J}$;

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(2) *weakly $\mathcal{J}$-pseudocontinous at a point* $x \in X$ if, for any open neighborhood V of $f(x)$, there exist open sets $G \subseteq Y$ and $U, W \subseteq X$, such that $f(x) \in G \subseteq V$, $x \in U$ and $[f^{-1}(G) \cap U]\Delta W \in \mathcal{J}$;

(3) *inner $\mathcal{J}$-pseudocontinuous* if, for any open subset V of Y, we have $f^{-1}(V)\backslash \operatorname{int}[f^{-1}(V)] \in \mathcal{J}$;

(4) *inner $\mathcal{J}$-pseudocontinuous at a point* $x \in X$ if, for any open neighborhood V of $f(x)$, there exist open sets $G \subseteq Y$ and $U \subseteq X$, such that $f(x) \in G \subseteq V$, $x \in U$ and $[f^{-1}(G)\backslash \operatorname{int}(f^{-1}(G))] \cap U \in \mathcal{J}$;

(5) *outer $\mathcal{J}$-pseudocontinuous* if, for any open subset V of Y, there exists an open subset W of X with $f^{-1}(V) \subseteq W$ and $W\backslash f^{-1}(V) \in \mathcal{J}$;

(6) *outer $\mathcal{J}$-pseudocontinuous at a point* $x \in X$ if, for each open neighborhood V of $f(x)$, there exists an open neighborhood U of x with $U\backslash f^{-1}(V) \in \mathcal{J}$.

In the case when $\mathcal{J}$ consists of all subsets of X with cardinalities $\leq \varkappa$, we shall say *weak* (*inner, outer*, repsectively) *$\varkappa$-pseudocontinuity* instead of weak (inner, outer, respectively) $\mathcal{J}$-pseudocontinuity.

Notions 1, 3, 5 and 6 were introduced in [**5**]. The authors of [**5**] also gave some local versions of weak and inner $\mathcal{J}$-pseudocontinuity stronger than our 2 and 4, but their versions, contrary to 2 and 4, do not allow one to replace any neighborhood of $f(x)$ by members of a local base of Y at $f(x)$.

It is easily seen that weak (inner, outer, respectively) $\mathcal{J}$-pseudocontinuity implies weak (inner, outer, respectively) $\mathcal{J}$-pseudocontinuity at each point and, in general, the converses do not hold (cf. Example 9). However, we have the following

Theorem 7. *Let $f : X \to Y$. If X is hereditarily $\varkappa$-Lindelöf, then the outer $\mathcal{J}$-pseudocontinuity of f is equivalent to the outer $\mathcal{J}$-pseudocontinuity of f at each point. If the graph $G(f)$ of f is hereditarily $\varkappa$-Lindelöf, then the weak* (*inner, respectively*) *$\mathcal{J}$-pseudocontinuity of f is equivalent to the weak* (*inner, respectively*) *$\mathcal{J}$-pseudocontinuity of f at each point.*

Proof. The part of our theorem which concerns outer $\mathcal{J}$-pseudocontinuity can be deduced from the proof of Proposition 1.3 of [**5**].

Let $G(f)$ be hereditarily $\varkappa$-Lindelöf. Suppose that f is weakly $\mathcal{J}$-pseudocontinuous at each point of X and consider an arbitrary open set $V \subseteq Y$. For any $x \in f^{-1}(V)$, choose open sets $G_x \subseteq Y$ and $U_x, W_x \subseteq X$ such that $x \in U_x$, $f(x) \in G_x \subseteq V$ and $[f^{-1}(G_x) \cap U_x]\Delta W_x \in \mathcal{J}$. There exists a set $A \subseteq f^{-1}(V)$ with $|A| \leq \varkappa$ and such that $f^{-1}(V) = \cup_{x \in A}[f^{-1}(G_x) \cap U_x]$. Put $W = \cup_{x\in A} W_x$. Then $f^{-1}(V)\Delta W \subseteq \cup_{x \in A}[(f^{-1}(G_x) \cap U_x)\Delta W_x]$, so $f^{-1}(V)\Delta W \in \mathcal{J}$.

Arguing similarly, we can check that the inner $\mathcal{J}$-pseudocontinuity of f at each point implies the inner $\mathcal{J}$-pseudocontinuity of f whenever $G(f)$ is hereditarily $\varkappa$-Lindelöf.

It is not known to the author whether the hereditary $\varkappa$-Lindelöf property of $G(f)$ can be replaced in Theorem 7 by the assumption that both X and Y be hereditarily $\varkappa$-Lindelöf. □

Theorem 8. *If a weakly $\varkappa$-pseudocontinuous mapping $f : X \to Y$ is outer $\varkappa$-pseudocontinuous at each point of X, then f is outer $\varkappa$-pseudocontinuous.*

Proof. Let V be an open subset of Y. There is an open set $U \subseteq X$ such that $|f^{-1}(V)\Delta U| \leq \varkappa$. Since f is outer $\varkappa$-pseudocontinuous at each point, for any $x \in f^{-1}(V)\setminus U$, there exists an open neighborhood U_x of x and a set $A_x \subseteq U_x$ with $|A_x| \leq \varkappa$ and $U_x\setminus A_x \subseteq f^{-1}(V)$. Put $W = U \cup \bigcup\{U_x : x \in f^{-1}(V)\setminus U\}$. Then, obviously, $f^{-1}(V) \subseteq W$. Moreover, $W\setminus f^{-1}(V) \subseteq [U\setminus f^{-1}(V)] \cup \bigcup\{A_x : x \in f^{-1}(V)\setminus U\}$, so $|W\setminus f^{-1}(V)| \leq \varkappa$. □

In the above theorem, the weak $\varkappa$-pseudocontinuity of f cannot be replaced by the inner $\varkappa$-pseudocontinuity of f at each point.

Example 9. Denote by X the set $[0, \omega_1)$ with the order topology, and let Y be the set $[0, \omega_1)$ with the discrete topology. Then the identity mapping $f : X \to Y$ is clearly outer and inner ω-pseudocontinuous at each point. Since there exists an open set in Y which is not representable in the form $U\setminus A$ where U is open in X and A is countable (cf. [**7**; page 120]), f fails to be outer ω-pseudocontinuous. In view of Theorem 8, f is not weakly ω-pseudocontinuous.

For a topology $\mathcal{T}$ on X, put $\mathcal{B}^1_{\mathcal{J}} = \{U\backslash A : U \in \mathcal{T} \text{ and } A \in \mathcal{J}\}$ and $\mathcal{B}^2_{\mathcal{J}} = \{U \subseteq X :$ there exist $G, H \in \mathcal{T}$ with $G \subseteq U \subseteq H$ and $H\backslash G \in \mathcal{J}\}$. For $k = 1, 2$, let $\mathcal{T}^k_{\mathcal{J}}$ be the topology on X generated by $\mathcal{B}^k_{\mathcal{J}}$. Evidently, if $(X, \mathcal{T})$ is hereditarily $\varkappa$-Lindelöf, then $\mathcal{B}^k_{\mathcal{J}} = \mathcal{T}^k_{\mathcal{J}}$ for $k = 1, 2$ (cf. [**2, 3, 9**]).

We can obtain the following generalizations of Propositions 1.8 and 1.9 of [**5**]:

Theorem 10. (i) *A mapping $f : (X, \mathcal{T}) \to Y$ is outer $\mathcal{J}$-pseudocontinuous at a point $x_0 \in X$ if and only if $f : (X, \mathcal{T}^1_{\mathcal{J}}) \to Y$ is continuous at x_0.*

(ii) *A mapping $f : (X, \mathcal{T}) \to Y$ is both outer and inner $\mathcal{J}$-pseudocontinuous at a point $x_0 \in X$ if and only if $f : (X, \mathcal{T}^2_{\mathcal{J}}) \to Y$ is continuous at x_0.*

Proof. Let V be any open neighborhood of $f(x_0)$ in Y. Assume that $f : (X, \mathcal{T}) \to Y$ is both inner and outer $\mathcal{J}$-pseudocontinuous at x_0. There exist sets $U_1, U_2 \in \mathcal{T}$ and an open set $V_0 \subseteq V$, such that $x_0 \in U_1 \cap U_2$, $f(x_0) \in V_0$, $[f^{-1}(V_0)\backslash \text{int}_{(X,\mathcal{T})} f^{-1}(V_0)] \cap U_1 \in \mathcal{J}$ and $U_2\backslash f^{-1}(V_0) \in \mathcal{J}$. Put $U = U_1 \cap U_2 \cap f^{-1}(V_0)$, $G = U_1 \cap U_2 \cap \text{int}_{(X,\mathcal{T})} f^{-1}(V_0)$ and $H = U_1 \cap U_2$. It is easily seen that $H\backslash G \in \mathcal{J}$; thus, $U \in \mathcal{T}^2_{\mathcal{J}}$. Since $x_0 \in U$ and $f(U) \subseteq V$, we have that $f : (X, \mathcal{T}^2_{\mathcal{J}}) \to Y$ is continuous at x_0.

Suppose now that $f : (X, \mathcal{T}^2_{\mathcal{J}}) \to Y$ is continuous at x_0. There exists $U \in \mathcal{T}^2_{\mathcal{J}}$ with $x_0 \in U$ and $f(U) \subseteq V$. We can find sets $G, H \in \mathcal{T}$ with $G \subseteq U \subseteq H$ and $H\backslash G \in \mathcal{J}$. Then $x_0 \in H$ and $H\backslash f^{-1}(V) \in \mathcal{J}$, so $f : (X, \mathcal{T}) \to Y$ is outer $\mathcal{J}$-pseudocontinuous at x_0. Moreover, the set $[f^{-1}(V)\backslash \text{int}_{(X,\mathcal{T})} f^{-1}(V)] \cap H$ belongs to $\mathcal{J}$ because it is contained in $H\backslash G$. Therefore $f : (X, \mathcal{T}) \to Y$ is inner $\mathcal{J}$-pseudocontinuous at x_0. Hence (ii) holds. The proof of (i) is similar. □

Theorem 11. *Let X_k, Y_k be topological spaces, and let $f_k : X_k \to Y_k$ for $k = 1, 2$. The following* (i) *and* (ii) *hold*:

(i) *If no nonempty open subset of X_1 is of cardinality $\leq \varkappa$, the product $f_1 \times f_2$ is weakly $\varkappa$-pseudocontinuous at a point $(x_1, x_2) \in X_1 \times$*

X_2, *and* f_1 *is outer* $\varkappa$*-pseudocontinuous at* x_1, *then* f_2 *is continuous at* x_2.

(ii) *If* $|X_1| > \varkappa$ *and* $f_1 \times f_2$ *is weakly* $\varkappa$*-pseudocontinuous, then* f_2 *is continuous.*

Proof. We shall show (i). The proof of (ii) is simpler.

Let V be any open neighborhood of $f_2(x_2)$ in Y_2. There exist open sets $G \subseteq Y_1 \times V$ and $U \subseteq X_1 \times X_2$, such that $(x_1, x_2) \in U$, $(f_1(x_1), f_2(x_2)) \in G$ and, for some $A, B \subseteq X_1 \times X_2$ with $|A \cup B| \leq \varkappa$, the set $[((f_1 \times f_2)^{-1}(G) \cap U)\backslash A] \cup B$ is open in $X_1 \times X_2$. Take open sets $V_k \subseteq Y_k$ and $U_k \subseteq X_k$ with $(f(x_1), f(x_2)) \in V_1 \times V_2 \subseteq G$ and $(x_1, x_2) \in U_1 \times U_2 \subseteq U$. Since f_1 is outer $\varkappa$-pseudocontinuous at x_1 and no nonempty open subset of X_1 is of cardinality $\leq \varkappa$, we have $|f_1^{-1}(V_1) \cap U_1| > \varkappa$. This, together with the inequality $|A \cup B| \leq \varkappa$, implies that there exists $x_0 \in f_1^{-1}(V_1) \cap U_1$ such that the set $W = \{x \in X_2 : (x_0, x) \in (f_1 \times f_2)^{-1}(G) \cap U\}$ equals $\{x \in X_2 : (x_0, x) \in [((f_1 \times f_2)^{-1}(G) \cap U)\backslash A] \cup B\}$; hence W is open in X_2. Clearly, $x_2 \in W$ and $f_2(W) \subseteq V$, which completes the proof of (i). □

Corollary 12. (i) *If no nonempty open subset of* X *is of cardinality* $\leq \varkappa$, *then* $f : X \to Y$ *is continuous at* $x_0 \in X$ *if and only if* $\mathrm{id}_X \times f : X \times X \to X \times Y$ *is weakly* $\varkappa$*-pseudocontinuous at* (x_0, x_0).

(ii) *If* $|X| > \varkappa$, *then* $f : X \to Y$ *is continuous if and only if* $\mathrm{id}_X \times f$ *is weakly* $\varkappa$*-pseudocontinuous.*

Observe that if Y is the Sorgenfrey line and $f : R \to Y$ is the identity mapping, then the diagonal mapping $(\mathrm{id}_R \Delta f) : R \to R \times Y$ is inner ω-pseudocontinuous and not continuous. Hence, in Corollary 12, the product $\mathrm{id}_X \times f$ cannot be replaced by the diagonal $\mathrm{id}_X \Delta f$.

Recall that a mapping $f : X \to Y$ is quasicontinuous at a point $x \in X$ if, for any open sets $U \subseteq X$ and $V \subseteq Y$ with $x \in U$ and $f(x) \in V$, there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$. If f is quasicontinuous at each point $x \in X$, then f is called quasicontinuous (cf. [**8**]).

Theorem 13. *Suppose that X is regular, $\mathcal{J}$ does not contain any nonempty open subset of X, and $f : X \to Y$ is quasicontinuous. If f is outer $\mathcal{J}$-pseudocontinuous at a point $x_0 \in X$, then f is continuous at x_0.*

Proof. Let U be an open neighborhood of $f(x_0)$. Take an open neighborhood V of $f(x_0)$ with $\mathrm{cl}\,(V) \subseteq U$. There exist an open neighborhood W of x_0 and a set $A \in \mathcal{J}$ such that $W \backslash A \subseteq f^{-1}(V)$. Suppose that $W \backslash f^{-1}(U) \neq \varnothing$. It follows from the quasicontinuity of f that there exists a nonempty open set $G \subseteq W \cap f^{-1}[Y \backslash \mathrm{cl}\,(V)]$. Then $G \subseteq A$, which is impossible. Therefore, $W \subseteq f^{-1}(U)$, which completes the proof. □

The requirement that f be quasicontinuous cannot be weakened to the quasicontinuity of f at x_0 in the above theorem since the characteristic function $f : R \to R$ of the set $\{1/n : n \in \mathbf{N}\}$ is quasicontinuous at 0 and outer ω-pseudocontinuous at 0 but not continuous at 0.

The assumption of regularity cannot be omitted in Theorem 13 since if Y denotes the set of reals with the topology $\mathcal{T} = \{U \subseteq Y$: there exist usual open sets $G, H \subseteq R$ with $G \subseteq U \subseteq H$ and $|H \backslash G| \leq \omega\}$, then the identity mapping from R onto Y is quasicontinuous and outer ω-pseudocontinuous but discontinuous at each point of R.

Theorem 14. *Suppose that $\mathcal{J}$ does not contain any nonempty open subset of X. If $f : X \to Y$ is inner $\mathcal{J}$-pseudocontinuous at a point $x_0 \in X$ and outer $\mathcal{J}$-pseudocontinuous at x_0, then f is quasicontinuous at x_0.*

Proof. Let V and W be open neighborhoods of $f(x_0)$ and x_0, respectively. It follows from 10(ii) that there exist subsets G, H and U of X such that G, H are open in X, $G \subseteq U \subseteq H$, $H \backslash G \in \mathcal{J}$, $x_0 \in U$ and $f(U) \subseteq V$. If $W \cap G$ were empty, then $H \cap W$ would be a nonempty open set belonging to $\mathcal{J}$. Hence, $W \cap G \neq \varnothing$, which implies the quasicontinuity of f at x_0. □

Corollary 15. *Suppose that $\mathcal{J}$ does not contain any nonempty open subset of X. If $f : X \overset{\text{onto}}{\to} Y$ is both inner and outer $\mathcal{J}$-pseudocontinuous at each point, then $d(Y) \leq d(X)$ and $c(Y) \leq c(X)$.*

Recall that a function is *somewhat-continuous* if whenever the inverse image of an open set is nonempty, it contains a nonempty open set (cf. [**8**]).

Without any difficulties, one can show that if an inner $\mathcal{J}$-pseudocontinuous mapping $f : X \to Y$ is such that $f^{-1}(V) \notin \mathcal{J}$ for any open subset V of Y with $V \cap f(X) \neq \varnothing$, then f is somewhat continuous (cf. [**8**]); therefore $d(Y) \leq d(X)$ and $c(Y) \leq c(X)$ whenever, in addition, $f(X) = Y$. Clearly, an outer ω-pseudocontinuous image of a second countable space need not be separable (cf. [**5**]).

Theorem 16. *Suppose that $\mathcal{J}$ contains no nonvoid open subset of X. If $f : X \overset{\text{onto}}{\to} Y$ is a weakly $\mathcal{J}$-pseudocontinuous mapping such that $f^{-1}(V) \notin \mathcal{J}$ for any nonempty open subset V of Y, then $c(Y) \leq c(X)$.*

Proof. Let $\{V_s : s \in S\}$ be a collection of pairwise disjoint nonempty open subsets of Y. For each $s \in S$, there exist $A_s, B_s \in \mathcal{J}$ such that the set $U_s = [f^{-1}(V_s) \backslash A_s] \cup B_s$ is open in X. As $U_s \cap U_t \subseteq B_s \cup B_t \in \mathcal{J}$ for $s \neq t$, $s, t \in S$, the sets U_s, $s \in S$, are pairwise disjoint. Thus, there exists a set $T \subseteq S$ such that $|T| \leq c(X)$ and, for any $s \in S \backslash T$, we have $U_s = \varnothing$. If $s \in S \backslash T$, then $f^{-1}(V_s) \subseteq A_s$, which is impossible. Hence, $T = S$ and $c(Y) \leq c(X)$. □

Recall that X is said to be weakly $\varkappa$-Lindelöf if, for any open cover $\{V_s : s \in S\}$ of X, there exists a set $T \subseteq S$ with $|T| \leq \varkappa$ and $\mathrm{cl}\,(\cup_{s \in T} V_s) = X$ (cf. [**6**; 2.34]).

Theorem 17. *Let X be a weakly $\varkappa$-Lindelöf space such that no nonempty open subset of X is of cardinality $\leq \varkappa$. If $f : X \overset{\text{onto}}{\to} Y$ is outer $\varkappa$-pseudocontinuous at each point of X, then Y is weakly $\varkappa$-Lindelöf.*

Proof. Let $\mathcal{T}$ be the original topology of X. Denote by $\mathcal{T}_\varkappa$ the smallest topology on X which contains all the sets $U\backslash A$ with $U \in \mathcal{T}$ and $|A| \le \varkappa$. In view of 10(i), it suffices to show that the space $(X, \mathcal{T}_\varkappa)$ is weakly $\varkappa$-Lindelöf.

Assume that $X = \cup_{s\in S}(U_s\backslash A_s)$ where $U_s \in \mathcal{T}$ and $|A_s| \le \varkappa$ for $s \in S$. There exists $T \subseteq S$ with $|T| \le \varkappa$ and $\mathrm{cl}_{(X,\mathcal{T})}(\cup_{s\in T}U_s) = X$. Suppose that there exists a nonempty set $U \in \mathcal{T}$ and a set $A \subseteq X$ with $|A| \le \varkappa$ and $U\backslash A \subseteq X\backslash \cup_{s\in T} (U_s\backslash A_s)$. Then $W = U \cap \cup_{s\in T}U_s$ is a nonempty open subset of $(X, \mathcal{T})$. Since $W \subseteq A \cup \cup_{s\in T}A_s$, we have $|W| \le \varkappa$, a contradiction. Hence, the set $\cup_{s\in T}(U_s\backslash A_s)$ is dense in $(X, \mathcal{T}_\varkappa)$, which completes the proof. □

Using similar arguments, one can show the following

Theorem 18. *If $f : X \overset{\text{onto}}{\to} Y$ is outer $\varkappa$-pseudocontinuous at each point of X, then $l(Y) \le l(X) + \varkappa$ and $hl(Y) \le hl(X) + \varkappa$.*

Definitions (cf. [**4**]). Let τ be an infinite cardinal number. A *family* $\mathcal{E}$ of subsets of X with $|\mathcal{E}| \le \tau$ is called:

(19) a *τ-pseudonet* in X if, for each open set $U \subseteq X$ and for $x \in U$, there exists $E \in \mathcal{E}$ such that $x \in E$ and $|E\backslash U| \le \tau$;

(20) a *weak τ-net* in X if, for each open set $U \subseteq X$, there exists a set $A \subseteq U$ such that $|A| \le \tau$ and, for each $x \in U\backslash A$, there exists $E \in \mathcal{E}$ with $x \in E \subseteq U$;

(21) a *weak τ-pseudonet* in X if, for each open set $U \subseteq X$, there exists a set $A \subseteq U$ such that $|A| \le \tau$ and, for each $x \in U\backslash A$, there exists $E \in \mathcal{E}$ with $x \in E$ and $|E\backslash U| \le \tau$.

Following [**4**], let us put

$$pn(X) = \min\{\tau \ge \omega : \text{ there exists a } \tau\text{-pseudonet in } X\},$$

$$wnw(X) = \min\{\tau \ge \omega : \text{ there exists a weak } \tau\text{-net in } X\},$$

$$wpn(X) = \min\{\tau \ge \omega : \text{ there exists a weak } \tau\text{-pseudonet in } X\}.$$

Theorem 22. *If $f : X \overset{\text{onto}}{\to} Y$ is weakly $\varkappa$-pseudocontinuous, then $wpn(Y) \le wpn(X) + \varkappa$.*

Proof. Suppose that $\mathcal{E}$ is a weak τ-pseudonet in X for some $\tau \geq \varkappa$. We shall show that the family $\{f(E) : E \in \mathcal{E}\}$ forms a weak τ-pseudonet in Y.

Let V be an open subset of Y. We can find sets $A, B \subseteq X$ such that $|A \cup B| \leq \varkappa$ and the set $U = [f^{-1}(V) \backslash A] \cup B$ is open in X. There exists $C \subseteq U$ such that $|C| \leq \tau$ and, for any $x \in U \backslash C$, there is $E_x \in \mathcal{E}$ with $|E_x \backslash U| \leq \tau$ and $x \in E_x$. Put $D = f(A \cup C) \cap V$. Of course, $|D| \leq \tau$. If $y \in V \backslash D$ and $x \in f^{-1}(y)$, then $x \in f^{-1}(V) \backslash (A \cup C)$, so there exists $E \in \mathcal{E}$ with $x \in E$ and $|E \backslash U| \leq \tau$. Then $y \in f(E)$ and $f(E) \backslash V \subseteq f(B) \cup f(E \backslash U)$, so $|f(E) \backslash V| \leq \tau$. □

Arguing similarly as in the above theorem, we can prove Theorems 23 and 24.

Theorem 23. *If* $f : X \overset{\text{onto}}{\to} Y$ *is inner* $\varkappa$*-pseudocontinuous, then* $wnw(Y) \leq wnw(X) + \varkappa$.

Theorem 24. *If* $f : X \overset{\text{onto}}{\to} Y$ *is outer* $\varkappa$*-pseudocontinuous, then* $pn(Y) \leq pn(X) + \varkappa$.

Corollary 25. *If* $f : X \overset{\text{onto}}{\to} Y$ *is outer* $\varkappa$*-pseudocontinuous at each point of* X*, then* $pn(Y) \leq pn(X) + \varkappa$.

Proof. It suffices to observe that, in view of [**4**; 1.5], $hl(X) \leq pn(X)$; hence f is outer $(pn(X) + \varkappa)$-pseudocontinuous by Theorem 7. □

Theorem 26. *A space* Y *has a weak* $\varkappa$*-pseudonet if and only if* Y *is a one-to-one weak* $\varkappa$*-pseudocontinuous image of a space of weight* $\leq \varkappa$.

Proof. Suppose that $\mathcal{E}$ is a weak $\varkappa$-pseudonet in Y. Let $X = Y$ be considered with the smallest topology containing $\mathcal{E}$. Of course, $w(X) \leq \varkappa$. Moreover, for any open subset U of Y, there exists a subfamily $\mathcal{E}^*$ of $\mathcal{E}$ with $|U \Delta \cup \{E : E \in \mathcal{E}^*\}| \leq \varkappa$. Hence, the identity mapping $f : X \to Y$ is weakly $\varkappa$-pseudocontinuous. Theorem 22 completes the proof. □

By slight modifications of the proof to Theorem 26, we obtain Theorems 27 and 28.

Theorem 27. *A space Y has a weak $\varkappa$-net if and only if Y is a one-to-one inner $\varkappa$-pseudocontinuous image of a space of weight $\leq \varkappa$.*

Theorem 28. *A space Y has a $\varkappa$-pseudonet if and only if Y is a one-to-one outer $\varkappa$-pseudocontinuous image of a space of weight $\leq \varkappa$.*

Lemma 29. *Suppose that $\mathcal{J}$ does not contain any nonempty open subset of X. If $f : X \to Y$ is outer $\mathcal{J}$-pseudocontinuous at each point of X and if H is a clopen subset of Y, then the set $f^{-1}(H)$ is clopen in X.*

Proof. Let $x_0 \in f^{-1}(H)$. There exists an open neighborhood U of x_0 with $U \backslash f^{-1}(H) \in \mathcal{J}$. Suppose that $U \backslash f^{-1}(H) \neq \varnothing$. There exist $x_1 \in U \cap f^{-1}(Y \backslash H)$ and an open neighborhood V of x_1, such that $V \backslash f^{-1}(Y \backslash H) \in \mathcal{J}$. Then $V \cap U \in \mathcal{J}$ because $V \cap U \subseteq [V \backslash f^{-1}(Y \backslash H)] \cup [U \backslash f^{-1}(H)]$. The contradiction obtained implies that $U \subseteq f^{-1}(H)$ and, consequently, $f^{-1}(H)$ is open. It follows from the arbitrariness of H that $f^{-1}(Y \backslash H)$ is open, so $f^{-1}(H)$ is clopen. □

With Lemma 29 in hand, we immediately deduce

Theorem 30. *Suppose that X is connected and that $\mathcal{J}$ contains no nonempty open subset of X. If $f : X \to Y$ is outer $\mathcal{J}$-pseudocontinuous at each point of X, then $f(X)$ is connected.*

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Institute of Mathematics, University of Lódz, UL. S. Banacha 22, 90-238 Łódź, Poland