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PRODUCTS OF SYMPLECTIC GROUPS ACTING ON ISOTROPIC SUBSPACES

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ABSTRACT. Let A be a finite dimensional commutative semisimple algebra over a field k, and let (V, B) be a finitely generated symplective space over A. We examine the action of the symplectic group $\mathrm{Sp}_A(V, B)$ on the set of B'-isotropic k-subspaces of V, where $B' = \phi \circ B$ is the k-symplectic form induced by a 'trace' map $\phi : A \to k$. The case of A being a field was studied earlier and here we consider the case where A has several simple components. The orbits are completely classified when $A = k \times k$ and for maximal B'-isotropic subspaces when $\dim_k A = 3$; the number of orbits of maximal B'-isotropic subspaces is infinite if $\dim_k A \geq 4$ and k is infinite.

1. Introduction. In [**6**] and [**3**] we studied the action on Grassmannians of products of general linear groups defined over extension fields of the base field. In [**4**] this work was extended to the case of a symplectic group defined over an extension field of the base field acting on isotropic subspaces of a symplectic space. The present paper examines the case of a product of symplectic groups acting on isotropic subspaces, i.e., we replace the extension field by a finite dimensional commutative semisimple algebra.

In more detail, the problem is the following. We have a finite dimensional commutative semisimple algebra A over a field k, a finitely generated faithful A-module V and an A-valued regular symplectic form B on V. By suitably choosing a k-linear functional $\phi : A \to k$ (see Section 2), we form the k-valued symplectic form $B' = \phi \circ B$ and look at the action of the symplectic group $\mathrm{Sp}_A(V, B)$ on B'-isotropic k-subspaces of V. The problem is to determine when the number of orbits is finite and to classify them. In group theoretic terms, if $A = k_1 \times \cdots \times k_p$ where each k_i/k is a finite extension of fields, this

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corresponds to looking at double coset spaces

$$P\backslash \mathrm{Sp}\,(2N,k)/(\mathrm{Sp}\,(2n_1,k_1)\times\cdots\times \mathrm{Sp}\,(2n_p,k_p))$$

for suitable integers n_i and N, where P is a maximal parabolic subgroup of $\mathrm{Sp}\,(2N,k)$. (We write $\mathrm{Sp}\,(2n)$ when the underlying space has dimension $2n$.) In [**4**] we considered the case $A = k_1$, that is, A is an extension field of k; the present paper examines the case of A having several simple components k_i. These double coset spaces are of importance in the theory of automorphic forms for the explicit construction of L-functions. In that context, Garrett [**1**] and Piatetski-Shapiro and Rallis [**5**] have obtained the number of orbits in the special cases $\dim_k A = 2$ or 3 and each $n_i = 1$. We generalize this to arbitrary dimensions n_i. (Piatetski-Shapiro and Rallis work with GSp groups instead of Sp, but it makes no difference in the results.)

Our main results are as follows. Assume the field k is infinite.

Theorem A. *A necessary and sufficient condition for the number of orbits for the action of* $\mathrm{Sp}_A(V,B)$ *on maximal* B'*-isotropic subspaces of* V *to be finite is* $\dim_k A \leq 3$.

Theorem B. (a) *If* $\dim_k A = 2$, *the number of orbits for the action of* $\mathrm{Sp}_A(V,B)$ *on* B'*-isotropic* k*-subspaces of* V *of dimension* d *is finite for every* d, $d \leq (1/2)\dim_k V$. *Moreover, this number depends on the degrees of the simple components* k_i *of* A *over* k *but not on the* k_i *or on the field* k *itself.*

(b) *If* $\dim_k A = 3$, *the number of orbits for the action of* $\mathrm{Sp}_A(V,B)$ *on* B'*-isotropic* k*-subspaces of* V *of dimension* d *is finite only in the case of maximal* B'*-isotropic subspaces and in the trivial cases of the zero subspace and of one-dimensional subspaces over* k. *When this number is finite, it depends on the degrees* $[k_i : k]$ *but not on the* k_i *or* k *itself.*

The sufficiency of the condition in Theorem A is part of Theorem B. The proof of the necessity of this condition is a variation on the proof of [**4**, Theorem A], making use of [**3**, Theorem 3.1], so we will not give any details. Theorem B summarizes [**4**, Theorem B] and results of Sections 4, 5 and 6.

In the case of a finite number of orbits when $\dim_k A = 2$ or 3, we actually determine the precise structure of the orbits and show that they can be classified in terms of certain integer invariants, which allows us to compute the exact number of orbits, and to give typical representatives for each orbit. This classification makes much use of the results of [**3**], where the analogous situation of the general linear group $\mathrm{GL}_A(V)$ acting on k-subspaces of V is considered. As an illustration, we list in the table below the different possibilities for A and in each case the asymptotic number of orbits of maximal B'-isotropic k-subspaces of V, where V is a free A-module of rank $2n$; F and L are respectively a quadratic and a cubic extension field of k. The exponent of n gives the number of independent parameters necessary to describe the orbits.

A	asymp. number of orbits
k	1
F	n
$k \times k$	n
L	$(n/2)^2$
$F \times k$	$(n/2)^3$
$k \times k \times k$	$(n/2)^4$

The cases where A is a field are treated in [**4**]; for the remaining cases, see the discussion following Theorem 4.3, Corollary 5.4, and Corollary 6.4.

In each of these cases, one should determine the precise form of the isotropy groups for each orbit and give more detailed information about open orbits, 'negligible' orbits, etc. This will appear later [**7**].

2. Notation and statement of the problem. We expand on the notation already introduced. The primitive idempotents of the algebra $A = k_1 \times \cdots \times k_p$ are the elements $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the ith position, $i = 1, \ldots, m$; they satisfy $e_1 + \cdots + e_m = 1$, $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. The A-module V satisfies $V = \oplus_i e_i V$, where each $e_i V$ is viewed as a vector space over k_i. The dimension vector of V is $\mathbf{dim}_A V = (\dim_{k_1} e_1 V, \ldots, \dim_{k_p} e_p V)$; if $p = 2$ or 3, we also call the dimension vector the bidimension or tridimension of V. Any A-bilinear form $B : V \times V \to A$ can be written uniquely in the form

$$B(x,y) = \sum_{i=1}^{p} e_i B_i(e_i x, e_i y)$$

for $x, y \in V$, where each B_i is a k_i-valued k_i-bilinear form on $e_i V$. Conversely, choosing the forms B_i determines the A-bilinear form B. (In formal terms, if R-**bil** is the category of R-modules equipped with bilinear forms, the categories A-**bil** and k_1-**bil** $\times \cdots \times k_m$-**bil** are equivalent.) The form B is symmetric or alternating or regular (regular = nondegenerate = nonsingular) if and only if each B_i satisfies the same property.

From now on, assume that B is symplectic, that is, regular alternating. The group $\mathrm{Sp}_A(V,B)$ of A-linear automorphisms of V, which preserve the form B, satisfies

$$\begin{aligned}\mathrm{Sp}_A(V,B) &\cong \mathrm{Sp}_{k_1}(e_1V, B_1) \times \cdots \times \mathrm{Sp}_{k_m}(e_mV, B_m)\\ &\cong \mathrm{Sp}\,(2n_1, k_1) \times \cdots \times \mathrm{Sp}\,(2n_m, k_m),\end{aligned}$$

where $2n_i = \dim_{k_i} e_i V$. For each i, fix a nonzero k-linear form $\phi_i : k_i \to k$ (for example, the trace map if k_i/k is separable); if $k_i = k$, one usually chooses ϕ_i to be the identity. Consider the k-linear form on A defined by $\phi(\Sigma \alpha_i e_i) = \Sigma \phi_i(\alpha_i)$, $\alpha_i \in k_i$. One easily checks that $B' = \phi \circ B : V \times V \to k$ is a regular alternating k-bilinear form on V, which makes (V, B') into a symplectic space over k. Clearly, $\mathrm{Sp}_A(V,B) \leq \mathrm{Sp}_k(V,B')$.

In this paper we propose to investigate the action of the symplectic group $\mathrm{Sp}_A(V) = \mathrm{Sp}_A(V,B)$ on the set $\mathrm{ISO}_d(V,B')$ of B'-isotropic (= totally isotropic) k-subspaces of V; in particular, if $\mathrm{ISO}_d(V,B')$ is the set of B'-isotropic subspaces of dimension d over k, we want to know when $\mathrm{ISO}_d(V,B')/\mathrm{Sp}_A(V)$ is finite. In group theoretic terms, if $\dim_k V = 2N = 2\Sigma_i n_i [k_i : k]$, the problem corresponds to looking at the double coset spaces

$$P_d \backslash \mathrm{Sp}\,(2N,k) / (\mathrm{Sp}\,(2n_1,k_1) \times \cdots \times \mathrm{Sp}\,(2n_m,k_m)),$$

where P_d is a maximal parabolic subgroup of $\mathrm{Sp}\,(2N,k)$ leaving invariant a d-dimensional B'-isotropic subspace of V. Our results will be phrased in geometric terms, but the interested reader can immediately translate them to obtain the cardinalities of the double coset spaces

$$|P_d\backslash \mathrm{Sp}\,(2N,k)/(\mathrm{Sp}\,(2n_1,k_1)\times\cdots\times \mathrm{Sp}\,(2n_m,k_m))| \\ = |\mathrm{ISO}_{\,d}(V,B')/\mathrm{Sp}_{\,A}(V)|,$$

as well as coset representatives.

Further terminology that we will use is as follows. We write $\mathbf{N}$ for the set of nonnegative integers. If R is a ring, the submodule of an R-module generated by a set S is written RS, or $\langle v_1,\dots,v_n\rangle_R$ if $S=\{v_1,\dots,v_n\}$. As in [**3**], a k-subspace W of V is the *direct sum over* A of the two k-subspaces W_1 and W_2, written $W=W_1\oplus_A W_2$, if $W=W_1\oplus W_2$ and $AW=AW_1\oplus AW_2$; this extends to more than two summands. The *A-component* of W, written $\mathrm{comp}_{\,A}W$, is the largest A-submodule of V contained in W.

Regarding the geometry, orthogonality with respect to B and B' will be denoted respectively by $\perp$ and $\perp'$; for vectors x and y in V and subsets $S,T\subseteq V$, we write, for instance: $x\perp y$, $S\perp' T$, $S^\perp$, $S^{\perp'}$ with the obvious meaning. If W,W_1,W_2 are k-subspaces of V, we write $W=W_1\perp_A W_2$ if $W=W_1\oplus_A W_2$ and $W_1\perp W_2$. The *B-radical* of W is defined by $\mathrm{rad}_{\,B}W=W\cap W^\perp$. The subspace W is called B-isotropic if $W\le W^\perp$, that is, $W=\mathrm{rad}_{\,B}W$. If $(U,\tilde B)$ is a symplectic space over some field, a *hyperbolic sequence* in $(U,\tilde B)$ is a sequence $v_1,v_1',\dots,v_r,v_r'$ of vectors of U satisfying $\tilde B(v_i,v_j')=\delta_{ij}$ and $\tilde B(v_i,v_j)=\tilde B(v_i',v_j')=0$; if that sequence also forms a basis for U, it is a *hyperbolic basis* of $(U,\tilde B)$.

3. Preliminaries. The following lemmas are proved as [**4**, Lemmas 4.1 and 4.2].

Lemma 3.1. (i) *Let $x\in V$, and let U be an A-submodule of V. Then $x\perp' U\iff x\perp U$.*

(ii) *For an A-submodule U of V, $U^{\perp'}=U^\perp$. In particular, an A-submodule of V is B'-isotropic if and only if it is B-isotropic, and is B'-nondegenerate if and only if it is B-nondegenerate* (*same as B-nonsingular*).

(iii) *If* $W \in \mathrm{ISO}\,(V, B')$, *then* $\mathrm{comp}_A W$ *is* B*-isotropic, i.e.,* $\mathrm{comp}_A W \le \mathrm{rad}_B W$. *If* W *is maximal isotropic in* (V, B') *then* $\mathrm{comp}_A W = \mathrm{rad}_B W = W^{\perp}$.

Lemma 3.2. (i) *If the* k*-subspaces* W, W_1, W_2 *of* V *satisfy* $W = W_1 \perp_A W_2$, *then there exist* B*-nondegenerate* A*-submodules* U_1 *and* U_2 *in* (V, B) *such that* $W_i \le U_i$, $i = 1, 2$, *and* $U_1 \perp U_2$.

(ii) *Let* $W \in \mathrm{ISO}\,(V, B')$, *and let* W_1 *be a* k*-subspace of* W. *If there is a* B*-nondegenerate* A*-submodule* U *of* V *containing* W_1 *as a maximal* B'*-isotropic subspace, then* $W = W_1 \perp_A Y$ *for some* k*-subspace* Y *of* W ($Y = W \cap U^{\perp}$ *will do*).

(iii) *Let* $W \in \mathrm{ISO}\,(V, B')$. *Then* $W = \mathrm{comp}_A W \perp_A Y$ *for some* k*-subspace* Y *of* W.

4. The case $A = k \times k$**.** In this section $A = k \times k$, V is an A-module of bidimension $(2n_1, 2n_2)$, and B is a regular A-valued symplectic form on V. We have

$$\begin{aligned} B(x, y) &= e_1 B_1(e_1 x, e_1 y) + e_2 B_2(e_2 x, e_2 y), \\ B'(x, y) &= B_1(e_1 x, e_1 y) + B_2(e_2 x, e_2 y) \end{aligned}$$

for $x, y \in V$. (In [**2**], the choice is $B' = B_1 - B_2$, but this only causes a minor sign change in the normalization of Theorem 4.1.)

Define the Sp-*type* of a subspace $W \in \mathrm{ISO}\,(V, B')$ to be the quadruple

$$\begin{aligned} \mathrm{type}_{\mathrm{Sp}}\,(W) = (&\dim_k(W \cap e_1 V), \dim_k(W \cap e_2 V), \\ &\dim_k\left(\frac{\mathrm{rad}_B W}{\mathrm{comp}_A W}\right), \frac{1}{2}\dim_k\left(\frac{W}{\mathrm{rad}_B W}\right)). \end{aligned}$$

In the course of the proof of Theorem 4.1 below, we will see that $\dim_k(W/\mathrm{rad}_B W)$ is always even, so that the parameters of $\mathrm{type}_{\mathrm{Sp}}\,(W)$ are all integers. The Sp-type of an isotrophic subspace in (V, B') is clearly invariant under the action of $\mathrm{Sp}_A(V, B)$, and we will show that it forms a complete set of invariants for this action.

Remark. As shown in [**3**], k-subspaces of V are completely characterized under the action of $\mathrm{GL}_A(V)$ by their GL-type, defined by

$\text{type}_{\text{GL}}(W) = (\dim_k(W \cap e_1 W), \dim_k(W \cap e_2 W), \dim_k(W/\text{comp}_A W))$, which is less discriminating than the Sp-type, as it should be. If $\text{type}_{\text{Sp}}(W) = (m_1, m_2, s, t)$, then $\text{type}_{\text{GL}}(W) = (m_1, m_2, s + 2t)$.

Theorem 4.1. *Let* $W \in \text{ISO}(V, B')$ *with* $\text{type}_{\text{Sp}}(W) = (m_1, m_2, s, t)$. *Then there exist hyperbolic sequences* $v_1^{(i)}, \tilde{v}_1^{(i)}, \dots, v_{m_i}^{(i)}, \tilde{v}_{m_i}^{(i)}$; $u_1^{(i)}, \tilde{u}_1^{(i)}, \dots, u_s^{(i)}, \tilde{u}_s^{(i)}$; $w_1^{(i)}, \tilde{w}_1^{(i)}, \dots, w_t^{(i)}, \tilde{w}_t^{(i)}$ *in* $(e_i V, B_i)$, $i = 1, 2$, *such that*

$$W = (W \cap e_1 W) \perp_A (W \cap e_2 W) \perp_A Y_1 \perp_A Y_2$$

with

$$W \cap e_i W = \langle v_1^{(i)}, \dots, v_{m_i}^{(i)} \rangle_k, \qquad i = 1, 2,$$

$$Y_1 = \bigoplus_{j=1}^{s} \langle u_j^{(1)} + u_j^{(2)} \rangle_k,$$

$$Y_2 = \bigoplus_{j=1}^{t} \langle w_j^{(1)} + w_j^{(2)}, \tilde{w}_j^{(1)} - \tilde{w}_j^{(2)} \rangle_k.$$

We have $\text{comp}_A W = (W \cap e_1 W) \perp_A (W \cap e_2 W)$ *and* $\text{rad}_B W = \text{comp}_A W \perp_A Y_1$.

Corollary 4.2. *Two* B'*-isotropic subspaces of* V *are in the same orbit for* $\text{Sp}_A(V, B)$ *if and only if they have the same* Sp*-type.*

Proof of Theorem . (a) The subspace $\text{comp}_A W = (W \cap e_1 W) \oplus (W \cap e_2 W)$ is B-isotropic, hence each $W \cap e_i W$ is B_i-isotropic and is equal to $\langle v_1^{(i)}, \dots, v_{m_i}^{(i)} \rangle_k$ for some hyperbolic sequence $v_1^{(i)}, \tilde{v}_1^{(i)}, \dots, v_{m_i}^{(i)}, \tilde{v}_{m_i}^{(i)}$ in $(e_i V, B_i)$. By Lemma 3.2 we may now assume that $\text{comp}_A W = 0$.

(b) Take a k-subspace Y such that $W = \text{rad}_B W \oplus Y$. Then $W = \text{rad}_B W \perp_A Y$ by [**3**, Lemma 4.1]. Let $u_1, \dots, u_2$ be a basis of $\text{rad}_B W$ over K, and put $u_j^{(i)} = e_i u_j$ for $i = 1, 2$. The vectors $u_1^{(i)}, \dots, u_s^{(i)}$ are k-independent and generate a B_i-isotropic subspace in $e_i V$. By Lemma 3.2 (i) we can find vectors $\tilde{u}_1^{(i)}, \dots, \tilde{u}_s^{(i)}$ B-orthogonal to Y such that $u_1^{(i)}, \tilde{u}_1^{(i)}, \dots, u_s^{(i)}, \tilde{u}_s^{(i)}$ is a hyperbolic sequence in $(e_i V, B_i)$. Since $\text{rad}_B Y = 0$, we may now assume that $\text{rad}_B W = 0$.

(c) If $W \neq 0$, choose a nonzero vector $w = w^{(1)} + w^{(2)}$ in W, with the obvious notation. Since $\text{rad}_B W = 0$, there exists a vector

$\tilde{w} = \tilde{w}^{(1)} - \tilde{w}^{(2)} \in W$ such that $B(w, \tilde{w}) = e_1 B_1(w^{(1)}, \tilde{w}^{(1)}) - e_2 B_2(w^{(2)}, \tilde{w}^{(2)}) \neq 0$. But $B'(w, \tilde{w}) = B_1(w^{(1)}, \tilde{w}^{(1)}) - B_2(w^{(2)}, \tilde{w}^{(2)}) = 0$. Multiplying $\tilde{w}$ by a scalar in k, we may assume that $B_1(w^{(1)}, \tilde{w}^{(1)}) = B_2(w^{(2)}, \tilde{w}^{(2)}) = 1$. The subspace $\langle w, \tilde{w}\rangle_k$ is maximal B'-isotropic in the B-nondegenerate A-submodule $\langle w, \tilde{w}\rangle_A$ (compare dimensions), which allows us to conclude the proof by an application of Lemma 3.2 (ii) and induction on t. □

Remarks. (1) If W is maximal B'-isotropic, then $\text{type}_{\text{Sp}}(W) = (n_1 - t, n_2 - t, 0, t)$ for $t = 0, \ldots, \min(n_1, n_2)$. So there are $1 + \min(n_1, n_2)$ orbits of maximal B'-isotropic subspaces of V. (In the case $n_1 = 1 \leq n_2$, this gives a total of two orbits for maximal isotropic subspaces which was obtained by Garrett [**1**]. Also, if $n_1 = n_2$, one recovers the result [**2**, page 8] that $\text{Sp}_A(V)$-orbits of maximal isotropic subspaces are characterized by the unique invariant $\dim_k(W \cap e_1 V) = \dim_k(W \cap e_2 V)$.)

(2) If $\text{type}_{\text{Sp}}(W) = (m_1, m_2, s, t)$, then

$$\dim_k W = m_1 + m_2 + s + 2t \quad \text{and} \quad \mathbf{dim}_A AW = (m_1 + s + 2t, m_2 + s + 2t).$$

A necessary and sufficient condition for a quadruple $(m_1, m_2, s, t) \in \mathbf{N}^4$ to be the Sp-type of some B'-isotropic subspace of V is $m_i + s + t \leq n_i$, $i = 1, 2$.

(3) By the preceding theorem, if a B'-isotropic subspace Y has $\text{rad}_B Y = 0$, the submodule AY is B-nondegenerate (and contains Y as a maximal B'-isotropic subspace). Now let W be any B'-isotropic subspace of V. As in the quadratic extension case in [**4**], any k-subspace Y of W such that AY is B-nondegenerate satisfies $A\,\text{rad}_B W \cap AY = 0$. A first consequence is that any k-subspace Y complementary to $\text{rad}_B W$ in W has B-nondegenerate A-span and satisfies $W = \text{rad}_B W \perp_A Y$.

(4) Another consequence of (3) is that the parameter $t = (1/2)\dim_k$ $(W/\text{rad}_B W)$ can be characterized as half the maximum dimension over k of a k-subspace of W with B-nondegenerate A-span. Also $m_i + s + t$, $i = 1, 2$, is half the minimum dimension over k of a B_i-nondegenerate subspace of $e_i V$ containing $e_i W$.

The set $\text{ISO}\,(V, B')$ is partially ordered by inclusion, and the group

$\mathrm{Sp}_A(V)$ acts on it by poset automorphisms. We can form the quotient poset $\mathrm{ISO}(V,B')/\mathrm{Sp}_A(V)$, whose elements are the orbits for the action of $\mathrm{Sp}_A(V)$, with the order relation defined by $\mathcal{O}_1 \le \mathcal{O}_2$ if there exist $x \in \mathcal{O}_1$, $y \in \mathcal{O}_2$ such that $x \le y$ in the original poset $\mathrm{ISO}(V,B')$. This quotient poset is ranked, with the d-th level equal to $\mathrm{ISO}_d(V,B')/\mathrm{Sp}_A(V)$. The next result follows as in [**4**].

Theorem 4.3. *The quotient poset* $\mathrm{ISO}(V,B')/\mathrm{Sp}_A(V,B)$ *is independent of the field* k. *It is isomorphic to the poset*

$$\mathcal{P}_{n_1,n_2} = \{(m_1,m_2,s,t) \in \mathbf{N}^4 \mid m_i + s + t \le n_i, i = 1,2\}$$

with the order relation $(m_1,m_2,s,t) \le (m_1',m_2',s',t')$ *given by the five inequalities*

$$\begin{aligned} m_i &\le m_i', \qquad i = 1,2,\\ t &\le t',\\ m_i + s + t &\le m_i' + s' + t', \qquad i = 1,2. \end{aligned}$$

Here are some properties of the poset $\mathcal{P}_{n_1,n_2}$. It is a ranked poset of height n_1+n_2 with the rank of (m_1,m_2,s,t) equal to m_1+m_2+s+t. It is not a meet-semilattice in general; the quadruples (1,2,0,0) and (1,1,1,0) of rank 3 both cover the quadruples (1,1,0,0) and (0,1,1,0) of rank 2. The $1+\min(n_1,n_2)$ maximal elements in $\mathcal{P}_{n_1,n_2}$ are the Sp-types of maximal isotropic subspaces in (V,B') as in Remark (1) above. If we write $\mathcal{P}^{(d)}_{n_1,n_2} = \mathrm{ISO}_d(V,B')/\mathrm{Sp}_A(V)$ for the d-th level and let

$$\begin{aligned} b_d &= |\{(m_1,m_2,s,t) \in \mathbf{N}^4 \mid m_1+m_2+s+2t = d\}|\\ &= \begin{cases} (1/24)(d+2)(d+4)(2d+3) & \text{for } d \text{ even},\\ (1/24)(d+1)(d+3)(2d+7) & \text{for } d \text{ odd}, \end{cases} \end{aligned}$$

then we have the obvious bound

$$|\mathcal{P}^{(d)}_{n_1,n_2}| \le b_d$$

with equality when $d \le \min(n_1,n_2)$. (Garrett [**1**] obtained $|\mathcal{P}^{(1)}_{1,1}| = 3$.) The exact expression for $|\mathcal{P}^{(d)}_{n_1,n_2}|$ for larger d is a bit messy and requires consideration of a lot of cases, so we will not write it down. If we restrict ourselves to $n_1 = n_2 = n$, i.e., when V is a free A-module, we get

$$|\mathcal{P}_{n,n}^{(d)}| = \begin{cases} b_d & \text{if } 0 \le d \le n, \\ (d-n)\binom{2n-d+2}{2} \\ \quad +\frac{1}{3}(4n-2d+3)\binom{n-d/2+2}{2} & \text{if } n \le d \le 2n,\ d \text{ even}, \\ (d-n)\binom{2n-d+2}{2} \\ \quad +\frac{1}{3}(4n-2d+7)\binom{n-\lceil d/2\rceil+2}{2} & \text{if } n \le d \le 2n,\ d \text{ odd}. \end{cases}$$

This formula yields the asymptotic number of orbits of maximal B'-isotropic k-subspaces of a free A-module as stated in the introduction. The total size of the poset for the general case of $n_1 \le n_2$ (the situation is symmetric in n_1 and n_2) is

$$|\mathcal{P}_{n_1,n_2}| = \binom{n_1+3}{3}\frac{2n_2-n_1+2}{2}.$$

5. The case $A = k \times k \times k$. In this section $A = k \times k \times k$, V is an A-module of tridimension $(2n_1, 2n_2, 2n_3)$ and B is an A-valued nondegenerate symplectic form on V. We have

$$B(x,y) = e_1B_1(e_1x, e_1y) + e_2B_2(e_2x, e_2y) + e_3B_3(e_3x, e_3y),$$
$$B'(x,y) = B_1(e_1x, e_1y) + B_2(e_2x, e_2y) + B_3(e_3x, e_3y)$$

for $x, y \in V$. We let $N = n_1 + n_2 + n_3$.

Theorem 5.1. *Suppose V is a faithful A-module.*

(a) $|\mathrm{ISO}_1(V, B')/\mathrm{Sp}_A(V, B)| = 7$.

(b) *If the field k is infinite, the number of orbits in* $\mathrm{ISO}_d(V, B')/\mathrm{Sp}_A(V, B)$, $d = 0, \dots, N$, *is finite exactly when* $d = 0, 1$ *or* N.

Proof. a) The subspaces in $\mathrm{ISO}_1(V,B')$ are all one-dimensional k-subspaces of V. It was shown in [**3**] that there are exactly 7 orbits for the action of the group $\mathrm{GL}_A(V)$ on such subspaces, and the subgroup $\mathrm{Sp}_A(V)$ acts transitively on each such orbit.

b) There is clearly a unique orbit when $d=0$. The case $d=1$ is part (a). If $1<d<N$, take hyperbolic bases $u_1^{(i)}, \tilde{u}_1^{(i)}, \dots, u_{n_i}^{(i)}, \tilde{u}_{n_i}^{(i)}$ of (e_iV, B_i), $i=1,2,3$, and set $(y_1,\dots,y_{N-3}) = (u_2^{(1)}, u_3^{(1)}, \dots, u_{n_1}^{(1)}; u_2^{(2)}, u_3^{(2)}, \dots, u_{n_2}^{(2)}; u_2^{(3)}, u_3^{(3)}, \dots, u_{n_3}^{(3)})$. It is an easy computation to check that, for different values of $a\in k$, the subspaces

$$\langle u_1^{(1)}+u_1^{(2)}+u_1^{(3)}, \tilde{u}_1^{(1)}+a\tilde{u}_1^{(2)}-(1+a)\tilde{u}_1^{(3)}, y_1,\dots,y_{d-2}\rangle_k \in \mathrm{ISO}_d(V,B')$$

are in different orbits for the action of $\mathrm{Sp}_A(V,B)$. Finally, the case of maximal B'-isotropic subspaces, that is, $d=N$, follows from Corollary 5.4 below. $\square$

For the remainder of this section we shall limit our attention to the action of $\mathrm{Sp}_A(V)$ on the set $\mathrm{ISO}_N(V,B')$ of maximal B'-isotropic subspaces. It is shown in [**3**] that, in the case $A=k\times k\times k$, the number of orbits for the action of the general linear group $\mathrm{GL}_A(V)$ on the set of all k-subspaces of V is always finite, the orbits being parameterized by eight integer parameters. Since the subgroup $\mathrm{Sp}_A(V)$ of $\mathrm{GL}_A(V)$ certainly preserves these eight parameters of a subspace, we first review the general linear situation (see [**3**] for more details).

In all that follows, (i,j,l) will always indicate a permutation of $(1,2,3)$. For a k-subspace W of V, we define, following the notation in [**3**],

$$\begin{aligned}
\Lambda_i(W) &= W\cap e_iW = W\cap e_iV,\\
\Lambda_{ij}(W) &= W\cap (e_i+e_j)W = W\cap(e_i+e_j)V,\\
\tilde{\Lambda}_{ij}(W) &= \Lambda_{ij}(W)\cap(\Lambda_{il}(W)+\Lambda_{jl}(W)),\\
\tilde{\Delta}(W) &= \tilde{\Lambda}_{12}(W)+\tilde{\Lambda}_{13}(W)+\tilde{\Lambda}_{23}(W),\\
\Delta(W) &= \Lambda_{12}(W)+\Lambda_{13}(W)+\Lambda_{23}(W).
\end{aligned}$$

The GL-type of W is the 8-tuple

$$\operatorname{type}_{\mathrm{GL}}(W) = \Bigg(\dim_k \Lambda_i(W)(i=1,2,3);\ \frac{1}{2}\dim_k\left(\frac{\tilde{\Delta}(W)}{\operatorname{comp}_A W}\right);\ \dim_k\left(\frac{\Lambda_{jl}(W)}{\tilde{\Lambda}_{jl}(W)}\right)(i=1,2,3);\ \dim_k\left(\frac{W}{\Delta(W)}\right)\Bigg).$$

The parameters of $\operatorname{type}_{\mathrm{GL}}(W) = (m_1, m_2, m_3; r; s_1, s_2, s_3; t)$ are integers that satisfy $m_i + r + s_j + s_l + t \le \dim_k e_i V$, $i = 1,2,3$, and $\dim_k W = m_1 + m_2 + m_3 + 2r + s_1 + s_2 + s_3 + t$. The following gives an explicit form for k-subspaces of a given GL-type.

Theorem 5.2 [**3**, Theorem 5.1]. *If the k-subspace W of V has* GL*-type $(m_1, m_2, m_3; r; s_1, s_2, s_3; t)$, then there exist k-independent vectors $v_1^{(i)}, \dots, v_{m_i}^{(i)}; x_1^{(i)}, \dots, x_r^{(i)}; y_1^{(i,j)}, \dots, y_{s_j}^{(i,j)}; y_1^{(i,l)}, \dots, y_{s_l}^{(i,l)}; z_1^{(i)}, \dots, z_t^{(i)}$ in $e_i V$, $i = 1,2,3$, such that*

$$W = \left(\bigoplus_{i=1}^{3} \Lambda_i(W)\right) \oplus_A X \oplus_A \left(\bigoplus_{i=1}^{3} Y_i\right) \oplus_A Z$$

with

$$\Lambda_i(W) = \langle v_1^{(i)}, \dots, v_{m_i}^{(i)}\rangle_k,$$

$$X = \bigoplus_{p=1}^{r} \langle x_p^{(1)} - x_p^{(2)}, x_p^{(1)} - x_p^{(3)}\rangle_k,$$

$$Y_i = \bigoplus_{p=1}^{s_i} \langle y_p^{(j,i)} + y_p^{(l,i)}\rangle_k,$$

$$Z = \bigoplus_{p=1}^{t} \langle z_p^{(1)} + z_p^{(2)} + z_p^{(3)}\rangle_k.$$

Turning to the case of maximal B'-isotropic subspaces of V, we will prove that all have GL-type of the form $(m_1, m_2, m_3; r; 2s_1, 2s_2, 2s_3; r)$ for some nonnegative integers satisfying $m_i + r + s_j + s_l = n_i$, $i = 1,2,3$. Moreover, any two maximal B'-isotropic subspaces having the same GL-type are in the same orbit for $\operatorname{Sp}_A(V)$. So the classes in

$\mathrm{ISO}_N(V,B')/\mathrm{Sp}_A(V)$ are characterized by four nonnegative integers, say r, s_1, s_2, s_3, satisfying $r + s_j + s_l \le n_i$, $i = 1,2,3$; if desired, one could define the Sp-type of a subspace $W \in \mathrm{ISO}_N(V,B')$ by $\mathrm{type}_{\mathrm{Sp}}(W) = (r, s_1, s_2, s_3)$ such that $\mathrm{type}_{\mathrm{GL}}(W) = (n_i - r - s_j - s_l (i = 1,2,3); r; 2s_1, 2s_2, 2s_3; r)$. In the notation of Theorem 5.2, if $W \in \mathrm{ISO}_N(V,B')$, the bases of the e_iV can be chosen so that X and Z are 'paired up' (to form the subspace U in the theorem below) with U maximal B'-isotropic in the B-nondegenerate submodule $AU = AX \oplus AZ$, and each Y_i is also maximal B'-isotropic in its B-nondegenerate A-span. The precise result is as follows.

Theorem 5.3. *Let W be a maximal B'-isotropic k-subspace of V. Then* $\mathrm{type}_{\mathrm{GL}}(W) = (m_1, m_2, m_3; r; 2s_1, 2s_2, 2s_3; r)$ *for some nonnegative integers satisfying $m_i + r + s_j + s_l = n_i$, $i = 1,2,3$, and there exist hyperbolic bases $v_1^{(i)}, \tilde{v}_1^{(i)}, \dots, v_{m_i}^{(i)}, \tilde{v}_{m_i}^{(i)}; x_1^{(i)}, \tilde{x}_1^{(i)}, \dots, x_r^{(i)}, \tilde{x}_r^{(i)}; y_1^{(i,j)}, \tilde{y}_1^{(i,j)}, \dots, y_{s_j}^{(i,j)}, \tilde{y}_{s_j}^{(i,j)}; y_1^{(i,l)}, \tilde{y}_1^{(i,l)}, \dots, y_{s_l}^{(i,l)}, \tilde{y}_{s_l}^{(i,l)}$ of (e_iV, B_i), $i = 1,2,3$, such that*

$$W = \left(\bigoplus_{i=1}^{3} \Lambda_i(W)\right) \perp_A U \perp_A \left(\bigoplus_{i=1}^{3} Y_i\right)$$

with

$$\Lambda_i(W) = \langle v_1^{(i)}, \dots, v_{m_i}^{(i)} \rangle_k,$$

$$U = \bigoplus_{p=1}^{r} \langle x_p^{(1)} - x_p^{(2)}, x_p^{(1)} - x_p^{(3)}, \tilde{x}_p^{(1)} + \tilde{x}_p^{(2)} + \tilde{x}_p^{(3)} \rangle_k,$$

$$Y_i = \bigoplus_{p=1}^{s_i} \langle y_p^{(j,i)} + y_p^{(l,i)}, \tilde{y}_p^{(j,i)} - \tilde{y}_p^{(l,i)} \rangle_k.$$

Moreover, we have

$$W^\perp = \mathrm{comp}_A W = \Lambda_1(W) \perp_A \Lambda_2(W) \perp_A \Lambda_3(W),$$

$$W \cap \Lambda_{ij}(W)^\perp = \Lambda_{il}(W) + \Lambda_{jl}(W),$$

$$W \cap \tilde{\Lambda}_{ij}(W)^\perp = W \cap \tilde{\Delta}(W)^\perp = \Delta(W),$$

$$W \cap \Delta(W)^\perp = \tilde{\Delta}(W).$$

Conversely, for any choice of hyperbolic bases of $(e_i V, B_i)$ *as above, the subspace* W *above is maximal* B'*-isotropic with* GL*-type* $(m_1, m_2, m_3; r; 2s_1, 2s_2, 2s_3; r)$.

Corollary 5.4. *Two maximal* B'*-isotropic subspaces of* V *are in the same orbit for* $\mathrm{Sp}_A(V, B)$ *if and only if they have the same* GL*-type. The number of these orbits is finite and independent of the field* k.

If $n_1 = n_2 = n_3 = n$, that is V is a free A-module, then the number of such orbits equals

$$|\mathrm{ISO}_{3n}(V, B')/\mathrm{Sp}_A(V)| = \left(\frac{n}{2}+1\right)^4 + \begin{cases} 0 & \text{for } n \text{ even,} \\ -1/16 & \text{for } n \text{ odd} \end{cases}$$
$$= \left\lfloor \left(\frac{n}{2}+1\right)^4 \right\rfloor \sim \frac{n^4}{16} \quad \text{as } n \to \infty.$$

This is obtained by counting the number of quadruples $(r; s_1, s_2, s_3) \in \mathbf{N}^4$ with $r + s_j + s_l \leq n$ (all $j \neq l$), using the same method as for the proof of [**3**, Proposition 5.8].

Remark. For $\mathbf{dim}_A V = (2, 2, 2)$, this gives a total of five orbits, which is a result of Piatetski-Shapiro and Rallis [**5**].

Proof of Theorem 5.3. The subspace $\mathrm{comp}_A W = \Lambda_1(W) \oplus \Lambda_2(W) \oplus \Lambda_3(W)$ is dealt with in the same manner as in Theorem 4.1. We now assume $\mathrm{comp}_A W = 0$ and let $\mathrm{type}_{\mathrm{GL}}(W) = (0, 0, 0; r; s_1, s_2, s_3; t)$. Then we have $\mathrm{rad}_B W = W^\perp = 0$ by Lemma 3.1 (iii), whence $AW = V$. From the two equations

$$\dim_k AW = 3r + 2s_1 + 2s_2 + 2s_3 + 3t = 2N,$$
$$\dim_k W = 2r + s_1 + s_2 + s_3 + t = N,$$

it follows that $r = t$.

If $r > 0$, consider the two B-orthogonal vectors $x_1^{(1)} - x_1^{(2)}$ and $x_1^{(1)} - x_1^{(3)}$ in $\tilde{\Delta}(W)$, according to the normalization of Theorem 5.2. Since $\mathrm{rad}_B W = 0$, there exists a vector $\tilde{x} = \tilde{x}^{(1)} + \tilde{x}^{(2)} + \tilde{x}^{(3)}$ in W

(with obvious notation) such that

$$B(x_1^{(1)} - x_1^{(2)}, \tilde{x}) = e_1 B_1(X_1^{(1)}, \tilde{x}^{(1)} - e_2 B_2(x_1^{(2)}, \tilde{x}^{(2)}) \neq 0.$$

Since

$$B'(x_1^{(1)} - x_1^{(2)}, \tilde{x}) = B_1(x_1^{(1)}, \tilde{x}^{(1)}) - B_2(x_1^{(2)}, \tilde{x}^{(2)}) = 0,$$

we may multiply $\tilde{x}$ by a scalar in k and assume that $B_1(x_1^{(1)}, \tilde{x}^{(1)}) = B_2(x_1^{(2)}, \tilde{x}^{(2)}) = 1$. Then

$$B'(x_1^{(1)} - x_1^{(3)}, \tilde{x}) = B_1(x_1^{(1)}, \tilde{x}^{(1)}) - B_3(x_1^{(3)}, \tilde{x}^{(3)}) = 0$$

implies $B_3(x_1^{(3)}, \tilde{x}^{(3)}) = 1$. The subspace $\langle x_1^{(1)} - x_1^{(2)}, x_1^{(1)} - x_1^{(3)}, \tilde{x}\rangle_k$ of W is maximal B'-isotropic in its B-nondegenerate A-span; so by Lemma 3.2 (ii) we can write $W = \langle x_1^{(1)} - x_1^{(2)}, x_1^{(1)} - x_1^{(3)}, \tilde{x}\rangle_k \perp_A W_1$ for some subspace W_1 necessarily of GL-type $(0,0,0; r-1; s_1, s_2, s_3; r-1)$. Repeating this process as many times as necessary, we may assume that $\text{type}_{\text{GL}}(W) = (0,0,0;0;s_1,s_2,s_3;0)$.

Write $W = Y_1 \oplus Y_2 \oplus Y_3$ in the notation of Theorem 5.2. It is easily checked that $W = Y_1 \perp_A Y_2 \perp_A Y_3$, because W is B'-isotropic. Each AY_i can be viewed as a module over $k \times k$ equipped with a nondegenerate $(k \times k)$-valued symplectic form and containing Y_i as a maximal B'-isotropic k-subspace with zero B-radical. Thus, one is reduced to the case $A = k \times k$ and the required normalization is given by Theorem 4.1. The equalities concluding the statement of the theorem follow by inspection of the final form of W. □

6. The case $\mathbf{A = F \times k}$ with $\mathbf{F/k}$ quadratic extension. In this section $A = F \times k$ where F/k is a quadratic field extension, V is an A-module of bidimension $(2n_1, 2n_2)$, $\phi : F \to k$ is a nonzero k-linear map, B_1 and B_2 are respectively an F-valued and a k-valued nondegenerate symplectic form on e_1V and e_2V, and

$$B(x,y) = e_1 B_1(e_1x, e_1y) + e_2 B_2(e_2x, e_2y),$$
$$B'(x,y) = \phi \circ B_1(e_1x, e_1y) + B_2(e_2x, e_2y)$$

for $x, y \in V$. We fix a basis $\{\alpha, \gamma\}$ of F over k such that $\phi(\alpha) = 1$ and $\phi(\gamma) = 0$. Also, let $N = 2n_1 + n_2$.

Theorem 6.1. *Suppose V is a faithful A-module.*

(a) $|\mathrm{ISO}_1(V,B')/\mathrm{Sp}_A(V,B)| = 3$.

(b) *If the field k is infinite, the number of orbits in $\mathrm{ISO}_d(V,B')/\mathrm{Sp}_A(V,B)$, $d=0,\dots,N$, is finite exactly when $d=0,1$ or N.*

Proof. The case $d=0$ or 1 is handled as in Theorem 5.1, using the results of [**3**]. For $1<d<N$, take hyperbolic bases $u_1^{(i)},\tilde{u}_1^{(i)},\dots,u_{n_i}^{(i)},\tilde{u}_{n_i}^{(i)}$ of (e_iV,B_i), $i=1,2$, and set $(y_1,\dots,y_{N-3}) = (\alpha u_2^{(1)},\dots,\alpha u_{n_1}^{(1)};\gamma u_2^{(1)},\dots,\gamma u_{n_1}^{(1)};u_2^{(2)},\dots,u_{n_2}^{(2)})$. Then, for different values of $a\in k$, the subspaces

$$\langle u_1^{(1)}+u_1^{(2)},(\alpha+a\gamma)\tilde{u}_1^{(1)}-\tilde{u}_1^{(2)},y_1,\dots,y_{d-2}\rangle_k \in \mathrm{ISO}_d(V,B')$$

are in different orbits for the action of $\mathrm{Sp}_A(V,B)$. The case $d=N$ follows from Corollary 6.4 below. □

For the remainder of this section we focus our attention on the action of $\mathrm{Sp}_A(V)$ on the set $\mathrm{ISO}_N(V,B')$ of maximal B'-isotropic subspaces. It is shown in [**3**] that, in the case $A=F\times k$, the number of orbits for the action of $\mathrm{GL}_A(V)$ on the set of all k-subspaces of V is always finite, the orbits being parametrized by six integer parameters. Since these parameters are preserved by the subgroup $\mathrm{Sp}_A(V)$, we first need to review the general linear situation (see [**3**] for more details).

For a k-subspace W of V, let $\Lambda_i(W)=W\cap e_iW$, $i=1,2$, and define its GL-type by

$$\begin{aligned}\mathrm{type}_{\mathrm{GL}}(W)=\Big(&\dim_F \mathrm{comp}_F\Lambda_1(W),\dim_k\Lambda_2(W);\\ &\dim_F\left(\frac{U}{\mathrm{comp}_F\Lambda_1(W)}\right),\dim_F\left(\frac{F\Lambda_1(W)}{U}\right),\\ &\dim_F\left(\frac{\mathrm{comp}_F e_1W}{U}\right),\dim_F\left(\frac{Fe_1W}{F\Lambda_1(W)+\mathrm{comp}_F e_1W}\right)\Big)\end{aligned}$$

with $U=F\Lambda_1(W)\cap\mathrm{comp}_F e_1W$. The parameters of $\mathrm{type}_{\mathrm{GL}}(W) = (m_1,m_2;r,s_1,s_2,t)$ are integers that satisfy $m_1+r+s_1+s_2+t\le n_1$, $m_2+r+2s_2+t\le n_2$ and $\dim_k W = 2m_1+m_2+2r+s_1+2s_2+t$.

The following result gives an explicit form for k-subspaces of a given GL-type.

Theorem 6.2 [**3**, Theorem 6.2]. *Let $\{1, \eta\}$ be a basis of F over k. If the k-subspace W of V has* GL*-type $(m_1, m_2; r, s_1, s_2, t)$, then there exist F-independent vectors $v_1^{(1)}, \dots, v_{m_1}^{(1)}; x_1^{(1)}, \dots, x_r^{(1)}; y_1, \dots, y_{s_1}; z_1^{(1)}, \dots, z_{s_2}^{(1)}; u_1^{(1)}, \dots, u_t^{(1)}$ in e_1V and k-independent vectors $v_1^{(2)}, \dots, v_{m_2}^{(2)}; x_1^{(2)}, \dots, x_r^{(2)}; z_1^{(2)}, \dots, z_{2s_2}^{(2)}; u_1^{(2)}, \dots, u_t^{(2)}$ in e_2V such that*

$$W = \mathrm{comp}_F \Lambda_1(W) \oplus_A \Lambda_2(W) \oplus_A X_1 \oplus_A Y \oplus_A Z \oplus_A X_2$$

with

$$\mathrm{comp}_F \Lambda_1(W) = \langle v_1^{(1)}, \dots, v_{m_1}^{(1)} \rangle_F, \qquad \Lambda_2(W) = \langle v_1^{(2)}, \dots, v_{m_2}^{(2)} \rangle_k,$$

$$X_1 = \bigoplus_{p=1}^{r} \langle x_p^{(1)}, \eta x_p^{(1)} + x_p^{(2)} \rangle_k, \qquad Y = \langle y_1, \dots, y_{s_1} \rangle_k,$$

$$Z = \bigoplus_{p=1}^{s_2} \langle z_p^{(1)} + z_{2p-1}^{(2)}, \eta z_p^{(1)} + z_{2p}^{(2)} \rangle_k, \qquad X_2 = \bigoplus_{p=1}^{t} \langle u_p^{(1)} + u_p^{(2)} \rangle_k.$$

Now for the case of maximal B'-isotropic subspaces of V. It will turn out that all have GL-type of the form $(m_1, m_2; r, 2s_1, 2s_2, r)$ for some nonnegative integers satisfying $m_1 + r + s_1 + s_2 = n_1, m_2 + r + 2s_2 = n_2$. Moreover, any two maximal B'-isotropic subspaces having the same GL-type are in the same orbit for $\mathrm{Sp}_A(V)$. So the classes in $\mathrm{ISO}_N(V, B')/\mathrm{Sp}_A(V)$ are characterized by three nonnegative integers, say r, s_1, s_2, satisfying $r + s_1 + s_2 \leq n_1$, $r + 2s_2 \leq n_2$; if desired, one could define the Sp-type of a subspace $W \in \mathrm{ISO}_N(V, B')$ by $\mathrm{type}_{\mathrm{Sp}}(W) = (r, s_1, s_2)$ such that $\mathrm{type}_{\mathrm{GL}}(W) = (n_1 - r - s_1 - s_2, n_2 - r - 2s_2; r, 2s_1, 2s_2, r)$. In the notation of Theorem 6.2, if $W \in \mathrm{ISO}_N(V, B')$, the bases of the e_iV can be chosen so that X_1 and X_2 are 'paired up' (to form the subspace X in the theorem below) with $AX_1 \oplus AX_2$ B-nondegenerate, and AY and AZ also B-nondegenerate. The precise result is as follows.

Theorem 6.3. *Let W be a maximal B'-isotropic k-subspace of V. Then $\mathrm{type}_{\mathrm{GL}}(W) = (m_1, m_2; r, 2s_1, 2s_2, r)$ for some nonnegative integers satisfying $m_1 + r + s_1 + s_2 = n_1$ and $m_2 + r +$*

$2s_2 = n_2$ and there exist hyperbolic bases $v_1^{(1)}, \tilde{v}_1^{(1)}, \dots, v_{m_1}^{(1)}, \tilde{v}_{m_1}^{(1)}$; $x_1^{(1)}, \tilde{x}_1^{(1)}, \dots, x_r^{(1)}, \tilde{x}_r^{(1)}; y_1, \tilde{y}_1, \dots, y_{s_1}, \tilde{y}_{s_1}$; $z_1^{(1)}, \tilde{z}_1^{(1)}, \dots, z_{s_2}^{(1)}, \tilde{z}_{s_2}^{(1)}$ of $(e_1 V, B_1)$ and $v_1^{(2)}, \tilde{v}_1^{(2)}, \dots, v_{m_2}^{(2)}, \tilde{v}_{m_2}^{(2)}$; $x_1^{(2)}, \tilde{x}_1^{(2)}, \dots, x_r^{(2)}, \tilde{x}_r^{(2)}$; $z_1^{(2)}, \tilde{z}_1^{(2)}, \dots, z_{2s_2}^{(2)}, \tilde{z}_{2s_2}^{(2)}$ of $(e_2 V, B_2)$ such that

$$W = \operatorname{comp}_F \Lambda_1(W) \perp_A \Lambda_2(W) \perp_A X \perp_A Y \perp_A Z$$

with

$$\operatorname{comp}_F \Lambda_1(w) = \langle v_1^{(1)}, \dots, v_{m_1}^{(1)} \rangle_F, \qquad \Lambda_2(W) = \langle v_1^{(2)}, \dots, v_{m_2}^{(2)} \rangle_k,$$

$$X = \bigoplus_{p=1}^{r} \langle \gamma x_p^{(1)}, \alpha x_p^{(1)} + x_p^{(2)}, \tilde{x}_p^{(1)} - \tilde{x}_p^{(2)} \rangle_k, \qquad Y = \bigoplus_{p=1}^{s_1} \langle y_p, \gamma \tilde{y}_p \rangle_k,$$

$$Z = \bigoplus_{p=1}^{s_2} (\langle z_p^{(1)} + z_{2p-1}^{(2)}, \gamma\alpha^{-1} z_p^{(1)} + z_{2p}^{(2)} \rangle_k \oplus \langle \alpha \tilde{z}_p^{(1)} - \tilde{z}_{2p-1}^{(2)}, \gamma \tilde{z}_p^{(1)} + \tau \tilde{z}_{2p}^{(2)} \rangle_k),$$

where $\tau = N_{F/k}(\gamma\alpha^{-1})$. Moreover, if π is the map $v \mapsto e_1 v$ from W to $e_1 V$, we have

$$W^\perp = \operatorname{comp}_A W = \operatorname{comp}_F \Lambda_1(W) \oplus \Lambda_2(W),$$

$$W \cap (\pi^{-1}(F\Lambda_1(W) \cap \operatorname{comp}_F e_1 W))^\perp = \pi^{-1}(F\Lambda_1(W) + \operatorname{comp}_F e_1 W),$$
$$W \cap (\pi^{-1}(F\Lambda_1(W))^\perp = \pi^{-1}(\operatorname{comp}_F e_1 W),$$
$$W \cap (\pi^{-1}(\operatorname{comp}_F e_1 W))^\perp = \pi^{-1}(F\Lambda_1(W)),$$
$$W \cap (\pi^{-1}(F\Lambda_1(W) + \operatorname{comp}_F e_1 W))^\perp = \pi^{-1}(F\Lambda_1(W) \cap \operatorname{comp}_F e_1 W).$$

Conversely, for any choice of hyperbolic bases of $(e_i V, B_i)$ as above, the subspace W above is maximal B'-isotropic with GL *-type $(m_1, m_2; r, 2s_1, 2s_2, r)$.*

Remark. In the special case where $\alpha = 1$, $\gamma^2 + a_1\gamma + a_0 = 0$ with a_0, a_1 in k, $\phi(1) = 1$, $\phi(\gamma) = 0$, the normalization for the summands in Z takes the form

$$\langle z^{(1)} + z_1^{(2)}, \gamma z^{(1)} + z_2^{(2)} \rangle_k \oplus \langle \tilde{z}^{(1)} - \tilde{z}_1^{(2)}, \gamma \tilde{z}^{(1)} + a_0 \tilde{z}_2^{(2)} \rangle_k.$$

Corollary 6.4. *Two maximal B'-isotropic subspaces of V are in the same orbit for* $\mathrm{Sp}_A(V,B)$ *if and only if they have the same* GL*-type. The number of these orbits is finite and independent of the fields k and F.*

The number $a(n_1,n_2) = |\mathrm{ISO}_N(V,B')/\mathrm{Sp}_A(V)|$ of orbits has the following value:

(1) if $n_2 \le n_1$,

$$a(n_1,n_2) = \frac{1}{8}(2n_1-n_2+2)\times\begin{cases}(n_2+2)^2 & \text{for } n_2 \text{ even},\\ (n_2+1)(n_2+3) & \text{for } n_2 \text{ odd;}\end{cases}$$

(2) if $n_1 \le n_2 \le 2n_1$,

$$\begin{aligned}a(n_1,n_2) &= \frac{1}{6}(n_2-n_1)(7n_1^2-5n_1n_2+n_2^2+18n_1-6n_2+11)\\ &\quad+\frac{1}{8}\times\begin{cases}(2n_1-n_2+2)^3 & \text{for } n_2 \text{ even},\\ (2n_1-n_2+3)(2n_1-n_2+2) & \\ \times(2n_1-n_2+1) & \text{for } n_2 \text{ odd;}\end{cases}\end{aligned}$$

(3) if $n_2 \ge 2n_1$,

$$a(n_1,n_2) = \binom{n_1+3}{3}.$$

In particular, if $n_1 = n_2 = n$, that is, if V is a free A-module, then the number of orbits equals

$$\begin{aligned}|\mathrm{ISO}_{3n}(V,B')/\mathrm{Sp}_A(V,B)| &= \begin{cases}\frac{1}{8}(n+2)^3 & \text{for } n \text{ even},\\ \frac{1}{8}(n+1)(n+2)(n+3) & \text{for } n \text{ odd}\end{cases}\\ &\sim \frac{n^3}{8} \quad \text{as } n\to\infty.\end{aligned}$$

Remark. For $\mathbf{dim}_A V = (2,2)$, this gives a total of three orbits, which was obtained by Piatetski-Shapiro and Rallis [**5**].

Proof of Theorem 6.3. The proof is parallel to the proof of Theorem 5.3, so we will be somewhat briefer. As previously, the proof reduces to the case $\operatorname{comp}_A W = \operatorname{comp}_F \Lambda_1(W) \oplus \Lambda_2(W) = 0$, which implies $AW = V$. If $\operatorname{type}_{\mathrm{GL}}(W) = (0,0;r,s_1,s_2,t)$, then the two equations

$$\dim_k AW = 3r + 2s_1 + 4s_2 + 3t = 2N,$$
$$\dim_k W = 2r + s_1 + 2s_2 + t = N$$

imply $t = r$.

Suppose $r > 0$. Applying Theorem 6.2 with $\eta = \alpha\gamma^{-1}$ and adjusting by an appropriate scalar in F, we can find in the subspace X_1 of W two vectors of the form $\gamma x_1^{(1)}$ and $\alpha x_1^{(1)} + x_1^{(2)}$ for some nonzero $x_1^{(1)} \in e_1 V$, $x_1^{(2)} \in e_2 V$. Since $\operatorname{rad}_B W = 0$, there exists a vector $\tilde{x}^{(1)} - \tilde{x}^{(2)}$ in W with $B(\gamma x_1^{(1)}, \tilde{x}^{(1)} - \tilde{x}^{(2)}) = e_1 \gamma B_1(x_1^{(1)}, \tilde{x}^{(1)}) \neq 0$. Since $B'(\gamma x_1^{(1)}, \tilde{x}^{(1)} - \tilde{x}^{(2)}) = \phi(\gamma B_1(x_1^{(1)}, \tilde{x}^{(1)})) = 0$, that is, $\gamma B_1(x_1^{(1)}, \tilde{x}^{(1)}) \in \ker\phi = k\gamma$, we may assume that $B_1(x_1^{(1)}, \tilde{x}^{(1)}) = 1$. Then

$$B'(\alpha x_1^{(1)} + x_1^{(2)}, \tilde{x}^{(1)} - \tilde{x}^{(2)}) = 0 \Rightarrow B_2(x_1^{(2)}, \tilde{x}^{(2)}) = 1.$$

Applying Lemma 3.2 (ii) to the subspace $\langle \gamma x_1^{(1)}, \alpha x_1^{(1)} + x_1^{(2)}, \tilde{x}^{(1)} - \tilde{x}^{(2)} \rangle_k$ of W and induction, we may assume that $r = t = 0$.

In the notation of Theorem 6.2, we now have $W = Y \oplus Z$ and, since W is B'-isotrophic, one easily checks that Y and Z have to be B-orthogonal, that is, $W = Y \perp_A Z$. By Lemma 3.2 (i), the subspaces Y and Z may be treated independently of each other. The normalization for Y follows from [**4**, Theorem 5.1], using the fact that $\operatorname{rad}_B Y = 0$. We now assume that $W = Z \neq 0$.

Because of the special form of Z, $AZ = e_1 Z + e_2 Z = V$, and the projection $Z \to e_1 Z = e_1 V$ is a k-linear isomorphism. So, given a hyperbolic pair $z_1^{(1)}, \tilde{z}_1^{(1)}$ in $(e_1 V, B_1)$, W contains a subspace of the form

$$\langle z_1^{(1)} + z_1^{(2)}, \gamma\alpha^{-1} z_1^{(1)} + z_2^{(2)} \rangle_k \oplus \langle \alpha \tilde{z}_1^{(1)} - \tilde{z}_1^{(2)}, \gamma \tilde{z}_1^{(1)} + \tau \tilde{z}_2^{(2)} \rangle_k$$

with $\tau = N_{F/k}(\gamma\alpha^{-1})$. Since the projection $Z \to e_2 Z = e_2 V$ is also an isomorphism, the vectors $z_1^{(2)}, \tilde{z}_1^{(2)}, z_2^{(2)}, \tilde{z}_2^{(2)}$ are k-independent in $e_2 V$. It remains to show that they form a hyperbolic sequence in $(e_2 V, B_2)$. We have

$$B'(z_1^{(1)} + z_1^{(2)}, \alpha \tilde{z}_1^{(1)} - \tilde{z}_1^{(2)}) = 0 \Rightarrow B_2(z_1^{(2)}, \tilde{z}_1^{(2)}) = 1,$$
$$B'(z_1^{(1)} + z_1^{(2)}, \gamma \tilde{z}_1^{(1)} + \tau \tilde{z}_2^{(2)}) = 0 \Rightarrow B_2(z_1^{(2)}, \tilde{z}_2^{(2)}) = 0,$$
$$B'(\gamma\alpha^{-1} z_1^{(1)} + z_2^{(2)}, \alpha \tilde{z}_1^{(1)} - \tilde{z}_1^{(2)}) = 0 \Rightarrow B_2(z_2^{(2)}, \tilde{z}_1^{(2)}) = 0.$$

Also, $(\gamma\alpha^{-1})^2 + \tau \in k\gamma\alpha^{-1}$ implies $\gamma^2\alpha^{-1} + \tau\alpha \in k\gamma = \ker\phi$; by applying ϕ we get

$$\tau = N_{F/k}(\gamma\alpha^{-1}) = -\phi(\gamma^2\alpha^{-1}).$$

From this it follows that

$$B'(\gamma\alpha^{-1}z_1^{(1)} + z_2^{(2)}, \gamma\tilde{z}_1^{(1)} + \tau\tilde{z}_2^{(2)}) = 0 \Rightarrow B_2(z_2^{(2)}, \tilde{z}_2^{(2)}) = 1.$$

Another application of Lemma 3.2 (ii) and induction allow us to conclude the normalization of W. The remaining assertions follow by inspection of the final form of W. □

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