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ALGEBRAIC VECTOR BUNDLES ON THE 2-SPHERE

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Barge and Ojanguren [**1**] have recently shown that there is a 1–1 correspondence between algebraic and topological vector bundles on the 2-sphere. This raises the problem of whether it is possible to give a purely algebraic classification of these bundles. I will give here an affirmative answer to this question. In particular, this gives a new algebraic proof that the tangent bundle of S^2 is nontrivial. An algebraic proof of this was previously given by Kong [**7**]. The present proof is considerably simpler but applies only to the 2-sphere whereas Kong's method applies to all even dimensional spheres.

Ideally, one would expect such an algebraic proof to apply to all real closed ground fields without the need to appeal to the Tarski principle (cf. [**8**]). While the present proof is purely algebraic, it does not meet this criterion since (in Section 2) it makes use of the fact that the additive and multiplicative groups of the real numbers have Archimedean orderings. I do not know if there is any easy way to avoid this difficulty.

I have included a number of remarks pointing out connections with topological results. These are not essential for the algebraic results presented here. I have also included an exposition of the theory of symplectic modules in an appendix for the convenience of those readers not familiar with this theory.

I do not know if there is a quaternionic analog of these results. See [**14**] for more information on this case.

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1. Known results. If R is a commutative ring we let $P_n(R)$ be the set of isomorphism classes of finitely generated projective modules of

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rank n. Define $P_n(R) \to P_{n+1}(R)$ by sending (P) to $(P \oplus R)$. If R is noetherian, the stability theorems of Bass and Serre (see Theorems A.5 and A.6) show that this map is onto for $n \geq \dim R$ and an isomorphism for $n > \dim R$, so that $P_n(R) \to \tilde{K}_0(R)$ is an isomorphisms for $n > \dim R$. Therefore, if R is a noetherian domain of dimension 2, it is enough to give $P_1(R) = \operatorname{Pic} R$, $\tilde{K}_0(R)$, and $P_2(R)$ in order to classify all finitely generated projective R-modules. If R is also an affine domain over $\mathbf{C}$, $P_n(R) \overset{\approx}{\to} \tilde{K}_0(R)$ for $n \geq 2$ by [**10**, Theorem 1] so it is enough to give $\operatorname{Pic} R$ and $\tilde{K}_0(R)$.

Now let $A = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, and let $B = \mathbf{C} \otimes_{\mathbf{R}} A$. By results of Claborn, Fossum, and Murthy [**13**, Part II] we have $\operatorname{Pic} A = 0$, $\tilde{K}_0(A) = \mathbf{Z}/2\mathbf{Z}$, and $\operatorname{Pic} B = \tilde{K}_0(B) = \mathbf{Z}$ where $\operatorname{Pic} B$ is generated by the invertible ideal $\mathfrak{p} = (x + iy, z - 1)$ of B (see Section 3). This ideal, considered as an A-module, also generates $\tilde{K}_0(A)$. The sequence

$$P_0(R) \to P_1(R) \to P_2(R) \to P_3(R) \to \cdots \to \tilde{K}_0(R)$$

thus takes the form

$$0 \to \mathbf{Z} \overset{=}{\to} \mathbf{Z} \overset{=}{\to} \cdots \overset{=}{\to} \mathbf{Z}$$

for $R = B$ and the form

$$0 \to 0 \to P_2(A) \twoheadrightarrow \mathbf{Z}/2\mathbf{Z} \overset{=}{\to} \cdots \overset{=}{\to} \mathbf{Z}/2\mathbf{Z}$$

for $R = A$. Therefore the following completes the classification of algebraic vector bundles on S^2.

Theorem 1.1 [**1**]. *$P_2(A) = \{A^2, \mathfrak{p}, \mathfrak{p}^2, \mathfrak{p}^3, \dots\}$, the given elements being distinct.*

In [**1**] it is shown by algebraic methods that $P_2(A) = \{A^2, \mathfrak{p}, \mathfrak{p}^2, \mathfrak{p}^3, \dots\}$. The distinctness follows by topological methods. The main result of the present paper will be an algebraic proof of the following.

Theorem 1.2. *$\mathfrak{p}^m \approx \mathfrak{p}^n$ as A-modules if and only if $m = \pm n$.*

This will be proved in Section 5 by reducing it to the following weaker result.

Theorem 1.3. *If $n \neq 0$, then $\mathfrak{p}^n$ is not free as an A-module.*

As a consequence of this we get an algebraic proof of the nontriviality of the tangent bundle of S^2.

Corollary 1.4. *Let T be the projective A-module defined by the unimodular row (x, y, z). Then T is not free.*

This follows from the well-known fact that $T \approx \mathfrak{p}^2$ (see Lemma 4.9).

Following a suggestion of the referee, I will begin with a proof of Theorem 1.3. This will make the proof of Corollary 1.4 more accessible to those not familiar with the results and methods of [**1**]. I have also included in Section 6 a brief account of the results needed from [**1**] to complete the proof of Theorem 1.1. In Section 7 I will show that the above results also hold for the localized ring A_S where S is the set of elements of A with no zeros on S^2.

2. Rouché's theorem. Let $f(t) \in \mathbf{C}(t)$ be a rational function of t. Write $f(t) = a\prod(t-\alpha_i)\prod(t-\beta_j)^{-1}$. If f has no zeros or poles on $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$, we let $\delta(f) = N - P$ where N is the number of f with $|\alpha_i| < 1$ and P is the number of j with $|\beta_j| < 1$.

Remark. $\delta(f)$ is just the degree of the map $f : S^1 \to \mathbf{C}^*$.

The following special case of Rouché's theorem can be proved in a purely algebraic way.

Theorem 2.1. *Let $f(s,t) \in \mathbf{C}(s,t)$ be a rational function of 2 variables. For any $a \in \mathbf{C}$ write $f_a(t) = f(a,t)$. Suppose $f(s,t)$ is defined, finite, and nonzero on the set $I \times S^1 = \{(s,t) \in \mathbf{R} \times \mathbf{C} \mid 0 \leq s \leq 1, |t| = 1\}$. Then $\delta(f_0) = \delta(f_1)$.*

Remark. This follows immediately from Cauchy's theorem since

$$\delta(f_a) = (2\pi i)^{-1} \int f_a'/f_a \, dt$$

is an integer and varies continuously with a.

The usual version of Rouché's theorem (for rational functions on the unit disc) follows immediately: If $f, g \in \mathbf{C}(t)$ are defined on S^1 and $|g(t)| < |f(t)|$ on S^1, then $\delta(f+g) = \delta(f)$. We need only consider $f(t) + sg(t)$.

The proof of Theorem 2.1 will be an algebraic version of the analytic proof described above. Define $S_n(f) = n^{-1}\Sigma f(\zeta)$ with the sum taken over all n-th roots of unity.

Lemma 2.2. *Let $f \in \mathbf{C}(t)$ have no zeros or poles on S^1. Let $\varepsilon > 0$ be given. Then there is an integer N such that $|S_n(tf'/f) - \delta(f)| < \varepsilon$ for all $n \geq N$.*

Proof. Let $f(t) = a\prod(t-\alpha_i)\prod(t-\beta_j)^{-1}$. Then $f'/f = \Sigma(t-\alpha_i)^{-1} - \Sigma(t-\beta_j)^{-1}$. Therefore it will suffice to show that, for large n, $S_n(t(t-\alpha)^{-1})$ is arbitrarily close to 0 if $|\alpha| < 1$ and arbitrarily close to 1 if $|\alpha| > 1$. But $(1-\alpha^n)S_n(t(t-\alpha)^{-1}) = n^{-1}\Sigma(\zeta(\zeta^n - \alpha^n)/(\zeta-\alpha)) = n^{-1}\Sigma(\zeta^n + \alpha\zeta^{n-1} + \cdots + \zeta\alpha^{n-1}) = 1$. Therefore $S_n(t(t-\alpha)^{-1}) = (1-\alpha^n)^{-1}$ which clearly has the required properties. □

Remark. We are using here the fact that the ordering of the multiplicative group $\mathbf{R}^*$ is Archimedean so that $|\alpha|^n$ becomes arbitrarily large if $|\alpha| > 1$ and arbitrarily small if $|\alpha| < 1$.

Lemma 2.3. *Let $g(s,t) \in \mathbf{C}(s,t)$ be a rational function which is defined and finite on $I \times S^1$. Then there is a constant $M \in \mathbf{R}$ such that $|g(r,t) - g(s,t)| \leq M|r-s|$ for all $r, s \in I$ and $t \in S^1$.*

Proof. We can write $g(r,t) - g(s,t) = h(r,s,t)(r-s)$ where $h(r,s,t) \in \mathbf{C}(r,s,t)$ is finite on $I \times I \times S^1$. Write $h(r,s,t) = p(r,s,t)/q(r,s,t)$ where p and q are polynomials with no common factor. Then q is never 0 on $I \times I \times S^1$ so there is a constant $\eta > 0$ such that $|q(r,s,t)| \geq \eta$ for $(r,s,t) \in I \times I \times S^1$. For a purely algebraic proof of this, see [**4**, 9.2] (see also the remark following Corollary 2.4). Since $I \times I \times S^1$ is bounded, we can find a constant C with $|p(r,s,t)| \leq C$ on $I \times I \times S^1$.

Therefore, we can take $M = C\eta^{-1}$. $\square$

Proof of Theorem 2.1. Lemma 2.3 applies to $g(s,t) = tf^{-1}\partial f/\partial t$. Let $\varepsilon > 0$. Since the ordering of the additive group $\mathbf{R}$ is Archimedean, we can find an integer n such that $0 \le r \le s \le 1$ and $|s-r| \le n^{-1}$ imply $|g(r,t) - g(s,t)| \le \varepsilon$ for all t in S^1. Fix such r and s, and choose N by Lemma 2.2 so that $|S_n(g_a) - \delta(f_a)| < \varepsilon$ for $a = r$ or s, and all $n \ge N$. Then $|\delta(f_r) - \delta(f_s)| \le 2\varepsilon + |S_n(g_r - g_s)| \le 3\varepsilon < 1$ if we choose $\varepsilon < 1/3$. Since $\delta(f_r)$ and $\delta(f_s)$ are integers, this implies that $\delta(f_r) = \delta(f_s)$. Therefore, $\delta(f_0) = \delta(f_{1/n}) = \delta(f_{2/n}) = \delta(f_{n/n}) = \delta(f_1)$. $\square$

In the applications given in this paper, we will only need the following special case of Theorem 2.1.

Corollary 2.4. *Let $f(x,y) \in \mathbf{C}[x,y]$ be a polynomial in two variables. Assume that f divides a polynomial of the form $1+\Sigma g_i(x,y)^2$ where the $g_i(x,y) \in \mathbf{R}[x,y]$. Let $t = x+iy$ and $w = x - iy$ and write $f(x,y) = h(t,w)$ with respect to these new variables. Then $\varphi(t) = h(t,t^{-1}) \in \mathbf{C}(t)$ has $\delta(\varphi) = 0$.*

Remark. Note that $h(t,w) = f((t+w)/2, (t-w)/2i)$. In the analytic case we could put $t = \exp(i\theta)$ getting $\varphi(\exp(i\theta)) = f(\cos\theta, \sin\theta)$. Thus the substitution used in Corollary 2.4 is just an algebraic version of the classical substitution $x = \cos\theta$, $y = \sin\theta$ used to integrate a function along the unit circle.

Proof. We have $h(t,w)H(t,w) = 1 + \Sigma g_i(x,y)^2$ for some H. If $t \in \mathbf{C}$ and $w = \bar{t}$, the corresponding values of x and y are real, so $h(t,\bar{t}) \neq 0$ for all t in $\mathbf{C}$. Define $\psi(s,t) = h(st, st^{-1})$. If $s \in \mathbf{R}$ and $|t| = 1$, then $st^{-1} = s\bar{t}$ so $\psi(s,t) = h(st, s\bar{t}) \neq 0$. Therefore, by Theorem 2.1, $\delta(\varphi) = \delta(\psi_1) = \delta(\psi_0) = 0$ since ψ_0 is a nonzero constant. $\square$

Remark. For Corollary 2.4 we can avoid the use of [**4**, 9.2] in the proof of Theorem 2.1. We must apply Lemma 2.3 to $g(s,t) = t\psi^{-1}\partial\psi/\partial t$ so, in the proof of Lemma 2.3, we can take $q(r,s,t) = t^m\psi(r,t)\psi(s,t)$ for some m. Now, for $(s,t) \in I \times S^1$, we have $\psi(s,t)H(st,st^{-1}) = 1 + \Sigma g_i(x,y)^2 \ge 1$ since x and y are real. If $|H(st,st^{-1})| \le D$ for $(s,t) \in I \times S^1$, we have $|\psi(s,t)| \ge D^{-1}$ for $(s,t) \in I \times S^1$ and we can choose $\eta = D^{-2}$ in the proof of Lemma 2.3.

3. The Picard group. Let $A = \mathbf{R}[x,y,z]/(x^2+y^2+z^2-1)$, and let $B = \mathbf{C}\otimes_{\mathbf{R}} A$. It is well known [**13**, Section 9] that $\operatorname{Pic} B = \mathbf{Z}$ generated by the invertible ideal $\mathfrak{p} = (x+iy, z-1)$. It will give another proof of this here which will then be modified to give the main result of this section. Let $u = 1-z$ and $v = 1+z$. Since $A = Au + Av$, we have localization squares

$$\begin{array}{ccc} A & \longrightarrow & A_u \\ \downarrow & & \downarrow \\ A_v & \longrightarrow & A_{uv} \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & B_u \\ \downarrow & & \downarrow \\ B_v & \longrightarrow & B_{uv} \end{array}$$

The following is an observation of Murthy. It is essentially an algebraic version of stereographic projection.

Lemma 3.1. *$A_u = \mathbf{R}[\xi,\eta]_{1+\xi^2+\eta^2}$ where $\xi = x/u$ and $\eta = y/u$.*

Proof. We have $A_u = \mathbf{R}[x,y,u,u^{-1}]/(x^2+y^2+(1-u)^2-1)$. The relation is equivalent to $x^2+y^2+u^2 = 2u$. Therefore $u^{-1} = (1/2)(1+\xi^2+\eta^2)$.

Note that $x = 2\xi(1+\xi^2+\eta^2)^{-1}$, $y = 2\eta(1+\xi^2+\eta^2)^{-1}$, and $z = (\xi^2+\eta^2-1)(1+\xi^2+\eta^2)^{-1}$.

Similarly, $A_v = \mathbf{R}[\xi',\eta']_{1+\xi'^2+\eta'^2}$ where $\xi' = x/v$ and $\eta' = y/v$. The image of $1+\xi'^2+\eta'^2$ in A_{uv} is $2(\xi^2+\eta^2)(1+\xi^2+\eta^2)^{-1}$ so $A_{uv} = \mathbf{R}[\xi,\eta]_{(1+\xi^2+\eta^2)(\xi^2+\eta^2)}$.

The same results hold for B_u, B_v, and B_{uv} with $\mathbf{R}$ replaced by $\mathbf{C}$. $\square$

We recall the following standard fact for the reader's convenience.

Lemma 3.2 [**2**, Chapter IX, Theorem 6.8].*If R is a regular commutative ring and S is a multiplicative set, then the natural map $\operatorname{Pic} R \to \operatorname{Pic} R_S$ is onto.*

It follows that $\operatorname{Pic} B_u = \operatorname{Pic} B_v = 0$ so the Mayer-Vietoris sequence for $\operatorname{Pic} B$ reduces to

$$U(B_u) \oplus U(B_v) \to U(B_{uv}) \xrightarrow{\partial} \operatorname{Pic} B \to 0.$$

Now $U(B_u) = \mathbf{C}^* \times \mathbf{Z}$ where the $\mathbf{Z}$ term is generated by $u = 2(1 + \xi^2 + \eta^2)^{-1}$. Similarly, $U(B_v) = \mathbf{C}^* \times \mathbf{Z}$ generated by v and $U(B_{uv}) = \mathbf{C}^* \times \mathbf{Z}^3$ where the three $\mathbf{Z}$ factors are generated by $\xi + i\eta$, $\xi - i\eta$, and $1 + \xi^2 + \eta^2$. Since $v = u(\xi^2 + \eta^2)$, it follows that $\operatorname{Pic} B = \mathbf{Z}$ generated by $\partial(\xi + i\eta)$.

The same argument applied to A shows that $\operatorname{Pic} A = 0$ since $\xi^2 + \eta^2$ is irreducible over $\mathbf{R}$.

Remark. It is easy to see that $\partial(\xi + i\eta) = \mathfrak{p} = (x + iy, 1 - z)$. In fact, $\mathfrak{p}_u = B_u$ since $u \in \mathfrak{p}$ and $\mathfrak{p}_v = B_v(x + iy)$ since $(x + iy)(x - iy) = uv$ and $u = 1 - z$. Therefore, we obtain $\mathfrak{p}$ by patching B_u and B_u via the isomorphism $(x + iy) : B_{uv} \approx B_{uv}$ so $\partial(x + iy) = \mathfrak{p}$. Since $\partial(u) = 0$ and $(\xi + i\eta)u = x + iy$, we have $\partial(\xi + i\eta) = \mathfrak{p}$.

Now let S be the multiplicative subset of A consisting of all elements of the form $1 + f_1^2 + f_2^2 + \cdots + f_n^2$. The following is the main result of this section.

Theorem 3.3. $\operatorname{Pic} B \to \operatorname{Pic} B_S$ *is an isomorphism.*

Remark. This would follow immediately from topological considerations since $B \subset B_S \subset C(S^2)$ gives $\operatorname{Pic} B \to \operatorname{Pic} B_S \to \operatorname{Pic} C(S^2) = H^2(S^2, \mathbf{Z})$ and the composition is known to be an isomorphism.

Remark. If $T = \{f \in A \mid f \text{ is never } 0 \text{ on } S^2\}$ then $A_S = A_T$ and $B_S = B_T$ by [**16**, Theorem 10.1]. This fact will not be needed here.

To prove Theorem 3.3, we consider the natural map of localization squares

$$\begin{array}{ccc} B & \longrightarrow & B_u \\ \downarrow & & \downarrow \\ B_v & \longrightarrow & B_{uv} \end{array} \quad \to \quad \begin{array}{ccc} B_S & \longrightarrow & B_{Su} \\ \downarrow & & \downarrow \\ B_{Sv} & \longrightarrow & B_{Suv} \end{array}$$

which gives us a commutative diagram

$$\begin{array}{ccccccc}
U(B_u) \oplus U(B_v) & \longrightarrow & U(B_{uv}) & \xrightarrow{\partial} & \operatorname{Pic} B & \longrightarrow & 0 \\
\downarrow \quad \downarrow & & \downarrow & & \downarrow & & \\
U(B_{Su}) \oplus U(B_{Sv}) & \longrightarrow & U(B_{Suv}) & \xrightarrow{\partial} & \operatorname{Pic} B_S & \longrightarrow & 0.
\end{array}$$

The element $\xi + i\eta$ of $U(B_{uv})$ maps to a generator of $\operatorname{Pic} B = \mathbf{Z}$. Since $\operatorname{Pic} B \to \operatorname{Pic} B_S$ is onto by Lemma 3.2, it will suffice to find a map $U(B_{Suv}) \to \mathbf{Z}$ sending $\xi + i\eta$ to 1, and annihilating the images of $U(B_{Su})$ and $U(B_{Sv})$.

Let $C = A/(z) = \mathbf{R}[x, y]/(x^2 + y^2 - 1)$. The natural map $A \to C$ extends to $A_{uv} \to C$. This sends u and v to 1, ξ and ξ' to x, and η and η' to y. It sends S to $T = \{1 + g_1^2 + g_2^2 + \cdots + g_n^2 \mid g_i \in C\}$. Let $D = \mathbf{C} \otimes_{\mathbf{R}} C = \mathbf{C}[t, t^{-1}]$ where $t = x + iy$. The elements of T are never 0 on $S^1 = \{t \in \mathbf{C} \mid |t| = 1\} = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ so the map δ of Section 2 defines a homomorphism $\delta : U(D_T) \to \mathbf{Z}$. The composition $U(B_{Suv}) \to U(D_T) \to \mathbf{Z}$ sends $\xi + i\eta$ to 1, so it is enough to check that it annihilates the images of $U(B_{Su})$ and $U(B_{Sv})$.

Let $V = \{1 + g_1^2 + g_2^2 + \cdots + g_n^2 \mid g_i \in \mathbf{R}[\xi, \eta]\}$ with the notation of Lemma 3.1. The image of S in A_u consists of elements of the form $w = 1 + h_1(x, y, z)^2 + h_2(x, y, z)^2 + \cdots + h_n(x, y, z)^2$. Recalling that $x = 2\xi(1 + \xi^2 + \eta^2)^{-1}$, $y = 2\eta(1 + \xi^2 + \eta^2)^{-1}$, and $z = (\xi^2 + \eta^2 - 1)(1 + \xi^2 + \eta^2)^{-1}$, we see that w can be written in the form $w = (1+\xi^2+\eta^2)^{-N}\{(1+\xi^2+\eta^2)^N + g_1(\xi, \eta)^2 + g_2(\xi, \eta)^2 + \cdots + g_m(\xi, \eta)^2\}$ and so lies in the multiplicative group generated by V. Therefore $\mathbf{C}[\xi, \eta]_V$ is a localization of B_{Su}. Since V clearly maps to T in D, it will suffice to show that the composition $U(\mathbf{C}[\xi, \eta]_V) \to U(D_T) \to \mathbf{Z}$ is 0 and similarly for the other map $U(\mathbf{C}[\xi', \eta']_{V'}) \to U(D_T) \to \mathbf{Z}$ corresponding to $U(B_v)$. In fact, these two maps are the same since ξ and ξ' both map to x, and η and η' both map to y in D.

An element of $U(\mathbf{C}[\xi, \eta]_V)$ has the form f/g where $g \in V$ and f divides an element of V. We must therefore show that a polynomial f dividing an element of V maps to an element φ of D_T such that $\delta(\varphi) = 0$. If we write $f(\xi, \eta) = h(\xi + i\eta, \xi - i\eta)$, then $\varphi(t) = h(t, t^{-1})$ so the required result follows from Corollary 2.4.

Corollary 3.4. $\mathfrak{p}_S^n$ *is not principal for* $n \neq 0$.

Remark. Theorem 3.3 is equivalent to the following elementary fact about the ring A: If $f, g \in A$ and $f^2 + g^2 = (1-z)^n t$ with $n > 0$, where $t \in A$ divides an element of S, then $1 - z$ divides f and g. To see this, observe that, since $(1-z) = \mathfrak{p}\bar{\mathfrak{p}}$, $(f + ig)B = \mathfrak{p}^a \bar{\mathfrak{p}}^b \mathfrak{a}$ where $\mathfrak{a}_S = B_S$. Since $(f - ig)B = \bar{\mathfrak{p}}^a \mathfrak{p}^b \bar{\mathfrak{a}}$, we see that $n = a + b$. Now $\mathfrak{p}^{a-b}$ becomes principal in B_S so $a = b \neq 0$ and $\mathfrak{p}\bar{\mathfrak{p}} = (1-z)$ divides $f + ig$. Conversely, if $\mathfrak{p}_S^n = (f + ig)B_S$, then $(f^2 + g^2)B_S = (1-z)^n B_S$ so $1 - z$ divides f and g giving the contradiction that $\bar{\mathfrak{p}}_S$, which divides $1 - z$, divides $\mathfrak{p}_S^n$.

4. The main theorem. Let A be a commutative R-algebra which is a domain. We begin by studying the endomorphism ring over A of an invertible ideal I of $B = \mathbf{C} \otimes_{\mathbf{R}} A$. If $f \in \operatorname{End}_A(I)$, we can write f uniquely as $f = g + h$ where g is B-linear and h is B-antilinear. For such a decomposition, we necessarily have $g(x) = (1/2)\{f(x) - if(ix)\}$ and $h(x) = (1/2)\{f(x) + if(ix)\}$ and it is easily checked that the functions g and h so defined have the required properties. Since I is an invertible ideal of B, $\operatorname{End}_B(I) = B$ and we can write $g(x) = \alpha x$ for some $\alpha \in B$. We can also factor h as $I \to \bar{I} \xrightarrow{h'} I$ where the first map is complex conjugation and h' is B-linear. Therefore $h' \in \operatorname{Hom}_B(\bar{I}, I) \approx \bar{I}^{-1} I$ and we can write $h'(x) = \gamma x$ so that $h(x) = \gamma \bar{x}$. The following lemma summarizes this discussion.

Lemma 4.1. *If I is an invertible ideal of B, there is an isomorphism* $B \oplus \bar{I}^{-1} I \xrightarrow{\approx} \operatorname{End}_A(I)$ *sending* (α, γ) *to* f *where* $f(x) = \alpha x + \gamma \bar{x}$.

If P is a finitely generated projective module and f is an endomorphism of P one can define the determinant of f to be that of $f \oplus 1_Q$ where $P \oplus Q$ is free [**6**]. Similarly, the trace of f is that of $f \oplus O_Q$. These definitions are easily checked to be independent of the choice of Q.

Lemma 4.2. *In Lemma* 4.1 *let* $\alpha = a + ib$ *and* $\gamma = c + id$ *with* $a, b, c, d \in A$. *Then* $\det(f) = a^2 + b^2 - c^2 - d^2$ *and* $\operatorname{tr}(f) = 2a$.

Proof. It is sufficient to calculate $\det(f)$ and $\operatorname{tr}(f)$ after tensoring with the quotient field K of A. Note that $K\otimes_A B = \mathbf{C}\otimes_{\mathbf{R}} K = K\otimes Ki$. The endomorphism of this induced by f is still given by the formula $f(x) = \alpha x + \gamma\bar{x}$. With respect to the base $1, i$ over K, this is given by the matrix

$$\begin{pmatrix} a+c & d-b \\ d+b & a-c \end{pmatrix}$$

which clearly has the stated properties. □

Lemma 4.3. *Let I be an invertible ideal of B. Suppose some $f \in \operatorname{End}_A(I)$ has $\det f = -1$. Then there are elements a and b of A such that $(\bar{I}^{-1}I)_{1+a^2+b^2}$ is principal over $B_{1+a^2+b^2}$.*

Proof. With the notation of Lemma 4.2 we have $a^2+b^2-c^2-d^2 = -1$ so that $c^2 + d^2 = 1 + a^2 + b^2$. Now $\gamma = c + id \in J = \bar{I}^{-1}I$ and $\bar{\gamma} = c - id \in \bar{J} = I^{-1}\bar{I} = J^{-1}$. Let $A' = A_{1+a^2+b^2}$, $B' = B_{1+a^2+b^2}$, $J' = J_{1+a^2+b^2}$, and apply the following lemma. □

Lemma 4.4. *Let J' be an invertible ideal of B' with $\bar{J}' = J'^{-1}$. If there is an element γ of J' with $\gamma\bar{\gamma} = u \in B'^*$, then $J' = B'\gamma$.*

Proof. Clearly $J' \supset B'\gamma$. Conversely, if $x \in J'$, then $x\bar{\gamma} \in J'\bar{J}' = J'J'^{-1} = B'$. Therefore $y = u^{-1}x\bar{\gamma} \in B'$ and $x = y\gamma$. □

Corollary 4.5. *If an invertible ideal I of B is decomposable as an A-module, then there are elements a and b of A such that $(\bar{I}^{-1}I)_{1+a^2+b^2}$ is principal over $B_{1+a^2+b^2}$.*

Proof. Suppose $I = P \oplus Q$ over A with P and Q nonzero. Then P and Q have rank 1. The endomorphism $f = 1_p \oplus (-1)_Q$ of $P \oplus Q$ has $\det(f) = -1$. □

Remark. Since this f has $\operatorname{tr}(f) = 0$, we have $a = 0$ so there is even an element b of A such that $(\bar{I}^{-1}I)_{1+b^2}$ is principal over B_{1+b^2}.

Lemma 4.6. *Let $S = \{1 + f_1^2 + f_2^2 + \cdots + f_n^2 \mid f_i \in A, n \geq 0\}$. Then* *in A_S, all elements of the form $1 + g_1^2 + g_2^2 + \cdots + g_m^2$, $g_i \in A_S$ are units.*

Proof. Write $g_i = h_i/s$ where $s = 1 + f_1^2 + f_2^2 + \cdots + f_n^2$ is in S. Then $1 + g_1^2 + g_2^2 + \cdots + g_m^2 = s^{-2}t$ where $t = s^2 + h_1^2 + h_2^2 + \cdots + h_m^2$. By multiplying out s^2 we see that $t \in S$. □

We can now give an algebraic proof of Theorem 1.1. Here $A = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and $\mathfrak{p} = (x + iy, z - 1)$ in $B = \mathbf{C} \otimes_{\mathbf{R}} A$ as in Sections 1 and 3. As in Section 3, let $S = \{1 + f_1^2 + f_2^2 + \cdots + f_n^2 \mid f_i \in A, n \geq 0\}$.

Theorem 4.7. *If $n \neq 0$, there is no automorphism f of $\mathfrak{p}_S^n$ over A_S with* $\det(f) = -1$. *The same is therefore true for $\mathfrak{p}^n$ over A.*

Proof. If there is such an f, then $(\bar{\mathfrak{p}}_S^n)^{-1}\mathfrak{p}_S^n$ becomes principal over B_S by Lemmas 4.3 and 4.6. Since $\bar{\mathfrak{p}} \approx \mathfrak{p}^{-1}$, $(\bar{\mathfrak{p}}_S^n)^{-1}\mathfrak{p}_S^n \approx \mathfrak{p}_S^{2n}$ which, by Corollary 3.4, is not principal unless $n = 0$.

Corollary 4.8. *The modules $\mathfrak{p}_S^n$ are not free over A_S for $n \neq 0$.*

It follows that $\mathfrak{p}^n$ is not free over A for $n \neq 0$ which proves Theorem 1.3.

Recall that the tangent bundle to S^2 corresponds to the projective A-module T defined by the unimodular row (x, y, z). The following is well known from topology.

Lemma 4.9. $T \approx \mathfrak{p}^2$.

Proof. The module T has 3 generators α, β, γ with the relation $x\alpha + y\beta + z\gamma = 0$. Set $u = x + iy$ and define $f : T \to \mathfrak{p}^2$ by $f(\alpha) = (1/2)\{u^2 - (z-1)^2\}$, $f(\beta) = (1/2i)\{u^2 + (z-1)^2\}$, and $f(\gamma) = u(z-1)$. Define a complex structure on T by setting $i\alpha = y\gamma - z\beta$, $i\beta = z\alpha - x\gamma$, $i\gamma = x\beta - y\alpha$. This is the usual complex structure on the tangent bundle of S^2 obtained by identifying S^2 with the complex projective line. It is easy to check that f is a map of

$B = \mathbf{C} \otimes_{\mathbf{R}} A$-modules, from which it follows by inspection that f is onto. Since T and $\mathfrak{p}^2$ are B-modules of rank 1, f must be an isomorphism. □

Corollary 4.10. *T is not free.*

Remark. Murthy has given a very simple algebraic proof that the unimodular row x, y, z is not completable over the ring $\mathbf{R}[x, y, z]_s$ where $s = (x^2 + y^2 + z^2)(1 + x^2 + y^2 + z^2)$. However, there seems to be no easy way to derive Corollary 4.10 from this.

5. Symplectic modules. The proof of Theorem 1.2 will make use of symplectic methods. For the reader's convenience, I have included an exposition of the required results in the appendix to this paper. Any unfamiliar notation or terminology used here can be found there.

Lemma 5.1. *Let A be an $\mathbf{R}$-algebra, and let $B = \mathbf{C} \otimes_{\mathbf{R}} A$. If M is a B-module, then $\Im : \mathrm{Hom}_B(M, B) \overset{\approx}{\to} \mathrm{Hom}_A(M, A)$.*

If $\varphi \in \mathrm{Hom}_B(M, B)$, write $\varphi(x) = f(x) + ig(x)$ with $f(x), g(x) \in A$. The map is given by $\Im(\varphi) = g$.

Proof. Since $\varphi(ix) = i\varphi(x)$, we see that $f(x) = g(ix)$. The inverse of our map is then easily seen to be given by $g \mapsto \psi$ with $\psi(x) = g(ix) + ig(x)$. □

We now consider the case $A = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and let $\mathfrak{p} = (x + iy, 1 - z) \subset B = \mathbf{C} \otimes_{\mathbf{R}} A$ as in Sections 1, 3, and 4. Let $\bar{\mathfrak{p}} = (x - iy, 1 - z)$ be its complex conjugate. Then $\mathfrak{p}\bar{\mathfrak{p}} = (1 - z)$ so $\mathfrak{p}^n\bar{\mathfrak{p}}^n = (1 - z)^n$. Since $\mathrm{Hom}_B(\mathfrak{p}^n, B) = \mathfrak{p}^{-n} = (1 - z)^{-n}\bar{\mathfrak{p}}^n$, we can define an A-isomorphism $\mathfrak{p}^n \overset{\approx}{\to} \mathrm{Hom}_A(\mathfrak{p}^n, A) = \mathrm{Hom}_B(\mathfrak{p}^n, B)$ by sending a to $(1-z)^{-n}\bar{a}$. This in turn defines a nondegenerate A-bilinear form $\langle , \rangle_n$ on $\mathfrak{p}^n$ with $\langle a, b\rangle_n = \Im[(1-z)^{-n}\bar{a}b]$. Clearly, $\langle a, a\rangle_n = 0$ since $a\bar{a}$ lies in A so $\langle , \rangle_n$ defines a symplectic structure on $\mathfrak{p}^n$. Let P_n be $\mathfrak{p}^n$ considered as an A-module with this symplectic structure.

Remark. Note that $\langle ia, ib\rangle_n = \langle a, b\rangle_n$. It is well known [**17**, Chapter 1, No. 2] that giving such a form is equivalent to giving a Hermitian form $(,) : M \times M \to B$ where $(a, b) = \langle ia, b\rangle + i\langle a, b\rangle$. It is easily seen, using Lemma 5.1, that $(,)$ is nondegenerate if and only if $\langle,\rangle$ is.

Since $P_0 = B$ has base $1, i$ as an A-module with $\langle 1, i\rangle_0 = 1$, we see that $P_0 \cong H$, the rank 2 hyperbolic A-module. Also, the A-isomorphism $\mathfrak{p}^n \overset{\approx}{\to} \mathfrak{p}^{-n}$ by $a \mapsto (1-z)^{-n}\bar{a}$ is easily checked to give an isomorphism $P_n^\vee \cong P_{-n}$ where $P_n^\vee = (P_n, -\langle,\rangle_n)$.

Lemma 5.2. *$P_{n+1} \perp P_{n-1} \cong P_n \perp P_n$ as symplectic A-modules for all $n \in \mathbf{Z}$.*

Proof. Let L be the quotient field of B, and let $\theta : L \oplus L \overset{\approx}{\to} L \oplus L$ by $\theta(a, b) = (a, b)M$ where M is the matrix given by

$$M = (1/\sqrt{2}) \begin{pmatrix} x+iy & -1 \\ 1-z & (x-iy)/(1-z) \end{pmatrix}.$$

An easy calculation shows that $\det M = 1$ so the inverse is given by

$$M^{-1} = (1/\sqrt{2}) \begin{pmatrix} (x-iy)/(1-z) & 1 \\ -1+z & x+iy \end{pmatrix}.$$

Since $x+iy,\ 1-z \in \mathfrak{p}$ and $1, (x-iy)/(1-z) \in \mathfrak{p}^{-1}$ we see that $\theta(\mathfrak{p}^n \oplus \mathfrak{p}^n) \subset \mathfrak{p}^{n+1} \oplus \mathfrak{p}^{n-1}$ and $\theta^{-1}(\mathfrak{p}^{n+1} \oplus \mathfrak{p}^{n-1}) \subset \mathfrak{p}^n \oplus \mathfrak{p}^n$. Therefore θ restricts to an isomorphism $\theta : \mathfrak{p}^n \oplus \mathfrak{p}^n \overset{\approx}{\to} \mathfrak{p}^{n+1} \oplus \mathfrak{p}^{n-1}$.

From the formula for M^{-1} we see immediately that $\overline{M}^{-1} = DM^T$ where M^T is the transpose of M and D is the diagonal matrix

$$D = \begin{pmatrix} (1-z)^{-1} & 0 \\ 0 & 1-z \end{pmatrix}.$$

If $(a, b)M = (c, d)$ and $(a', b')M = (c', d')$, then $(\bar{c}, \bar{d})D(c', d')^T = (\bar{a}, \bar{b})\overline{M}DM^T(a', b')^T = (\bar{a}, \bar{b})(a', b')^T = \bar{a}a' + \bar{b}b'$ showing that $\bar{a}a' + \bar{b}b' = \bar{c}c'(1-z)^{-1} + \bar{d}d'(1-z)$. Multiplying this by $(1-z)^{-n}$ and taking imaginary parts gives $\langle a, a'\rangle_n + \langle b, b'\rangle_n = \langle c, c'\rangle_{n+1} + \langle d, d'\rangle_{n-1}$ showing that θ preserves the symplectic structure.

Now let $p_n = [P_n] - [P_0] \in \tilde{K}Sp_0(A)$. Then $p_{n+1} + p_{n-1} = p_n + p_n$ and $p_0 = 0$ so we conclude that $p_n = np_1$ for all n.

Since $\operatorname{Pic} A = 0$, we have an isomorphism $\varphi : \Lambda^2 P \approx A$ for any rank 2 projective A-module P. Therefore we can give P a symplectic structure by taking $\langle p, q\rangle = \varphi(p \wedge q)$. It is easy to check that this is nondegenerate by localizing to make P free. It also follows from the fact that $\Lambda^2 P \cong A$ that if $\langle , \rangle$ is one symplectic form on P, any other one must have the form $\langle , \rangle' = u\langle , \rangle$ where $u \in A^* = \mathbf{R}^*$. If $r \in \mathbf{R}^*$, the isomorphism $P \to P$ sending x to $r^{-1}x$ transforms $\langle , \rangle$ to $r^2\langle , \rangle$. Therefore, there are at most two symplectic structures on P up to isomorphism, namely, $(P, \langle , \rangle)$ and $(P, \langle , \rangle)^\vee = (P, -\langle , \rangle)$. In particular, the two symplectic structures on the A-module $\mathfrak{p}^n$ are P_n and P_{-n}. $\square$

We can now prove Theorem 1.2. If $\mathfrak{p}^m \approx \mathfrak{p}^n$, then $P_m \cong P_n$ or $P_m \cong P_{-n}$. Replacing n by $-n$ if necessary we can assume that $P_m \cong P_n$. Therefore, $p_m = p_n$ in $\tilde{K}Sp_0(A)$ so that $p_{m-n} = p_m - p_n = 0$. By Corollary A.11, it follows that $P_{m-n} \cong P_0 \cong H$ and so $\mathfrak{p}^{m-n}$ is free over A. This contradicts Theorem 1.3.

6. Results of Barge and Ojanguren. I will recall here some results from [**1**] in order to give more explicit versions of certain arguments for the case of S^2. The results we need do not involve the group $W^{lf}(A)$ used in [**1**] and, in particular, we can avoid the use of [**1**, Proposition 2.1] here. As in [**1**] we let $W^-(A)$ be the Witt group of symplectic A-modules. The definition of this is recalled at the end of the appendix.

Remark. The arguments of [**1**] seem to require that $1/2$ lies in A and that all maximal ideals of A have height 2. These conditions are clearly satisfied for two-dimensional affine domains over $\mathbf{R}$.

The following lemma contains the results we need from [**1**].

Lemma 6.1. *Let A be a smooth affine domain over $\mathbf{R}$ of dimension two. Let $\mathcal{P}$ be a set of symplectic A-modules of rank 2 such that for any maximal ideal $\mathfrak{M}$ of A there is an epimorphism $P \twoheadrightarrow \mathfrak{M}$ with $P \in \mathcal{P}$. Then $W^-(A)$ is generated by the classes $[P]$ for $P \in \mathcal{P}$.*

Proof. We refer to [**1**, Section 3, Theorem 3.6] for the definition of $\rho(M,\eta) \in W^-(A)$ where M has finite length and $\eta : M \overset{\approx}{\to} \hat{M} = \operatorname{Ext}^2_A(M,A)$ is symmetric. The argument of [**1**, p. 626, last paragraph] shows that $W^-(A)$ is generated by the elements $\rho(A/\mathfrak{M},\eta)$ where $\mathfrak{M}$ runs over the maximal ideals of A. Let $\mathfrak{M}$ be such an ideal. Find $P \in \mathcal{P}$ with an epimorphism $f : P \twoheadrightarrow \mathfrak{M}$. Since $\operatorname{rk} P = 2$, the alternating form on P gives an isomorphism $\Lambda^2 P \to A$ (easily seen by localizing). Form the Koszul complex

$$0 \to \Lambda^2 P \overset{g}{\to} P \overset{f}{\to} A \to A/\mathfrak{M} \to 0$$

where $g(p \wedge q) = f(p)q - f(q)p$. Since $A_{\mathfrak{M}}$ is regular of dimension 2, this localizes to the usual Koszul complex. As in [**1**, p. 622, first paragraph], we get a resolution

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{g}{\longrightarrow} & P & \overset{f}{\longrightarrow} & A & \longrightarrow & A/\mathfrak{M} & \longrightarrow & 0 \\
 & & \Big\Downarrow & & \Big\downarrow \psi & & \Big\Downarrow & & \Big\downarrow \varphi & & \\
0 & \longrightarrow & A & \overset{-f^*}{\longrightarrow} & P^* & \overset{g^*}{\longrightarrow} & A & \longrightarrow & (A/\mathfrak{M})^\wedge & \longrightarrow & 0
\end{array}$$

of the type considered in [**1**, Theorem 3.3]. This shows that $\rho(A/\mathfrak{M},\varphi) = [P]$ in $W^-(A)$. If $c \in A - \mathfrak{M}$, then $x \mapsto c^{-1}x$ clearly gives an isomorphism $(A/\mathfrak{M},\eta) \approx (A/\mathfrak{M}, c^2\eta)$ as noted in [**1**, Proposition 2.1]. Therefore, if $A/\mathfrak{M} = \mathbf{C}$, $\rho(A/\mathfrak{M},\eta)$ is independent of η, while if $A/\mathfrak{M} = \mathbf{R}$, there are two possible values: $\rho(A/\mathfrak{M},\eta)$ and $\rho(A/\mathfrak{M},-\eta)$. By [**1**, Lemma 3.7], $\rho(A/\mathfrak{M},-\eta) = -\rho(A/\mathfrak{M},\eta)$. Therefore, in either case, $\rho(A/\mathfrak{M},\eta) = \pm\rho(A/\mathfrak{M},\varphi) = \pm[P]$, and the lemma follows. □

Now let $A = \mathbf{R}[x,y,z]/(x^2+y^2+z^2-1)$, and let $B = \mathbf{C} \otimes_{\mathbf{R}} A$.

Lemma 6.2. *If $A/\mathfrak{M} = \mathbf{C}$, there is an epimorphism $A^2 \to \mathfrak{M}$.*

Proof. Let x, y, z map to α, β, γ in $A/\mathfrak{M} = \mathbf{C}$. At least one of these, say α, does not lie in $\mathbf{R}$. Find $r, s \in \mathbf{R}$ with $\beta + r\alpha$ and $\gamma + s\alpha$ in $\mathbf{R}$. Let $f = y + rx - (\beta + r\alpha)$ and $g = z + sx - (\gamma + s\alpha)$. Then $A/(f,g) = \mathbf{R}[x]/(h(x))$ where $\deg h \le 2$. Since $A/(f,g)$ maps onto $A/\mathfrak{M} = \mathbf{C}$, it is clear that $\mathfrak{M} = (f,g)$. □

Let $\mathfrak{p}$ be the invertible ideal $(x+iy, z-1)$ of B. Then $\mathfrak{p}$ generates $\operatorname{Pic} B = \mathbf{Z}$. Now $\mathfrak{p}\bar{\mathfrak{p}} = (z-1)$ so $\mathfrak{p}^{-1} \approx \bar{\mathfrak{p}}$ which is isomorphic to $\mathfrak{p}$ as an A-module by complex conjugation. Therefore, the two generators of $\operatorname{Pic} B$, $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are isomorphic as A-modules.

Lemma 6.3. *If $A/\mathfrak{M} = \mathbf{R}$, there is an epimorphism $\mathfrak{p} \twoheadrightarrow \mathfrak{M}$.*

Proof. Let x, y, z map to α, β, γ in $A/\mathfrak{M} = \mathbf{R}$. Since $\alpha^2+\beta^2+\gamma^2 = 1$, we can find an orthogonal matrix over $\mathbf{R}$ sending (α, β, γ) to (0,0,1). By a corresponding linear change of coordinates, we can assume that $A = \mathbf{R}[x', y', z']/(x'^2 + y'^2 + z'^2 - 1)$ with $\mathfrak{M} = (x', y', z' - 1)$. Let $\mathfrak{p}' = (x' + iy', z' - 1)$. Then $\mathfrak{p}'$ as well as $\mathfrak{p}$ generates $\operatorname{Pic} B$. By the remarks preceding the lemma, we see that $\mathfrak{p}' \approx \mathfrak{p}$ as an A-module. The map $\Re : B \to A$ taking $a_1 + a_2 i$ to a_1 is easily seen to map $\mathfrak{p}'$ onto $\mathfrak{M}$. □

Recall that in Section 5 we defined $p_n = [P_n] - [P_0] \in \tilde{K}Sp_0(A)$.

Corollary 6.4. *The map $\mathbf{Z} \to \tilde{K}Sp_0(A)$ by $n \mapsto p_n$ is an isomorphism.*

Proof. We saw in Section 5 that this map is a monomorphism so it will suffice to show that p_1 generates $\tilde{K}Sp_0(A)$. By [**1**, Section 7] or the remark below, $\tilde{K}Sp_0(A) \overset{\approx}{\to} W^-(A)$. Apply Lemma 6.1 with $\mathcal{P} = \{P_0, P_1\}$ noting that $P_1 \approx \mathfrak{p}$ and $P_0 \cong H \approx A^2$ as A-modules, and that $P_0 \cong H$ represents 0 in $\tilde{K}Sp_0(A)$ and $W^-(A)$. □

Remark. The fact that $\tilde{K}Sp_0(A) \to W^-(A)$ is an isomorphism can also be seen as follows. If Q is symplectic, then $Q \oplus Q^\vee \cong H(Q)$ by Lemma A.3. In particular, $H(\mathfrak{p}) \cong P_1 \perp P_1^\vee \cong P_1 \perp P_{-1} \cong P_0 \perp P_0 \cong H \perp H$ by Lemma 5.2 and the remarks preceding it. Since $[\mathfrak{p}]$ generates $\tilde{K}_0(A)$, it follows that $H : \tilde{K}_0(A) \to \tilde{K}Sp_0(A)$ is zero so $\tilde{K}Sp_0(A) \to W^-(A)$ is an isomorphism by the observation at the end of the appendix.

The proof of Theorem 1.1 now follows easily. Consider the map

$PSp_2(A) \to P_2(A)$ which forgets the symplectic structure. Since $\operatorname{Pic} A = 0$, we have $\Lambda^2 P \approx A$ for any rank 2 projective A-module P. As observed in Section 5, it follows that P has a symplectic structure, showing that our map is onto. By Corollary A.11, $PSp_2(A) \overset{\approx}{\to} \tilde{K}Sp_0(A)$ so $PSp_2(A) = \{P_n \mid n \in \mathbf{Z}\}$ by Corollary 6.4. Since $\mathfrak{p}^n \approx \mathfrak{p}^{-n}$ as an A-module by complex conjugation, we have $P_2(A) = \{\mathfrak{p}^n \mid n \geq 0\}$, the elements being distinct by Theorem 1.2.

7. The real coordinate ring. As above, let $A = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, let $B = \mathbf{C} \otimes_{\mathbf{R}} A$, and let $S = \{1 + f_1^2 + f_2^2 + \cdots + f_n^2 \mid f_i \in A\}$. In [**16**, Theorem 11.1] it is shown that the elements of $P_n(A_S)$ are in 1–1 correspondence with real vector bundles of rank n on S^2 (and similarly for any S^n). The stable case was proved earlier by Moore [**9**]. I will show here how to give an algebraic proof for this classification of projective A_S-modules.

Remark. The maximal ideals of A_S correspond to those of A with $A/\mathfrak{M} = \mathbf{R}$, e.g., by [**16**, Theorem 10.4, Lemma 10.6], so that A_S can be regarded as the "real" coordinate ring of S^2.

Theorem 7.1. *For any $s \in S$, the maps $P_n(A) \to P_n(A_s)$, $P_n(B) \to P_n(B_s)$, $K_0(A) \to K_0(A_s)$, $K_0(B) \to K_0(B_s)$, $KSp_0(A) \to KSp_0(A_s)$ are all isomorphisms.*

The corresponding results for A_S follow immediately by taking colimits.

Corollary 7.2. *The maps $P_n(A) \to P_n(A_S)$, $P_n(B) \to P_n(B_S)$, $K_0(A) \to K_0(A_S)$, $K_0(B) \to K_0(B_S)$, $KSp_0(A) \to KSp_0(A_S)$ are all isomorphisms.*

We first recall a classical observation of Serre.

Lemma 7.3. *If R is a commutative regular domain of dimension ≤ 2 and S is a multiplicative set in R, then $P_n(R) \to P_n(R_S)$ and $K_0(R) \to K_0(R_S)$ are onto.*

In fact, if P is a finitely generated projective R_S-module, write $P = M_S$ where M is a finitely generated R-module. Then $Q = M^{**}$ (where $N^* = \mathrm{Hom}\,(N, R)$) gives the required projective lifting [**12**, Lemma 3.1].

That $P_1(B) \to P_1(B_s)$ is an isomorphism now follows from Theorem 3.3. Since $P_1(B) \to P_n(B)$ is onto, so is $P_1(B_S) \to P_n(B_S)$ and this is injective since $P_1(B_S) \to K_0(B_S)$ is split by the determinant map. The assertions of the theorem for B follow immediately.

Lemma 7.4. *If $s \in S$, $K_0(A) \overset{\cong}{\to} K_0(A_s)$.*

Proof. Since A is regular, the localization sequence [**2**, Chapter IX, Theorem 6.3] gives

$$G_0(A/As) \to K_0(A) \to K_0(A_s) \to 0.$$

Now $G_0(A/As)$ is generated by $[A/\mathfrak{p}]$ for prime ideals $\mathfrak{p}$ of A of height 1 which contain s and $[A/\mathfrak{M}]$ for maximal ideals $\mathfrak{M}$ of A which contain s. Since A is a UFD, the ideals $\mathfrak{p}$ are principal so $[A/\mathfrak{p}]$ maps to 0 in $K_0(A)$. If $s \in \mathfrak{M}$, then the definition of S clearly implies that $A/\mathfrak{M} = \mathbf{C}$. By Lemma 6.2, we have a Koszul resolution

$$0 \to A \to A^2 \to A \to A/\mathfrak{M} \to 0$$

showing that $[A/\mathfrak{M}]$ maps to 0 in $K_0(A)$. □

Corollary 7.5. *If $s \in S$, $P_n(A) \overset{\cong}{\to} P_n(A_s)$ for $n \geq 3$.*

The case $n = 0$ is trivial, and the case $n = 1$ is clear by Lemma 7.3 since $\mathrm{Pic}\,(A) = 0$.

Now consider the diagram

$$\begin{array}{ccc} \tilde{K}_0(A) & \xrightarrow{H} & \tilde{K}Sp_0(A) \\ \downarrow & & \downarrow \\ \tilde{K}_0(A_s) & \xrightarrow{H} & \tilde{K}Sp_0(A_s). \end{array}$$

Since the top arrow is trivial by [**1**, Section 7] (see the remark after Corollary 6.4) and the left vertical map is onto by Lemma 7.3, the bottom arrow must also be trivial so that $\tilde{K}Sp_0(A_s) \overset{\cong}{\to} W^-(A_s)$. Let $(p_n)_s = [(P_n)_s] - [(P_0)_s] \in \tilde{K}Sp_0(A_s)$ denote the image of p_n under the map $\tilde{K}Sp_0(A) \to \tilde{K}Sp_0(A_s)$. Since Lemmas 6.2 and 6.3 localize, we see, by Lemma 6.1, that $\tilde{K}Sp_0(A_s)$ is generated by $(p_1)_s$ as in Corollary 6.4. It follows that $\tilde{K}Sp_0(A) \to \tilde{K}Sp_0(A_s)$ is onto and the elements of $\tilde{K}Sp_0(A_s)$ are the $n(p_1)_s = (p_n)_s$ for $n \in \mathbf{Z}$. If the map $\mathbf{Z} = \tilde{K}Sp_0(A) \to \tilde{K}Sp_0(A_s)$ is not injective, then, as in the proof of Theorem 1.2, some $(P_n)_s$ will be hyperbolic and therefore free, contradicting Corollary 4.8. This shows that $\mathbf{Z} \overset{\approx}{\to} \tilde{K}Sp_0(A) \overset{\approx}{\to} \tilde{K}Sp_0(A_s)$ by the map taking n to $(p_n)_s$.

Since $\operatorname{Pic} A_s = 0$, $PSp_2(A_s) = \tilde{K}Sp_0(A_s)$ maps onto $P_s(A_s)$ as in the proof of Theorem 1.1 in Section 6, showing that $P_2(A_s) = \{A_s^2, \mathfrak{p}_s, \mathfrak{p}_s^2, \mathfrak{p}_s^3, \dots\}$. In order to show that these elements are distinct, we must first examine the action of the group of units on KSp_0 which is defined as follows. If $u \in R^*$ and $(P, \langle , \rangle)$ is a symplectic R-module, define $u \circ (P, \langle , \rangle) = (P, u\langle , \rangle)$. This preserves direct sums and hyperbolic modules and so induces actions of R^* on $KSp_0(R)$, $\tilde{K}Sp_0(R)$ and $W^-(R)$.

Now let R be a smooth two-dimensional affine domain over $\mathbf{R}$. Let M be an R-module of finite length, and let $\varphi : M \to \hat{M}$ be symmetric and nondegenerate. If $a \in R$ induces an automorphism $a : M \overset{\approx}{\to} M$, define $a \circ (M, \varphi) = (M, a\varphi)$. This is again symmetric and nondegenerate. Note that $a^2 \circ (M, \varphi) \approx (M, \varphi)$ via the map $M \to M$ sending x to $a^{-1}x$. The construction of [**1**, Section 3] defines an element $\rho(M, \varphi)$ in $W^-(R)$.

Lemma 7.6. $\rho(M, u\varphi) = u \circ \rho(M, \varphi)$.

Proof. By [**1**, Theorem 3.3], $\rho(M, \varphi)$ is represented by a symplectic module K which fits into a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P^* & \xrightarrow{\ t\ } & K & \xrightarrow{\ s\ } & P & \xrightarrow{\ \alpha\ } & M & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \psi} & & \Vert & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & P^* & \xrightarrow{\ -s^*\ } & K^* & \xrightarrow{\ t^*\ } & P & \xrightarrow{\ \beta\ } & \hat{M} & \longrightarrow & 0
\end{array}$$

where $\psi(x) = \langle x, -\rangle$ and $\beta \in \mathrm{Hom}\,(P, \hat{M}) \approx \mathrm{Ext}^2(M, P^*)$ corresponds to the extension class of the top row. Replacing t by $t' = u^{-1}t$ changes this extension class to $\beta' = u\beta$, and we get a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P^* & \xrightarrow{t'} & K & \xrightarrow{s} & P & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ & & \Big\Downarrow & & \Big\downarrow u\psi & & \Big\Downarrow & & \Big\downarrow u\varphi & & \\ 0 & \longrightarrow & P^* & \xrightarrow{-s^*} & K^* & \xrightarrow{t'^*} & P & \xrightarrow{\beta'} & \hat{M} & \longrightarrow & 0. \end{array}$$

The lemma follows immediately from this. $\square$

Now let A and s be as in Theorem 7.1. If $\mathfrak{M}$ is a maximal ideal of A_s with $A_s/\mathfrak{M} = \mathbf{R}$ and $u \in A_s^*$, define $\mathrm{sgn}_{\mathfrak{M}}(u) = \mathrm{sgn}\,(u \bmod \mathfrak{M})$. If $\mathfrak{M}$ corresponds to the point $p \in S^2$, then $\mathrm{sgn}_{\mathfrak{M}}(u) = \mathrm{sgn}\,(u(p))$.

Lemma 7.7. $\mathrm{sgn}_{\mathfrak{M}}$ *is independent of* $\mathfrak{M}$.

Proof. This is clear by continuity, but a simple algebraic proof can be given as follows. If $p, q \in S^2$, take a plane section through p and q, reducing the problem to the case of S^1. Choose coordinates on S^1 such that $p, q \neq (-1, 0)$ and parametrize S^1 by $(x, y) = ((1-t^2)/(1+t^2), 2t/(1+t^2))$. Then $u = f(t)/(1+t^2)^N$ where $f(t)$ has no real zeros and so is of constant sign, showing that $\mathrm{sgn}\,u(p) = \mathrm{sgn}\,u(q)$. The fact that f is of constant sign can be checked algebraically by factoring into quadratic factors and completing the square. $\square$

Lemma 7.8. *Let P be a rank 2 symplectic A_s-module. Let $u \in A_s^*$ with* $\mathrm{sgn}\,(u) = 1$. *Then $u \circ P \cong P$ as a symplectic module.*

Proof. Identify $PSp_2(A_s) = \tilde{K}Sp_0(A_s) = W^-(A_s)$. By the Bertini argument of [**1**, p. 626, last paragraph], we have $P = \rho(A/I, \varphi)$ as a symplectic module, where $I = \cap \mathfrak{M}_i$ the $\mathfrak{M}_i$ being distinct maximal ideals. By Lemma 7.6, $u \circ P = \rho(A/I, u\varphi)$. If $A/\mathfrak{M}_i = \mathbf{R}$, the fact that $\mathrm{sgn}\,(u) = 1$ shows that we can write $u \equiv a_i^2 \bmod \mathfrak{M}_i$. The same is trivially true if $A/\mathfrak{M}_i = \mathbf{C}$. By the Chinese remainder theorem we can write $u \equiv a^2 \bmod I$ so $(A/I, u\varphi) = (A/I, a^2\varphi) \approx (A/I, \varphi)$. Therefore $u \circ P = \rho(A/I, u\varphi) = \rho(A/I, \varphi) = P$.

Corollary 7.9. *If P is a rank 2 projective A_s-module which is not free, then*

$$\operatorname{Aut}(P) \stackrel{\det}{\to} A_s^* \stackrel{\operatorname{sgn}}{\to} \{\pm 1\} \to 1$$

is exact.

Proof. If $u \in A_s^*$ and $\operatorname{sgn}(u) = 1$, Lemma 7.8 gives us an automorphism f of P with $f : (P, \langle , \rangle) \cong (P, u\langle , \rangle)$. Clearly $\det(f) = u$, showing that $\operatorname{im}(\det) \supset \ker(\operatorname{sgn})$. This inclusion cannot be proper otherwise $\operatorname{im}(\det) = A_s^*$ contradicting Theorem 4.7.

We can now show that the indicated elements of $P_2(A_s) = \{A_s^2, \mathfrak{p}_s, \mathfrak{p}_s^2, \mathfrak{p}_s^3, \dots\}$ are distinct. Lemma 7.8 implies that there are at most two nonisomorphic symplectic structures on $\mathfrak{p}_s^n$, namely, $(P_n)_s$ and $(P_{-n})_s \cong (P_n)_s^\vee$. If $\mathfrak{p}_s^m \approx \mathfrak{p}_s^n$, it follows that $(P_m)_s \cong (P_n)_s$ or $(P_m)_s \cong (P_{-n})_s$. Therefore, $(p_m)_s = (p_n)_s$ or $(p_{-n})_s$ which implies that $m = \pm n$. □

Appendix

For the reader's convenience, I will include here a brief outline of Bass's theory of symplectic modules which is used in Sections 5, 6 and 7. A much more general discussion can be found in [**3**]. All rings considered here will be commutative.

A symplectic A-module here will mean a finitely generated projective A-module P with an alternating bilinear form $\langle , \rangle : P \times P \to A$ which is nondegenerate, i.e., $P \stackrel{\approx}{\to} P^* = \operatorname{Hom}_A(P, A)$ by the map $p \mapsto \langle p, - \rangle$. I will write $P \cong Q$ to indicate an isomorphism preserving the symplectic form $\langle , \rangle$.

The orthogonal sum $P \perp Q$ of two symplectic modules is the direct sum $P \oplus Q$ with $\langle (p, p'), (q, q') \rangle = \langle p, q \rangle + \langle p', q' \rangle$.

Lemma A.1. *Let P be a symplectic A-module, and let $Q \subset P$ be a submodule which is symplectic with respect to the restriction $\langle , \rangle \mid Q$. Define $Q^\perp = \{p \in P \mid \langle p, Q \rangle = 0\}$. Then $Q^\perp$ is also symplectic and $P = Q \perp Q^\perp$.*

Proof. $Q \cap Q^\perp = 0$ since Q is symplectic and $\langle Q \cap Q^\perp, Q \rangle = 0$. If

$p \in P$, there is a $q \in Q$ such that $\langle p, -\rangle \mid Q = \langle q, -\rangle$ so $p - q \in Q^\perp$ showing that $P = Q \oplus Q^\perp$. Since $\langle Q^\perp, Q\rangle = 0$, the isomorphism $P \overset{\approx}{\to} P^*$ is the direct sum of $Q \to Q^*$ and $Q^\perp \to Q^{\perp *}$ so these are also isomorphisms and $Q^\perp$ is also symplectic. □

If M is a finitely generated projective A-module, we define the hyperbolic module $H(M)$ to be the symplectic module $H(M) = M \oplus M^*$ with $\langle (a, \varphi), (b, \psi)\rangle = \psi(a) - \varphi(b)$. In particular, I will write H for $H(A)$, the free A module with basis e, f and $\langle e, f\rangle = 1$. By H^n I will mean $H(A^n) = H \perp H \perp \cdots \perp H$.

Lemma A.2. *If P is a symplectic A-module, there is a bilinear form $b : P \times P \to A$ such that $\langle x, y\rangle = b(x, y) - b(y, x)$.*

Proof. Since P is a finitely generated projective A-module, we can find a finitely generated projective A-module Q with $P \oplus Q$ free. Let $F = P \perp P \perp H(Q)$. Then F is symplectic and is free as an A-module since $F = P \oplus P \oplus Q \oplus Q^*$ but $P \approx P^*$ so $P \oplus Q^* \approx P^* \oplus Q^*$ is free. Let $\{e_i\}$ be a basis for F, and let $\langle e_i, e_j\rangle = a_{ij}$ so that $\langle \Sigma x_i e_i, \Sigma y_j e_j\rangle = \Sigma a_{ij} x_i y_j$. Note that $a_{ii} = 0$ and $a_{ji} = -a_{ij}$. Let $b_{ij} = a_{ij}$ if $i < j$ and $b_{ij} = 0$ otherwise. Set $b(x, y) = \Sigma b_{ij} x_i y_j$. Then $\langle x, y\rangle = b(x, y) - b(y, x)$ for all $x, y \in F$, and restricting b to P gives the required form.

Since $P \approx P^*$ we can define the dual $P^\vee$ of P by simply changing the sign of the alternating form $(P, \langle , \rangle)^\vee = (P, -\langle , \rangle)$. □

Lemma A.3. *If P is a symplectic A-module, then $P \perp P^\vee \cong H(P)$.*

Of course, we ignore the symplectic structure on P in defining $H(P)$.

Proof. Let b be as in Lemma A.2 and define $\theta : P \perp P^\vee \to H(P) = P \oplus P^*$ by $\theta(x, y) = \langle x + y, b(y, -) + b(-, x)\rangle$. Then $\ker\theta \subset D := \{(-x, x) \mid x \in P\}$. Note $D \approx P$ as an A-module. Now $\theta(-x, x) = (0, b(x, -) + b(-, -x)\rangle = (0, \langle x, -\rangle)$ so θ induces the canonical isomorphism $P \approx D \overset{\approx}{\to} P^*$. Therefore, $\ker\theta = 0$ and $0 \oplus P^* \subset \operatorname{im}\theta$. Since $\operatorname{im}\theta$ clearly projects onto P, we see that θ is also onto. An easy calculation shows that θ preserves the symplectic form. □

Corollary A.4. *If P is a symplectic A-module, there is a symplectic A-module Q with $P \perp Q \cong H^n$ for some $n < \infty$.*

Proof. Let $P \oplus P' \approx A^n$ be a free A-module. Then we can choose $Q = P^\vee \perp H(P')$ since $P \perp P^\vee \perp H(P') \cong H(P) \perp H(P') = H(P \oplus P') \cong H^n$. □

If P is a finitely generated projective A-module and $\mathfrak{p}$ is a prime ideal of A, then $P_\mathfrak{p}$ is free, and we denote its rank by $\mathrm{rk}_\mathfrak{p} P$. We write $\mathrm{rk}\, P \geq r$ if $\mathrm{rk}_\mathfrak{p} P \geq r$ for all $\mathfrak{p}$. Recall that an element x of P is called unimodular if there is a homomorphism $P \to A$ such that $x \mapsto 1$.

Definition. The projective stable range $\mathrm{psr}\,(A)$ is defined as follows: $\mathrm{psr}\,(A) \leq r$ if whenever P is a finitely generated projective A-module of rank $\geq r$ and $(a, x) \in A \oplus P$ is unimodular, we can find $y \in P$ such that $x + ay$ is unimodular.

The following two theorems summarize the well known stability theorems of Bass and Serre. Proofs may be found in [**2**] and [**5**]. A short exposition is given in [**15**].

Theorem A.5. (Bass). *If A is noetherian,* $\mathrm{psr}\,(A) \leq 1 + \dim\,(A)$.

More generally, $\mathrm{psr}\,(A) \leq 1 + \dim m - \mathrm{spec}\,(A)$.

Theorem A.6. *Let P be a finitely generated projective A-module such that* $\mathrm{rk}\, P \geq \mathrm{psr}\,(A)$. *Then*

(1) *$P \approx A \oplus P'$ for some P'.*

(2) *If $A \oplus P \approx A \oplus P$, then $P \approx Q$.*

The main result proved here is the following theorem of Bass [**3**] which gives a symplectic analog of Theorem A.6.

Theorem A.7. (Bass). *Let P be a symplectic A-module.*

(1) *If $\operatorname{rk} P \geq \operatorname{psr}(A)$, then $P \cong P' \perp H$ for some symplectic A-module P'.*

(2) *If $\operatorname{rk} P \geq \operatorname{psr}(A) - 1$ and $P \perp H \cong Q \perp H$, then $P \cong Q$.*

The importance of this for the study of projective modules is that the rank bound in (2) is improved by 1 over that in Theorem A.5(2).

Proof. (1) Since $(1, 0) \in A \oplus P$ is unimodular, there is a $y = 0+1y \in P$ which is unimodular. Since $\langle , \rangle$ is nondegenerate, we can find an $x \in P$ such that $\langle x, y\rangle = 1$. Now $Q = Ax + Ay$ is free on x and y since $ax + by = 0$ implies $a = \langle ax + by, y\rangle = 0$ and similarly $b = 0$. Clearly $Q \cong H$ and (1) follows from Lemma A.1.

(2) Bass defines a symplectic transvection of a symplectic module M to be an automorphism of the form

$$\sigma(x) = \sigma_{u,v,\alpha}(x) = x + \langle u, x\rangle v + \langle v, x\rangle u + \alpha\langle u, x\rangle u$$

where $\alpha \in A$, $u, v \in M$, and $\langle u, v\rangle = 0$. It is easy to check that $\langle \sigma x, \sigma y\rangle = \langle x, y\rangle$ and that the inverse of $\sigma = \sigma_{u,v,\alpha}$ is given by $\sigma^{-1} = \sigma_{u,-v,-\alpha}$.

Here we will take $M = P \perp H = P \oplus A \oplus A$, writing the elements of M as (p, b, a) with $b, a \in A$, and we will consider only symplectic transvections with $v = (y, 0, 0)$ for some $y \in P$ and with either (I) $u = (0, 1, 0)$ or (II) $u = (0, 0, -1)$. These have the form

(I) $\sigma(p, b, a) = (p + ay, b + \langle y, p\rangle + \alpha a, a)$.

(II) $\tau(p, b, a) = (p + by, b, a - \langle y, p\rangle + \beta b)$.

Suppose that $\theta : Q \perp H \cong P \perp H$. Let $\theta(0, 0, 1) = (p, b, a)$ which is unimodular since (0,0,1) is. Since $\operatorname{rank}(P \oplus A) \geq \operatorname{psr}(A)$, we can find a $y \in P$ and $\alpha \in A$ with $(p + ay, b + \alpha a)$ unimodular in $P \oplus A$. Let σ be the corresponding symplectic transvection of type (I) taking (p, b, a) to $(p + ay, b + \langle y, p\rangle + \alpha a, a) = (p', b', a)$. Note that (p', b') is also unimodular since we can transform it to $(p + ay, b + \alpha a)$ by an ordinary transvection of the form $(x, \xi) \mapsto (x, \xi - \langle y, x\rangle)$.

Now since (p', b') is unimodular, we can clearly find a symplectic transvection τ of type (II) transforming (p', b', a) to $(p'', b', 1)$. We can then find a symplectic transvection σ' of type (I) transforming $(p'', b', 1)$ to (0,0,1). So if $\theta' = \sigma'\tau\sigma\theta : Q \perp H \cong P \perp H$, then $\theta'(0,0,1) = (0,0,1)$. Let $\theta'(0,1,0) = (p_1, b_1, a_1)$. Since $\langle (0,1,0),(0,0,1)\rangle = 1$, we have $b_1 = \langle (p_1, b_1, a_1), (0,0,1)\rangle = 1$. A symplectic transvection τ' of type (II) now takes (p_1, b_1, a_1) to (0,1,0) and clearly fixes (0,0,1). It follows that $\theta'' = \tau'\theta' : Q \perp H \cong P \perp H$ sends H isomorphically onto H and therefore sends $Q = H^\perp$ in $Q \perp H$ isomorphically onto $P = H^\perp$ in $P \perp H$.

If A is a field, $\operatorname{psr}(A) = 1$ and (1) implies that every symplectic module is isomorphic to H^n for some n since it is obviously impossible to have a symplectic module of rank 1. In particular, the rank of any symplectic module is even. For any A, if $\mathfrak{p}$ is a prime ideal of A, let $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and define $\operatorname{rk}_\mathfrak{p}(P) = \dim_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes_A P)$ which is even since $k(\mathfrak{p}) \otimes_A P$ is a symplectic module over the field $k(\mathfrak{p})$. As in [**2**, Chapter IX, Section 3], we get a function $\mathfrak{p} \mapsto \operatorname{rk}_\mathfrak{p}(P)$ in $2H_0(A)$ where $H_0(A)$ is the ring of continuous $\mathbf{Z}$-valued functions on $\operatorname{Spec} A$. When A is a domain, $H_0(A) = \mathbf{Z}$. □

Define $KSp_0(A)$ to be the Grothendieck group with generators $[P]$ for each symplectic module P and relations $[P \perp Q] = [P] + [Q]$. Just as in ordinary K-theory, we have the following lemma [**15**, Corollary 6.2].

Lemma A.8. (1) *Any element of* $KSp_0(A)$ *has the form* $[P] - n[H]$.

(2) $[P] = [Q]$ *in* $KSp_0(A)$ *if and only if* $P \perp H^n \cong Q \perp H^n$ *for some* $n < \infty$.

In fact, (1) follows by expressing any element as $[Q]-[P]$ and applying Corollary A.4 to P, and (2) follows by writing out $[P]-[Q] = 0$ in terms of the defining relations $[P \perp Q] = [P] + [Q]$.

We have a map $\operatorname{rk} : KSp_0(A) \to 2H_0(A)$ sending $[P]$ to the function $\mathfrak{p} \mapsto \operatorname{rk}_\mathfrak{p}(P)$, and we define $\tilde{K}Sp_0(A)$ to be the kernel of this map.

Let $PSp_n(A)$ be the set of isomorphism classes of symplectic A-modules of rank n. Define $PSp_n(A) \to PSp_{n+2}(A)$ by sending P to $P \perp H$. Define $PSp_{2n}(A) \to \tilde{K}Sp_0(A)$ by sending P to $[P] - n[H]$.

Corollary A.9. $\operatorname{colim} PSp_{2n}(A) \overset{\approx}{\to} \tilde{K}Sp_0(A)$.

Proof. The ontoness follows from Lemma A.8 (1) and the injectivity from Lemma A.8 (2). □

As a special case of Theorem A.7, we have

Corollary A.10. (1) *If* $\operatorname{psr} A \leq 2n+1$, *then*

$$PSp_{2n}(A) \overset{\approx}{\to} PSp_{2n+2}(A) \overset{\approx}{\to} \cdots \overset{\approx}{\to} \tilde{K}Sp_0(A).$$

(2) *If* $\operatorname{psr} A \leq 2n$, *then* $PSp_{2n-2}(A) \twoheadrightarrow PSp_{2n}(A)$ *is onto.*

Corollary A.11. *If A is noetherian of dimension* 2, *then* $PSp_2(A) \overset{\approx}{\to} \tilde{K}Sp_0(A)$.

This special case of [**3**, Chapter IV, 4.11.2, 4.16] was used in [**11**, Theorems 1.1, 1.2].

We can define a map $H : K_0(A) \to KSp_0(A)$ by sending $[P]$ to $[H(P)]$. The cokernel of this map is, by definition, the symplectic Witt group $W^-(A)$ used in [**1**]. Since the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{H} & KSp_0(A) \\ \downarrow & & \downarrow \\ H_0(A) & \xrightarrow{2} & 2H_0(A) \end{array}$$

is commutative, we see that $W^-(A)$ is also the cokernel of $H : \tilde{K}_0(A) \to \tilde{K}Sp_0(A)$. In particular, $\tilde{K}Sp_0(A) \to W^-(A)$ is an isomorphism if and only if $H : \tilde{K}_0(A) \to \tilde{K}Sp_0(A)$ is zero.

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