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CONGRUENCE NETWORKS FOR STRONG SEMILATTICES OF REGULAR SIMPLE SEMIGROUPS

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1. Introduction and summary. Normal cryptogroups (or normal bands of groups) form the class of semigroups which are strong semilattices of completely simple semigroups. We consider here the more general class of semigroups which are strong semilattices of regular simple semigroups. We denote the latter by $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ where Y is a semilattice, for each $\alpha \in Y$, S_α is a regular simple semigroup, and for $\alpha \geq \beta$, $\varphi_{\alpha,b} : S_\alpha \to S_\beta$ is a homomorphism. These homomorphisms satisfy the usual conditions and determine the multiplication of S. This is the semigroup on whose lattice of congruences $\mathcal{C}(S)$ we consider certain operators.

A congruence ρ on S can be expressed by means of a congruence aggregate $(\xi; \rho_\alpha)$ where $\xi \in \mathcal{C}(Y)$ and $\rho_\alpha \in \mathcal{C}(S_\alpha)$ are congruences satisfying certain conditions, and we write $\rho \sim (\xi; \rho_\alpha)$. We call $\operatorname{gl} \rho = \xi$ and $\operatorname{loc} \rho = (\rho_\alpha)$ the global and the local of ρ. These induce the global relation $\mathcal{G}$ and the local relation $\mathcal{L}$ on $\mathcal{C}(S)$ by

$$\lambda \mathcal{G} \rho \iff \operatorname{gl} \lambda = \operatorname{gl} \rho, \qquad \lambda \mathcal{L} \rho \iff \operatorname{loc} \lambda = \operatorname{loc} \rho.$$

Our "global and local operators" are induced by the greatest and the least elements of the equivalence classes of $\mathcal{G}$ and $\mathcal{L}$ as follows:

ρG and ρg are the greatest and the least elements $\mathcal{G}$-related to ρ, respectively,

ρL and ρl are the greatest and the least elements $\mathcal{L}$-related to ρ, respectively.

These produce the four operators G, g, L and l on $\mathcal{C}(S)$. We are interested in the semigroup generated by $A = \{G, g, L, l\}$. This semigroup will be represented by generators and relations.

As for general regular semigroups, we define $E(S)$ to be the set of idempotents of S,

$$\ker \rho = \{a \in S \mid a \rho e \text{ for some } e \in E(S)\}$$

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to be the kernel of ρ and $\operatorname{tr}\rho = \rho|_{E(S)}$ to be the trace of ρ. These induce the kernel relation $\mathcal{K}$ and the trace relation $\mathcal{T}$ on $\mathcal{C}(S)$ by

$$\lambda\mathcal{K}\rho \iff \ker\lambda = \ker\rho, \qquad \lambda\mathcal{T}\rho \iff \operatorname{tr}\lambda = \operatorname{tr}\rho.$$

Our "kernel and trace operators" are induced by the greatest and the least elements of the equivalence classes of $\mathcal{K}$ and $\mathcal{T}$ as follows:

ρK and ρk are the greatest and the least congruences $\mathcal{K}$-related to ρ, respectively,

ρT and ρt are the greatest and the least congruences $\mathcal{T}$-related to ρ, respectively.

Similarly as above, we are interested in the set $\Gamma = \{K, k, T, t\}$ of operators on $\mathcal{C}(S)$ and the semigroup generated by it. To this end, one must first represent the value of the congruence aggregates under these operators again in terms of congruence aggregates. Since this creates considerable difficulties for the operators K and T, for them we restrict our attention to the case when each S_α is completely simple, that is, S is a normal cryptogroup. In this case we can characterize the semigroup sought in terms of generators and relations $\langle\Gamma, \Sigma\rangle$.

A similar analysis can be found for Clifford semigroups (semilattices of groups) in [**7**], and it will be seen that we arrive here at the same set Σ of relations as in the case of Clifford semigroups. Also, in [**6**], we performed a similar analysis for completely simple semigroups; in that case we applied the operators in the semigroup generated by Γ to a fixed congruence on S and characterized the sublattice of $\mathcal{C}(S)$ generated by the resulting set of congruences. Already in the case of Clifford semigroups in [**7**], we stopped at the partially ordered set making up the semigroup generated by Γ since the lattice generated seemed quite out of reach. For example, the latter lattice is not modular whereas, in the case of completely simple semigroups, it is even distributive. Further similar investigations may be found in Pastijn-Trotter [**3**] and Petrich-Reilly [**8**].

The description of congruences on S in terms of congruence aggregates is taken from [**5**]; several other results from that paper are of fundamental importance for our discussion here. We further relegate the discussion of $\langle\Gamma, \Sigma\rangle$ to [**7**] since it turns out to be the same semigroup as there.

Section 2 contains a general definition of the global and the local of a congruence. Basic concepts and results concerning strong semilattices of (regular simple) semigroups taken from [**5**] can be found in Section 3. A sequence of lemmas in Section 4 leads to a representation of the semigroup generated by G, g, L and l by means of generators and relations and the related network. For a congruence aggregate for S, the values of k and t are computed in Section 5. These values are calculated for T and K in Section 6 for the case when S is a normal cryptogroup.

2. Preamble. Since we will discuss some new concepts for congruences on very special semigroups, it seems in order to first give a general definition of the notions. All undefined symbols and terminology can be found in [**4**].

Let S be any semigroup, $\mathcal{C}(S)$ be its congruence lattice and η be the least semilattice congruence on S. For any $\rho \in \mathcal{C}(S)$, define

$$\operatorname{gl} \rho = (\rho \vee \eta)/\eta$$

the *global* of ρ, and

$$\operatorname{loc} \rho = (\rho|_{a\eta}) \in \prod_{a \in S} \mathcal{C}(a\eta)$$

the *local* of ρ. On $\mathcal{C}(S)$, define the relations $\mathcal{G}$ and $\mathcal{L}$ by

$$\lambda \mathcal{G} \rho \iff \operatorname{gl} \lambda = \operatorname{gl} \rho, \qquad \lambda \mathcal{L} \rho \iff \operatorname{loc} \lambda = \operatorname{loc} \rho.$$

Clearly, for any $\lambda, \rho \in \mathcal{C}(S)$, we have

$$\lambda \mathcal{G} \rho \iff \lambda \vee \eta = \rho \vee \eta, \qquad \lambda \mathcal{L} \rho \iff \lambda \wedge \eta = \rho \wedge \eta.$$

Evidently, one could use other fundamental congruences on S instead of η to arrive to possibly new fruitful equivalences on $\mathcal{C}(S)$. If S is regular, recall that $\eta = \mathcal{D}^* = \mathcal{J}^*$.

We denote by $E(S)$ the set of idempotents of S. The equality and the universal relations on any set X are denoted by ε and ω, respectively. However, we write sometimes ε_X and ω_X, for emphasis, or ε_α and ω_α when $X = S_\alpha$. For any regular semigroup, μ and σ denote the greatest idempotent and the least group congruences, respectively. For S_α, we also write μ_α instead of μ.

We shall freely use the fact that on a regular semigroup, a congruence is uniquely determined by its kernel and trace, see, e.g., ([**2**, Corollary 2.11]).

3. Strong semilattices of semigroups and their congruences. We fix the following notation for the entire paper.

Let Y be a semilattice. For each $\alpha \in Y$, let S_α be a semigroup and assume that $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$. For any $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism, and assume that $\varphi_{\alpha,\alpha}$ is the identity mapping on S_α and $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ whenever $\alpha > \beta > \gamma$. On the set $S = \cup_{\alpha \in Y} S_\alpha$ define a multiplication by: for $a \in S_\alpha$, $b \in S_\beta$,

$$ab = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}).$$

Then S is a semigroup called a strong semilattice Y of semigroups S_α with structure homomorphisms $\varphi_{\alpha,\beta}$ denoted by $[Y; S_\alpha, \varphi_{\alpha,\beta}]$, or simply a *strong semilattice of semigroups*.

We call an element $(\rho_\alpha) \in \prod_{\alpha \in Y} \mathcal{C}(S_\alpha)$ such that

$$\text{(1)} \qquad a, b \in S_\alpha, \qquad a\rho_\alpha b, \qquad \alpha > \beta \;\Rightarrow\; a\varphi_{\alpha,\beta}\,\rho_\beta\, b\varphi_{\alpha,\beta}$$

a *local congruence* on S. If, in addition, $\xi \in \mathcal{C}(Y)$ and

$$\text{(2)} \qquad a, b \in S_\alpha, \qquad \alpha > \beta, \qquad a\varphi_{\alpha,\beta}\,\rho_\beta\, b\varphi_{\alpha,\beta},\ \alpha\xi\beta \;\Rightarrow\; a\rho_\alpha b,$$

then $(\xi; \rho_\alpha)$ is called a *congruence aggregate* for S. In such a case, define a relation $\rho_{(\xi;\rho_\alpha)}$ on S by: for $a \in S_\alpha$, $b \in S_\beta$,

$$a\rho_{(\xi;\rho_\alpha)}b \iff \alpha\,\xi\,\beta,\ a\varphi_{\alpha,\alpha\beta}\,\rho_{\alpha\beta}\, b\varphi_{\beta,\alpha\beta}.$$

Denote by $\mathcal{LC}(S)$ and $\mathcal{CA}(S)$ the sets of all local congruences and congruence aggregates for S, respectively, ordered by (componentwise) inclusion.

We now assume that, for each $\alpha \in Y$, S_α is regular and simple.

Theorem 3.1 ([**5**, Theorem 4.2]). *For every* $(\xi;\rho_\alpha)\in\mathcal{CA}(S)$, *we have* $\rho_{(\xi;\rho_\alpha)}\in\mathcal{C}(S)$. *Conversely, let* $\rho\in\mathcal{C}(S)$, *define* ξ *on* Y *by*

$$\alpha\,\xi\,\beta \iff a\rho u,\ v\rho b \qquad \text{for some } a\in S_\alpha,\ u,v\in S_{\alpha\beta},\ \beta\in S_\beta$$

and, for each $\alpha\in Y$, *define* $\rho_\alpha=\rho|_{S_\alpha}$. *Then* $(\xi;\rho_\alpha)\in\mathcal{CA}(S)$ *and* $\rho=\rho(\xi;\rho_\alpha)$.

The mapping $(\xi;\rho_\alpha)\to\rho_{(\xi;\rho_\alpha)}$ *is a lattice isomorphism of* $\mathcal{CA}(S)$ *onto* $\mathcal{C}(S)$.

We will write $\rho\sim(\xi;\rho_\alpha)$ for the above correspondence and will identify the two concepts when convenient to do so. Next we determine the lattice operations on $\mathcal{CA}(S)$. If $\varphi:U\to V$ is a homomorphism of semigroups and $\rho\in\mathcal{C}(V)$, define a relation $\rho\varphi^{-1}$ by

$$x\rho\varphi^{-1}y \iff x\varphi\rho y\varphi \qquad (x,y\in U).$$

Clearly $\rho\varphi^{-1}\in\mathcal{C}(U)$.

Theorem 3.2 ([**5**, Theorem 4.4]). *For* $(\xi;\rho_\alpha),(\xi';\rho'_\alpha)\in\mathcal{CA}(S)$, *we have*

(i) $(\xi;\rho_\alpha)\wedge(\xi';\rho'_\alpha)=(\xi\wedge\xi';\rho_\alpha\wedge\rho'_\alpha)$,

(ii) $(\xi;\rho_\alpha)\vee(\xi';\rho'_\alpha)=(\xi\vee\xi';\vee\{(\rho_\beta\vee\rho'_\beta)\varphi^{-1}_{\alpha,\beta}|\beta\leq\alpha\ \xi\vee\xi'\beta\})$.

Since each S_α is simple and the equivalence relation on S whose classes are the S_α's is a semilattice congruence, this equivalence coincides both with the Green relation $\mathcal{J}$ and with the least semilattice congruence η on S. Now slightly modifying the concepts introduced in the preceding section, we arrive at the following definitions.

For $\rho\sim(\xi;\rho_\alpha)$, we call

$$\operatorname{gl}\rho=\operatorname{gl}(\xi;\rho_\alpha)=\xi, \qquad \operatorname{loc}\rho=\operatorname{loc}(\xi;\rho_\alpha)=(\rho_\alpha)$$

the *global* and the *local* of both ρ and $(\xi;\rho_\alpha)$, respectively. These two concepts give rise to the following relations on $\mathcal{C}(S)$ and $\mathcal{CA}(S)$:

$$\lambda\mathcal{G}\rho \iff \operatorname{gl}\lambda=\operatorname{gl}\rho, \qquad \lambda\mathcal{L}\rho \iff \operatorname{loc}\lambda=\operatorname{loc}\rho.$$

In order to study the structure of these relations, we need the following constructs.

Let $\xi \in \mathcal{C}(Y)$. For each $\alpha \in Y$, define a relation $\xi\theta$ on S_α by

$$a\xi\theta b \iff a\varphi_{\alpha,\beta} = b\varphi_{\alpha,\beta} \qquad \text{for some } \beta \le \alpha\xi\beta.$$

Next let $(\rho_\alpha) \in \mathcal{LC}(S)$. For any $\alpha \ge \beta$ define

$$\alpha\zeta\beta \iff (a, b \in S_\alpha,\ a\varphi_{\alpha,\beta}\, \rho_\beta\, b\varphi_{\alpha,\beta} \ \Rightarrow\ a\rho_\alpha b).$$

On Y define a relation $(\rho_\alpha)\kappa$ by

$$\alpha(\rho_\alpha)\kappa\beta \iff \text{for every } \gamma \in Y,\ \alpha\gamma\zeta\alpha\beta\gamma,\ \beta\gamma\zeta\alpha\beta\gamma.$$

Theorem 3.3 ([**5**, 4.5–4.8]). *The mapping*

$$\mathrm{gl}\ : (\xi; \rho_\alpha) \to \xi \qquad ((\xi; \rho_\alpha) \in \mathcal{CA}(S))$$

is a homomorphism of $\mathcal{CA}(S)$ onto $\mathcal{C}(Y)$ which induces $\mathcal{G}$. For any $\delta = (\xi; \rho_\alpha) \in \mathcal{CA}(S)$, we have $\delta\mathcal{G} = [\delta g, \delta G]$ where $\delta g = (\xi; \xi\theta)$ and $\delta G = (\xi; \omega_\alpha)$.

The mapping

$$\mathrm{loc}\ : (\xi; \rho_\alpha) \to (\rho_\alpha) \qquad ((\xi, \rho_\alpha) \in \mathcal{CA}(S))$$

is a $\wedge$-homomorphism of $\mathcal{CA}(S)$ onto $\mathcal{LC}(S)$ which induces $\mathcal{L}$. For any $\delta = (\xi; \rho_\alpha) \in \mathcal{CA}(S)$, we have $\delta\mathcal{L} = [\delta l, \delta L]$ where $\delta l = (\varepsilon; \rho_\alpha)$ and $\delta L = ((\rho_\alpha)\kappa; \rho_\alpha)$.

Note that for any $\delta \in \mathcal{CA}(S)$, $\delta = \delta g \vee \delta l = \delta G \wedge \delta L$ and, as a consequence, $\mathcal{G} \cap \mathcal{L} = \varepsilon$. Hence, every congruence is uniquely determined by its global and its local.

4. The semigroup generated by G, g, L and l. By means of the results of the preceding section, using a sequence of lemmas we are able here to represent the semigroup evoked above in terms of generators and relations. In addition, we characterize the associated partially ordered set of congruences of the form ρ, ρG, ρg, ρL, $\rho l, \dots$ for a fixed congruence ρ on S which we term the network of ρ.

Notice that $\mathcal{J} \sim (\varepsilon; \omega_\alpha)$ and that, for any $\xi = (\xi; \rho_\alpha) \in \mathcal{CA}(S)$, using Theorems 3.2 and 3.3, we have

$$(\xi; \rho_\alpha) \wedge (\varepsilon; \omega_\alpha) = (\varepsilon; \rho_\alpha) = \xi l,$$
$$(\xi; \rho_\alpha) \vee (\varepsilon; \omega_\alpha) = (\xi; \omega_\alpha) = \xi G.$$

Transferring all of the above to congruences, we can write: for any $\rho \in \mathcal{C}(S)$,

$$\rho l = \rho \wedge \mathcal{J}, \qquad \rho G = \rho \vee \mathcal{J}.$$

We provide next an interesting characterization of a congruence aggregate.

Lemma 4.1. *Let $\xi \in \mathcal{C}(Y)$ and $(\rho_\alpha) \in \mathcal{LC}(S)$. Then $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$ if and only if $\xi \subseteq (\rho_\alpha)\kappa$.*

Proof. Necessity. Let $\alpha > \beta$ and $\alpha \, \xi \, \beta$. Then $a\varphi_{\alpha,\beta}\rho_\beta b\varphi_{\alpha,\beta}$ implies $a\rho_\alpha b$ by the definition of a congruence aggregate. Hence $\xi \subseteq \zeta$ in the definition of $(\rho_\alpha)\kappa$. Also, for $\alpha, \beta, \gamma \in Y$ such that $\alpha \, \xi \, \beta$, we have

$$\alpha\gamma \geq \alpha\beta\gamma, \qquad \beta\gamma\xi\alpha\beta\gamma, \qquad \beta\gamma \geq \alpha\beta\gamma, \qquad \beta\gamma\xi\alpha\beta\gamma,$$

and hence $\alpha\gamma\zeta\alpha\beta\gamma$ and $\beta\gamma\zeta\alpha\beta\gamma$ so that $\alpha(\rho_\alpha)\kappa\beta$.

Sufficiency. Let $\alpha \geq \beta$, $\alpha \, \xi \, \beta$ and $a\varphi_{\alpha,\beta}\rho_\beta b\varphi_{\alpha,\beta}$. Then $\alpha(\rho_\alpha)\kappa\beta$ which implies that $\alpha\rho_\alpha b$. Therefore $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$. $\square$

It follows from Theorem 3.3 that θ maps $\mathcal{C}(Y)$ into $\mathcal{LC}(S)$ and κ maps $\mathcal{LC}(S)$ into $\mathcal{C}(Y)$. We will need further properties of these functions.

Lemma 4.2. *Let $\xi \in \mathcal{C}(Y)$ and $(\rho_\alpha) \in \mathcal{LC}(S)$.*

(i) *θ preserves inclusion.*

(ii) *$\xi \subseteq \xi\theta\kappa$.*

(iii) *$\xi\theta = \xi\theta\kappa\theta$.*

(iv) *$\varepsilon\theta = (\varepsilon_\alpha)$.*

(v) $(\rho_\alpha)\kappa\theta \subseteq (\rho_\alpha)$.

(vi) $(\omega_\alpha)\kappa = \omega$.

Proof. (i) This follows directly from the definition of θ.

(ii) This follows from

$$(\xi;\xi\theta) \leq (\xi;\xi\theta)L = (\xi\theta\kappa;\xi\theta).$$

(iii) By parts (ii) and (i), we have $\xi\theta \subseteq \xi\theta\kappa\theta$. Also,

$$(\xi;\xi\theta)Lg = (\xi\theta\kappa;\xi\theta)g = (\xi\theta\kappa;\xi\theta\kappa\theta) \leq (\xi;\xi\theta)L = (\xi\theta\kappa;\xi\theta)$$

so that $\xi\theta\kappa\theta \subseteq \xi\theta$ and equality prevails.

(iv) Indeed, $(\varepsilon;\varepsilon_\alpha)g = (\varepsilon;\varepsilon\theta) = (\varepsilon;\varepsilon_\alpha)$.

(v) This follows from

$$(\varepsilon;\rho_\alpha)L = ((\rho_\alpha)\kappa;\rho_\alpha) \geq ((\rho_\alpha)\kappa;\rho_\alpha)g = ((\rho_\alpha)\kappa;(\rho_\alpha)\kappa\theta).$$

(vi) Indeed, $(\omega;\omega_\alpha)L = ((\omega_\alpha)\kappa;\omega_\alpha) = (\omega;\omega_\alpha)$. $\square$

The next lemma lists some constants in the iteration of the operators G, g, L, l.

Lemma 4.3. *Let $\rho \in \mathcal{C}(S)$, $\rho \sim (\xi;\rho_\alpha)$. Denote by ψ the least simple congruence on S and by ζ the greatest congruence on S whose meet with $\mathcal{J}$ is the equality relation.*

(i) $\rho GL = \omega \sim (\omega;\omega_\alpha)$.

(ii) $\rho GLg = \psi \sim (\omega;\omega\theta)$.

(iii) $\rho lG = \rho Gl = \mathcal{J} \sim (\varepsilon;\omega_\alpha)$.

(iv) $\rho GLgl = \psi \wedge \mathcal{J} \sim (\varepsilon;\omega\theta)$.

(v) $\rho lg = \varepsilon \sim (\varepsilon;\varepsilon_\alpha)$.

(vi) $\rho lgL = \zeta \sim ((\varepsilon_\alpha)\kappa;\varepsilon_\alpha)$

(vii) $\rho lgLG = \zeta \vee \mathcal{J} \sim ((\varepsilon_\alpha)\kappa;\omega_\alpha)$.

(viii) $\rho gL = \rho gLg \sim (\zeta\theta\kappa;\zeta\theta)$.

Proof. Indeed,

(i) $(\xi;\rho_\alpha)GL = (\xi;\omega_\alpha)L = (\omega;\omega_\alpha) \sim \omega$ by Lemma 4.2(vi).

(ii) $(\xi;\rho_\alpha)GLg = (\omega;\omega\theta) \sim \psi$.

(iii) $(\xi;\rho_\alpha)lG=(\varepsilon;\rho_\alpha)G=(\varepsilon;\omega_\alpha)$, $(\xi;\rho_\alpha)Gl=(\xi;\omega_\alpha)l=(\varepsilon;\omega_\alpha) \sim \mathcal{J}$.

(iv) $(\xi;\rho_\alpha)GLgl = (\varepsilon;\omega\theta) \sim \psi \wedge \mathcal{J}$.

(v) $(\xi;\rho_\alpha)lg = (\varepsilon;\rho_\alpha)g = (\varepsilon;\varepsilon_\alpha) \sim \varepsilon$ by Lemma 4.2 (iv).

(vi) $(\xi;\rho_\alpha)lgL = ((\varepsilon_\alpha)\kappa;\varepsilon_\alpha) \sim \zeta$.

(vii) $(\xi;\rho_\alpha)lgLG = ((\varepsilon_\alpha)\kappa;\varepsilon_\alpha) \sim \zeta \vee \mathcal{J}$.

(viii) $(\xi;\rho_\alpha)gL = (\xi;\xi\theta)L = (\xi\theta\kappa;\xi\theta) = (\xi\theta\kappa;\xi\theta\kappa\theta) = (\xi;\rho_\alpha)gLg$ by Lemma 4.2 (iii). □

In view of Lemma 4.3 (i)–(vii), we may drop ρ in those expressions and use the following notation:

$$\varepsilon = lg, \qquad \zeta = lgL, \qquad \zeta \vee \mathcal{J} = lgLG, \qquad \mathcal{J} = lG,$$
$$\omega = GL, \qquad \psi = GLg, \qquad \psi \wedge \mathcal{J} = GLgl.$$

Note that the second line is obtained from the first by the transformation $G \leftrightarrow l$, $L \leftrightarrow g$. Finally, let

$$A = \{G, g, L, l\}, \qquad B = \{\varepsilon, \psi, \mathcal{J}, \zeta, \psi \wedge \mathcal{J}, \zeta \vee \mathcal{J}, \omega\}.$$

Lemma 4.4. *Operators A satisfy the following relations*

$$\begin{aligned} R = \{&\text{(i)} && G^2 = gG = G, \qquad g^2 = Gg = g,\\ &&& l^2 = Ll = l, \qquad L^2 = lL = L,\\ &\text{(ii)} && LGL = GLG = GL,\\ &&& glg = lgl = lg,\\ &\text{(iii)} && gLg = gL,\\ &\text{(iv)} && lG = Gl\}. \end{aligned}$$

Proof. The argument is essentially the same as in the proof of ([**7**, Lemma 6) if we substitute

$$K \to L, \qquad k \to l, \qquad T \to G, \qquad t \to g \tag{3}$$

and use Lemma 4.3. □

All this leads to the first principal result of the paper.

Theorem 4.5. *The set*

$$Q = \{L, LG, Lg, LgL, Lgl, LgLG, l, g, gl, gL, gLG, G\} \cup B$$

is a system of representatives for the congruence on A^+ generated by the relations R. These can be given the multiplication of representatives, thereby providing an isomorphic copy of $\langle A, R\rangle$.

Proof. The argument here is again literally the same as in the proof of [**7**, Theorem 1] under the substitution (3). All the above has a faithful analogue in the discussion prior to the result cited. The part of that proof concerning nonrelatedness of any two different elements of Ω carries over to this case since the example used there refers to a Clifford semigroup. But in a Clifford semigroup, the association (3) becomes identity and thus the example in [**7**] can serve our purpose here as well. □

Transformation (3) can be used to obtain the $\mathcal{D}$-structure of Q from that of Ω in ([**7**, Proposition 1]).

Similarly as in [**7**], we may deduce the following result.

Theorem 4.6. *Let S be a strong semilattice of regular simple semigroups. The semigroup $Q(S)$ generated by the operators G, g, L and l on $\mathcal{C}(S)$ is a homomorphic image of Q. For the semigroup S in ([**7**, Example 1]), we have $Q(S) \cong Q$.*

5. Kernels and traces for strong semilattices of semigroups. We continue with the same notation and now embark upon the same type of analysis relative to kernels and traces instead of globals and locals. In the generality of S considered heretofore, we represent here, for any $\zeta \in \mathcal{CA}(S)$, ζk and ζt as a congruence aggregate.

Let T be a regular semigroup and $\rho \in \mathcal{C}(T)$. Then

$$\ker \rho = \{a \in T \mid a\,\rho\, e \text{ for some } e \in E(T)\}, \qquad \operatorname{tr} \rho = \rho|_{E(T)}$$

are, respectively, the *kernel* and the *trace* of ρ. The relations on $\mathcal{C}(T)$ defined by

$$\lambda\mathcal{K}\rho \iff \ker\lambda = \ker\rho, \qquad \lambda\mathcal{T}\rho \iff \operatorname{tr}\lambda = \operatorname{tr}\rho$$

are, respectively, the kernel and the trace relations on $\mathcal{C}(T)$.

For any relation τ on T, τ^* denotes the congruence on T generated by τ. If τ is an equivalence relation on T, then τ^0 denotes the greatest congruence on T contained in τ. In fact,

$$a\tau^0 b \iff xay\,\tau\,xby \qquad \text{for all } x, y \in T^1 \qquad (a, b \in T).$$

If $\varnothing \neq A \subseteq T$, letting $\tau = (A \times A) \cup [(T\backslash A) \times (T\backslash A)]$, we set $\pi_A = \tau^0$ (π_A is the principal congruence on T relative to A).

Theorem 5.1 ([**2**, Theorem 3.2]). *For any $\rho \in \mathcal{C}(T)$, we have*

$$\rho\mathcal{K} = [\rho k, \rho K], \qquad \rho\mathcal{T} = [\rho t, \rho T]$$

where, with $\tau = \operatorname{tr}\rho$,

$$\rho k = \{(a, a^2) \mid a \in \ker\rho\}^*, \qquad \rho K = \pi_{\ker\rho},$$
$$\rho t = \tau^*, \qquad \rho T = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0.$$

We are interested in the semigroup generated by K, k, T and t as operators on $\mathcal{C}(S)$. To simplify our discussion, we will occasionally identify a congruence ρ with its congruence aggregate. We thus may speak of $\ker(\xi; \rho_\alpha)$ or $\operatorname{tr}(\xi; \rho_\alpha)$, etc., for $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$, and importantly notice that

$$\ker(\xi; \rho_\alpha) = \bigcup_{\alpha \in Y} \ker\rho_\alpha.$$

Thus, the information concerning the kernel is contained in the local of ρ. For the trace, we have the following situation.

Lemma 5.2. *Let $(\xi; \rho_\alpha)$, $(\xi'; \rho'_\alpha) \in \mathcal{CA}(S)$. Then $\operatorname{tr}(\xi; \rho_\alpha) \subseteq \operatorname{tr}(\xi'; \rho'_\alpha)$ if and only if $\xi \subseteq \xi'$ and $\operatorname{tr}\rho_\alpha \subseteq \operatorname{tr}\rho'_\alpha$ for every $\alpha \in Y$.*

Proof. Let $\rho \sim (\xi; \rho_\alpha)$ and $\rho' \sim (\xi'; \rho'_\alpha)$.

Necessity. Let $\alpha\,\xi\,\beta$. Then, for $e \in E(S_\alpha)$, we have $e\rho e\varphi_{\alpha,\alpha\beta}$ and thus $e\rho' e\varphi_{\alpha,\alpha\beta}$ which yields $\alpha\,\xi'\alpha\beta$. By symmetry, we also have $\beta\xi'\alpha\beta$ and thus $\alpha\,\xi'\beta$. Hence $\xi \subseteq \xi'$. Trivially, $\operatorname{tr}\rho_\alpha \subseteq \operatorname{tr}\rho'_\alpha$ for all $\alpha \in Y$.

Sufficiency. Let $e \in E(S_\alpha)$, $f \in E(S_\beta)$, $e\rho f$. Then $\alpha\,\xi\,\beta$ and $e\varphi_{\alpha,\alpha\beta}\rho_{\alpha\beta} f\varphi_{\beta,\alpha\beta}$. The hypothesis implies that $\alpha\,\xi'\,\beta$ and

$e\varphi_{\alpha,\alpha\beta}\rho'_{\alpha\beta} f\varphi_{\beta,\alpha\beta}$ so that $e\rho' f$. Therefore $\operatorname{tr}\rho \subseteq \operatorname{tr}\rho'$. □

Clearly, Lemma 5.2 remains valid with equality written instead of inclusion everywhere. Our first order of business is, for a given $\rho \sim (\xi'; \rho_\alpha)$, to find the congruence aggregates for ρK, ρk, ρT and ρt.

In order to treat ρk and ρt, we need some preparation. For any $(\rho_\alpha) \in \mathcal{LC}(S)$ and any operator P on congruences, in the case that $(\rho_\alpha P) \in \mathcal{LC}(S)$, we say that P is *compatible*.

Lemma 5.3. *Operators k and t are compatible.*

Proof. Let $(\rho_\alpha) \in \mathcal{LC}(S)$ and $\alpha > \beta$.

First, let $a\rho_\alpha kb$. By Theorem 5.1, there exists a sequence

$$a = u_1 y_1 v_1, \qquad u_i z_i v_i = u_{i+1} y_{i+1} v_{i+1}, \qquad u_n z_n v_n = b$$

with $u_i, v_i \in S^1_\alpha$, $y_i, z_i \in S_\alpha$, and either $z_i = y_i^2$, $y_i \in \ker\rho_\alpha$ or $y_i = z_i^2$, $z_i \in \ker\rho_\alpha$ for $i = 1, 2, \dots, n-1$. Now applying $\varphi_{\alpha,\beta}$ to all elements of the above sequence, with $1\varphi_{\alpha,\beta} = 1$, we obtain an analogous sequence in S_β and hence $a\varphi_{\alpha,\beta}\rho_\beta kb\varphi_{\alpha,\beta}$. Therefore, k is compatible.

Next let $a\rho_\alpha tb$. Then there is a sequence as above with the only modification being the conditions on y_i and z_i, which in this case are $y_i, z_i \in E(S_\alpha)$ and $y_i\rho_\alpha z_i$. All this is again carried by $\varphi_{\alpha,\beta}$ to S_β. Therefore, also t is compatible. □

Theorem 5.4. *For any $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$, we have $(\xi; \rho_\alpha)k = (\varepsilon; \rho_\alpha)$.*

Proof. By Lemma 5.3, $(\varepsilon;\rho_\alpha k)\in\mathcal{CA}(S)$, and also

$$\ker(\xi;\rho_\alpha)=\bigcup_{\alpha\in Y}\ker\rho_\alpha=\bigcup_{\alpha\in Y}\ker\rho_\alpha k=\ker(\varepsilon;\rho_\alpha k).$$

Let $(\eta;\lambda_\alpha)\in\mathcal{CA}(S)$ be such that $\ker(\eta;\lambda_\alpha)=\ker(\xi;\rho_\alpha)$. Then $\ker\lambda_\alpha=\ker\rho_\alpha$ and thus $\rho_\alpha k\subseteq\lambda_\alpha$ for every $\alpha\in Y$. But then $(\varepsilon;\rho_\alpha k)\subseteq(\eta;\lambda_\alpha)$ which proves the minimality of $(\varepsilon;\rho_\alpha k)$. □

We will often use the following notation. Let $\xi\in\mathcal{C}(Y)$ and $(\rho_\alpha)\in\mathcal{LC}(S)$. For each $\alpha\in Y$, define a relation ρ_α^ξ on S_α by

$$a\rho_\alpha^\xi b\iff a\varphi_{\alpha,\gamma}\rho_\gamma b\varphi_{\alpha,\gamma}\qquad\text{for some }\gamma\le\alpha\,\xi\,\gamma.$$

Note that $\varepsilon_\alpha^\xi=\xi\theta$ in the notation introduced in Section 3.

Lemma 5.5. *For any $\xi\in\mathcal{C}(Y)$ and $(\rho_\alpha)\in\mathcal{LC}(S)$, we have $(\xi;\rho_\alpha^\xi)\in\mathcal{CA}(S)$.*

Proof. Clearly ρ_α^ξ is reflexive and symmetric. Let $a\rho_\alpha^\xi b$ and $b\rho_\alpha^\xi c$ with

$$a\varphi_{\alpha,\gamma}\,\rho_\gamma\,b\varphi_{\alpha,\gamma},\qquad b\varphi_{\alpha,\delta}\,\rho_\delta\,c\varphi_{\alpha,\delta},\qquad\gamma,\delta\le\alpha,\qquad\gamma\,\xi\,\alpha\,\xi\,\delta.$$

Since $(\rho_\alpha)\in\mathcal{LC}(S)$, we obtain

$$a\varphi_{\alpha,\gamma\delta}\,\rho_{\gamma\delta}\,b\varphi_{\alpha,\gamma\delta}\,\rho_{\gamma\delta}\,c\varphi_{\alpha,\gamma\delta},\qquad\gamma\delta\le\alpha\,\xi\,\gamma\delta,$$

and thus $a\rho_\alpha^\xi c$ and ρ_α^ξ is also transitive.

Next let $a\rho_\alpha^\xi b$ with the above notation and $c\in S_\alpha$. Then

$$(ac)\varphi_{\alpha,\gamma}=(a\varphi_{\alpha,\gamma})(c\varphi_{\alpha,\gamma})\rho_\gamma(b\varphi_{\alpha,\gamma})(c\varphi_{\alpha,\gamma})=(bc)\varphi_{\alpha,\gamma},$$

and thus $ac\rho_\alpha^\xi bc$. By symmetry, we conclude that $\rho_\alpha^\xi\in\mathcal{C}(S_\alpha)$.

Continuing with the same notation for $a\rho_\alpha^\xi b$, let $\alpha>\beta$. Then $a\varphi_{\alpha,\beta\gamma}\rho_{\beta\gamma}b\varphi_{\alpha,\beta\gamma}$ with $\beta\gamma\le\beta=\beta\alpha\xi\beta\gamma$, and hence

$$(a\varphi_{\alpha,\beta})\varphi_{\beta,\beta\gamma}\eta_{\beta\gamma}(b\varphi_{\alpha,\beta})\varphi_{\beta,\beta\gamma},$$

so that $a\varphi_{\alpha,\beta}\rho_\beta^\xi b\varphi_{\beta,\gamma}$. Consequently, $(\rho_\alpha^\xi) \in \mathcal{LC}(S)$.

Next, let $a, b \in S_\alpha$, $\alpha > \beta$, $a\varphi_{\alpha,\beta}\rho_\beta^\xi b\varphi_{\alpha,\beta}$, $\alpha\xi\beta$. Then

$$(\alpha\varphi_{\alpha,\beta})\varphi_{\beta,\gamma}\rho_\gamma(b\varphi_{\alpha,\beta})\varphi_{\beta,\gamma}$$

for some $\gamma \leq \beta\xi\gamma$. It follows that $a\varphi_{\alpha,\gamma}\rho_\gamma b\varphi_{\alpha,\gamma}$ with $\gamma \leq \alpha\,\xi\,\gamma$ and thus $a\rho_\alpha^\xi b$. Therefore, $(\xi; \rho_\alpha^\xi) \in \mathcal{CA}(S)$.

Theorem 5.6. *For any* $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$, *we have* $(\xi; \rho_\alpha)t = (\xi; (\rho_\alpha t)^\xi)$.

Proof. In view of Lemma 5.3, we have $(\rho_\alpha t) \in \mathcal{LC}(S)$ and hence by Lemma 5.5, we get $(\xi; (\rho_\alpha t)^\xi) \in \mathcal{CA}(S)$. Let $\lambda \sim (\xi; (\rho_\alpha t)^\xi) = (\xi; \lambda_\alpha)$.

Note that the notation $(\rho_\alpha t)^\xi$ means: apply the operator ξ to the congruence $\rho_\alpha t$ on S_α and thus $(\rho_\alpha t)^\xi$ in this context may not be an element of $\prod_{\alpha \in Y} \mathcal{C}(S_\alpha)$.

Let $a\lambda_\alpha b$. Then $a\varphi_{\alpha,\gamma}\rho_\gamma tb\varphi_{\alpha,\gamma}$ for some $\gamma \leq \alpha\xi\gamma$. Hence $a\varphi_{\alpha,\gamma}\rho_\gamma b\varphi_{\alpha,\gamma}$ which together with $\gamma\xi\alpha$ implies that $a\rho_\alpha b$. Therefore, $\lambda_\alpha \subseteq \rho_\alpha$ and thus $\lambda \subseteq \rho$. Next, let $e, f \in E(S_\alpha)$ and $e\rho_\alpha f$. Then $e\rho_\alpha tf$ and thus $e\lambda_\alpha f$. Hence, $\operatorname{tr}\rho_\alpha \subseteq \operatorname{tr}\lambda_\alpha$ and equality prevails. Now Lemma 5.2 gives that $\operatorname{tr}\lambda = \operatorname{tr}\rho$.

Finally, let $\theta \in \mathcal{C}(S)$ be such that $\operatorname{tr}\theta = \operatorname{tr}\rho$. Then, by Lemma 5.2, $\theta \sim (\xi; \theta_\alpha)$ with $\operatorname{tr}\theta_\alpha = \operatorname{tr}\rho_\alpha$ for every $\alpha \in Y$. It follows that $\theta_\alpha t = \rho_\alpha t$ for every $\alpha \in Y$. Now let $a\lambda_\alpha b$. Then $a\varphi_{\alpha,\gamma}\rho_\gamma tb\varphi_{\alpha,\gamma}$ for some $\gamma \leq \alpha\xi\gamma$. It follows that $a\varphi_{\alpha,\gamma}\theta_\gamma tb\varphi_{\alpha,\gamma}$ whence $a\varphi_{\alpha,\gamma}\theta_\gamma b\varphi_{\alpha,\gamma}$ which together with $\gamma\xi\alpha$ yields $a\theta_\alpha b$. Therefore, $\lambda_\alpha \subseteq \theta_\alpha$ so that $\lambda \subseteq \theta$, establishing minimality of λ. $\square$

It seems natural to attempt a similar argument for describing the congruence aggregates for ρK and ρT. First, Example 6.8 in the next section shows that K is not compatible. The deeper reason why this may not be possible is the form of ρK and ρT. For ρK and ρT are of the form $(\quad)^\circ$, which, by its expression bears upon the whole semigroup T rather than only upon parameters intimately related to ρ, as this is the case with ρk and ρt which are of the general form $(\quad)^*$. We will partly overcome this difficulty in the next section by imposing the additional hypothesis that each S_α be completely simple.

6. The congruences ρT and ρK for normal cryptogroups. A completely regular semigroup V is the union of its subgroups. If the Green relation $\mathcal{H}$ is a congruence on S, then V is a cryptogroup. If, in addition, $V/\mathcal{H}$ is a normal band (that is, satisfies the identity $axya = axya$), then V is a normal cryptogroup (also called a normal band of groups). According to ([**4**, IV. 4.3]), normal cryptogroups coincide with strong semilattices of completely simple semigroups. For any $a \in V$, a^{-1} denotes the inverse of a in the maximal subgroup of V containing a and $a^0 = aa^{-1}(= a^{-1}a)$.

We thus specialize the case of S in the previous sections to be of the form $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ where S_α is completely simple for every $\alpha \in Y$. We may further assume that $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$ is a Rees matrix semigroup with P_α normalized at some element $1_\alpha \in I_\alpha \cap \Lambda_\alpha$. We denote the identity of G_α by e_α.

Let $V = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. An admissible triple (r, N, π) for S consists of a partition r of I, a normal subgroup N of G and a partition π of Λ satisfying the conditions:

$$irj \;\Rightarrow\; p_{\lambda i}p_{\lambda j}^{-1} \in N, \qquad \lambda\pi\mu \;\Rightarrow\; p_{\lambda i}p_{\mu i}^{-1} \in N.$$

If so, then $\rho = \rho_{(r,N,\pi)}$ defined by

$$(i, g, \lambda)\rho(j, h, \mu) \iff irj, \qquad gh^{-1} \in N, \; \lambda\pi\mu \tag{4}$$

is a congruence on V, to be denoted by $\rho \sim (r, N, \pi)$. Conversely, every congruence on V can be so represented uniquely. Giving admissible triples componentwise order, we obtain an isomorphic copy of $\mathcal{C}(V)$. We may thus consider the operators K, k, T and t as acting directly on the set $\mathcal{AT}(V)$ of admissible triples for V.

For $\zeta = (r, N, \pi) \in \mathcal{AT}(V)$, define r_N, $\overline{r\pi}$ and π_N by

$$ir_Nj \iff p_{\lambda i}p_{\lambda j}^{-1} \in N \qquad \text{for all } \lambda \in \Lambda \qquad (i, j \in I),$$

$\overline{r\pi}$ is the normal subgroup of G generated by the set

$$\{p_{\lambda i}p_{\lambda j}^{-1} \mid irj, \lambda \in \Lambda\} \cup \{p_{\lambda i}p_{\mu i}^{-1} \mid i \in I, \lambda\pi\mu\},$$
$$\lambda\pi_N\mu \iff p_{\lambda i}p_{\mu i}^{-1} \in N \qquad \text{for all } i \in I \qquad (\lambda, \mu \in \Lambda).$$

Lemma 6.1. *For* $\zeta = (r, N, \pi) \in \mathcal{AT}(V)$, *we have*

$$\zeta K = (r_N, N, \pi_N), \qquad \zeta k = (\varepsilon, N, \varepsilon),$$
$$\zeta T = (r, G, \pi), \qquad \zeta t = (r, \overline{r\pi}, \pi).$$

Proof. Straightforward. □

Let also $V' = \mathcal{M}(I', G', \Lambda'; P')$ (normalization is not needed here). Any homomorphism $\varphi : V \to V'$ can be constructed as follows. Let

$$\xi : I \to I', \qquad u : I \to G', \qquad \omega : G \to G',$$
$$v : \Lambda \to G', \qquad \eta : \Lambda \to \Lambda'$$

be functions with $u : i \to u_i$, ω a homomorphism, $v : \lambda \to v_\lambda$. Then

$$\varphi : (i, g, \lambda) \to (i\xi, u_i(g\omega)v_\lambda, \lambda\eta) \qquad ((i, g, \lambda) \in V). \tag{5}$$

For a full discussion of these constructions, we refer to ([**1**, III.4]).

Therefore, for normal cryptogroups we have two essential advantages over the case considered previously: (1) each S_α can be taken to be a Rees matrix semigroup with normalized sandwich matrix with a convenient form of congruences in terms of admissible triples and (2) the structure homomorphisms $\varphi_{\alpha,\beta}$ can be given a more explicit form. This will make it possible to handle ρK and ρT. We begin with the easier case of ρT which follows the same general form as ρt.

The above notation is fixed throughout this and the next sections.

Lemma 6.2. *Operator* T *is compatible.*

Proof. Let $(\rho_\alpha) \in \mathcal{LC}(S)$, $\alpha > \beta$, $\varphi = \varphi_{\alpha,\beta}$ as given in (5), $\rho_\gamma \sim (r_\gamma, N_\gamma, \pi_\gamma)$ for $\gamma \in \{\alpha, \beta\}$, $(i, g, \lambda)\rho_\alpha T(j, h, \mu)$. By Lemma 6.1, we have $ir_\alpha j$ and $\lambda\pi_\alpha\mu$. But then $(i, e_\alpha, \lambda)\rho_\alpha(j, e_\alpha, \mu)$, see (4), which implies that $(i, e_\alpha, \lambda)\varphi\rho_\beta(j, e_\alpha, \mu)\varphi$. In view of both (5) and (4), we conclude that $i\xi r_\beta j\xi$ and $\lambda\eta\pi_\beta\mu\eta$. It follows that $(i, g, \lambda)\varphi\rho_\beta T(j, h, \mu)\varphi$, as required. □

Recall the convention that $(\rho_\alpha T)^\xi$ means: apply ξ to the congruence $\rho_\alpha T$.

Theorem 6.3. *For any $(\xi;\rho_\alpha) \in \mathcal{CA}(S)$, we have $(\xi;\rho_\alpha)T = (\xi;(\rho_\alpha T)^\xi)$.*

Proof. By Lemma 6.2, we have $(\rho_\alpha T) \in \mathcal{LC}(S)$ so by Lemma 5.5, we get $(\xi;(\rho_\alpha T)^\xi) \in \mathcal{CA}(S)$. Let $\lambda_\alpha = (\rho_\alpha T)^\xi$ and $\lambda \sim (\xi;\lambda_\alpha)$.

Let $a, b \in S_\alpha$ with $a\rho_\alpha b$. Then $a\rho_\alpha Tb$ so that $a\lambda_\alpha b$. Hence $\rho \subseteq \lambda$. Next let $e, f \in E(S_\alpha)$ and $e\lambda_\alpha f$. Then $e\varphi_{\alpha,\gamma}\rho_\gamma Tf\varphi_{\alpha,\gamma}$ for some $\gamma \leq \alpha\xi\gamma$. But then $e\varphi_{\alpha,\gamma}\rho_\gamma f\varphi_{\alpha,\gamma}$ which together with $\gamma\,\xi\,\alpha$ implies $e\rho_\alpha f$. Thus, $\operatorname{tr}\lambda_\alpha \subseteq \operatorname{tr}\rho_\alpha$ and equality prevails. Now Lemma 5.2 implies that $\operatorname{tr}\lambda = \operatorname{tr}\rho$.

Finally, let $\theta \in \mathcal{C}(S)$ be such that $\operatorname{tr}\theta = \operatorname{tr}\rho$ and let $a\,\theta\,b$. Then $a, b \in S_\alpha$ for some $\alpha \in Y$ and with $\theta \sim (\xi;\theta_\alpha)$, we have $\theta_\alpha T = \rho_\alpha T$ so that $a\theta_\alpha Tb$ and thus $a\rho_\alpha Tb$. But then $a\lambda_\alpha b$ which proves that $\theta \subseteq \lambda$ and establishes the maximality of λ. □

For treating ρK we need some preparation.

Lemma 6.4. *Let V be a completely regular semigroup, $a, b \in V$ and $\rho \in \mathcal{C}(V)$. If $ab \in \ker\rho$, then $ba \in \ker\rho$.*

Proof. Let $ab \in \ker\rho$. Then

$$\begin{aligned} ba &= b(ab)a(ba)^{-1}\rho b(ab)^0 a(ba)^{-1} = b(ab)^0(ab)b^{-1}(ba)^{-1} \\ &= b(ab)b^{-1}(ba)^{-1} = (ba)(ba)^{-1} = (ba)^0 \end{aligned}$$

so $ba \in \ker\rho$. □

Lemma 6.5. *Let V be a completely simple semigroup, $a, b \in V$ and $\rho \in \mathcal{C}(V)$. Assume that $xa \in \ker\rho$ if and only if $xb \in \ker\rho$ for all $x \in S$. Then also $a \in \ker\rho$ if and only if $b \in \ker\rho$.*

Proof. Let $a \in \ker\rho$. Then $a^0a \in \ker\rho$, so by hypothesis, $a^0b \in \ker\rho$. Hence, $(a^0b)b^0 \in \ker\rho$ and thus, by Lemma 6.4, we get $b^0a^0b \in \ker\rho$.

Again the hypothesis implies that $b^0a^0a \in \ker\rho$, that is, $b^0a \in \ker\rho$. Finally,

$$b = (b^0a)^0 b\rho b^0 ab\rho b^0 a^0 b\rho b^0 (a^0b)^0 = b^0$$

so that $b \in \ker\rho$. The opposite implication follows by symmetry. $\square$

For $(\rho_\alpha) \in \prod_{\alpha\in Y} \mathcal{C}(S_\alpha)$ and for each $\alpha \in Y$ define a relation ρ_α^π by

$$a\rho_\alpha^\pi b \iff a\varphi_{\alpha,\beta}\rho_\beta b\varphi_{\alpha,\beta} \qquad \text{for all } \beta \le \alpha \qquad (a, b \in S_\alpha).$$

Also, let $(\rho_\alpha)\pi = (\rho_\alpha^\pi)$.

Lemma 6.6. *For $(\rho_\alpha) \in \prod_{\alpha\in Y} \mathcal{C}(S_\alpha)$, we have $(\rho_\alpha)\pi \in \mathcal{LC}(S)$.*

Proof. Let $\alpha \in Y$ and $\lambda_\alpha = \rho_\alpha^\pi$. Clearly λ_α is reflexive and symmetric. If $a\lambda_\alpha b$ and $b\lambda_\alpha c$, then for any $\beta \le \alpha$, we have

$$a\varphi_{\alpha,\beta}\, \rho_\beta\, b\varphi_{\alpha,\beta}\, \rho_\beta\, c\varphi_{\alpha,\beta}$$

and thus $a\lambda_a c$. Therefore, λ_α is also transitive. Assuming $a\lambda_\alpha b$ and $c \in S_\alpha$, we get for any $\beta \le \alpha$,

$$(ac)\varphi_{\alpha,\beta} = (a\varphi_{\alpha,\beta})(c\varphi_{\alpha,\beta})\rho_\beta(b\varphi_{\alpha,\beta})(c\varphi_{\alpha,\beta}) = (bc)\varphi_{\alpha,\beta}$$

and thus $ac\lambda_\alpha bc$. By symmetry, we conclude that $\lambda_\alpha \in \mathcal{C}(S_\alpha)$.

Now let $a\lambda_\alpha b$ and $\beta \le \alpha$. For any $\gamma \le \beta$, we get $a\varphi_{\alpha,\gamma}\rho_\gamma b\varphi_{\alpha,\gamma}$ whence $(a\varphi_{\alpha,\beta})\varphi_{\beta,\gamma}\rho_\gamma(b\varphi_{\alpha,\beta})\varphi_{\beta,\gamma}$. Therefore, $a\varphi_{\alpha,\beta}\lambda_\beta b\varphi_{\alpha,\beta}$ which proves that $(\lambda_\alpha) \in \mathcal{LC}(S)$. $\square$

Recall the convention that $(\rho_\alpha K)^\pi$ means: apply π to the congruence $\rho_\alpha K$.

Theorem 6.7. *For any $(\xi; \rho_\alpha) \in \mathcal{CA}(S)$, we have $(\xi; \rho_\alpha)K = ((\rho_\alpha K)\pi\kappa; (\rho_\alpha K)^\pi)$.*

Proof. Since $(\rho_\alpha K) \in \prod_{\alpha\in Y} \mathcal{C}(S_\alpha)$, by Lemma 6.6 we have $(\rho_\alpha K)\pi \in \mathcal{LC}(S)$. Letting $\eta = (\rho_\alpha K)\pi\kappa$ and $\lambda_\alpha = (\rho_\alpha K)^\pi$, by Theorem 3.3 we obtain $(\eta;\lambda_\alpha) \in \mathcal{CA}(S)$. Let $\rho \sim (\xi;\rho_\alpha)$ and $\lambda \sim (\eta;\lambda_\alpha)$. It remains to show that $\rho K = \lambda$.

For any $\alpha \in Y$ and $a \in S_\alpha$, we have

$$\begin{aligned} a \in \ker \rho_\alpha &\iff a\rho_\alpha a^0 \\ &\iff a\varphi_{\alpha,\beta}\rho_\beta a^0\varphi_{\alpha,\beta} \qquad \text{for all } \beta \le \alpha \\ &\iff a\varphi_{\alpha,\beta}\rho_\beta K a^0\varphi_{\alpha,\beta} \qquad \text{for all } \beta \le \alpha \\ &\iff a\lambda_\alpha a^0 \\ &\iff a \in \ker \lambda_\alpha \end{aligned}$$

so that $\ker \rho_\alpha = \ker \lambda_\alpha$. It follows that

$$\ker \rho = \bigcup_{\alpha\in Y} \ker \rho_\alpha = \bigcup_{\alpha\in Y} \ker \lambda_\alpha = \ker \lambda.$$

Let $\rho K \sim (\zeta;\theta_\alpha)$. Since $\ker \lambda = \ker \rho$, it follows that $\lambda \subseteq \rho K$ and hence $\eta \subseteq \zeta$ and $\lambda_\alpha \subseteq \theta_\alpha$ for all $\alpha \in Y$. Let $a\theta_\alpha b$. Then $a\rho K b$ and thus, by Lemma 6.4,

$$(6) \qquad xa \in \ker \rho \iff xb \in \ker \rho \qquad \text{for all } x \in S^1.$$

Let $\beta \le \alpha$. We wish to show that $a\varphi_{\alpha,\beta}\rho_\beta K b\varphi_{\alpha,\beta}$. So, let $x \in S_\beta$ and assume that $x(a\varphi_{\alpha,\beta}) \in \ker \rho_\beta$. Then $xa = x(a\varphi_{\alpha,\beta}) \in \ker \rho$, and hence by (6), we get $xb \in \ker \rho$. But then $x(b\varphi_{\alpha,\beta}) = xb \in \ker \rho_\beta$. By symmetry and Lemma 6.5, we conclude that $a\varphi_{\alpha,\beta}\rho_\beta K b\varphi_{\alpha,\beta}$. It follows that $a\lambda_\alpha b$ which proves that $\theta_\alpha \subseteq \lambda_\alpha$ and equality prevails. By maximality of η, we must have $\zeta \subseteq \eta$. Therefore, $\zeta = \eta$ which then gives $\lambda = \rho K$, as required. $\square$

The following observations shed further light on the nature of operators K, k, T and t.

Example 6.8. This is to show that K is not compatible. Let $S_i = \mathcal{M}(I_i; G_i; \Lambda_i; P_i)$ for $i = 0, 1$.

$$I_0 = I_1 = \{1,2\}, \qquad \Lambda_0 = \{1,2\}, \qquad \Lambda_1 = \{1\}, \qquad G_0 = G_1 = \{e,a\}$$

$$P_0 = \begin{bmatrix} e & e \\ e & a \end{bmatrix}, \qquad P_1 = [\, e \quad e \,]$$

$$\varphi : (i, g, 1) \to (i, g, 1) \qquad (i = 1, 2,\ g \in G_1)$$

which gives the semigroup. Let $\rho_i = \varepsilon_{S_i}$ for $i = 0, 1$. Clearly $(\rho_0, \rho_1) \in \mathcal{LC}(\mathcal{A})$. It follows easily that $\rho_1 K = \varepsilon_{S_1}$ and $\rho_0 K \sim (\omega_{I_0}, \{e\}, \omega_{\Lambda_0})$. In particular, $(1, e, 1)\rho_1 K(2, e, 1)$ but $(1, e, 1)\varphi\rho_0 K(2, e, 1)\varphi$ does not hold. Therefore K is not compatible.

Lemma 6.9. *For any $\rho \in \mathcal{C}(S)$, we have $\rho K = \rho \wedge \mu$ and $\rho T = \rho \vee \mu$. Operators k, T and t are order preserving but K is not.*

Proof. Let $\rho \sim (\xi; \rho_\alpha)$ and observe that $\mu \sim (\varepsilon; \mu_\alpha)$ and that the above formulae are valid in a completely simple semigroup. Now Theorems 3.2, 5.4 and 6.3 yield

$$\begin{aligned}
(\xi; \rho_\alpha) \wedge (\varepsilon; \mu_\alpha) &= (\varepsilon; \rho_\alpha \wedge \mu_\alpha) = (\varepsilon; \rho_\alpha k) \sim \rho k, \\
(\xi; \rho_\alpha) \vee (\varepsilon; \mu_\alpha) &= (\xi; \vee\{(\rho_\beta \vee \mu_\beta)\varphi^{-1}_{\alpha,\beta} \mid \beta \le \alpha\xi\beta\}) \\
&= (\xi; \vee\{(\rho_\beta T)\varphi^{-1}_{\alpha,\beta} \mid \beta \le \alpha\xi\beta\}) \\
&= (\xi; (\rho_\alpha T)^\xi) \sim \rho T.
\end{aligned}$$

This also implies that k and T preserve order.

Also let $\lambda \sim (\eta; \lambda_\alpha)$ and assume that $(\xi; \rho_\alpha) \le (\eta; \lambda_\alpha)$. Then $\xi \subseteq \eta$ and $\rho_\alpha \subseteq \lambda_\alpha$ for all $\alpha \in Y$. Let $a, b \in S_\alpha$ be such that $a\,\rho t\,b$. Then $a\varphi_{\alpha,\gamma}\rho_\gamma t b\varphi_{\alpha,\gamma}$ for some $\gamma \le \alpha\xi\gamma$. By Lemma 5.3, this implies that $a\varphi_{\alpha,\gamma}\lambda_\gamma t b\varphi_{\alpha,\gamma}$. Since also $\gamma \le \alpha\eta\gamma$, it follows that $a\,\lambda\, t b$. Therefore, $(\xi; \rho_\alpha)t \le (\eta; \lambda_\alpha)t$ as required. $\square$

That K does not preserve order was illustrated in ([**7**, Example 1]) on an instance of a Clifford semigroup.

7. The semigroup generated by K, k, T and t for a normal cryptogroup. With the expressions we have obtained for ρk and ρt in Section 5 for a strong semilattice of regular simple semigroups, and for ρT and ρk in Section 6 for strong semilattices of completely simple semigroups, we are now able to characterize the semigroup generated by the operators K, k, T and t. To this end, we first obtain certain relations satisfied by these operators.

Lemma 7.1. *For any $\rho \in \mathcal{C}(S)$, we have $\rho T K = \omega$.*

Proof. Let $\rho \sim (\xi; \rho_\alpha)$, $\theta_\alpha = (\rho_\alpha T)^\xi$, $(\lambda_\alpha) = (\theta_\alpha K)\pi$ so that $(\xi; \rho_\alpha)TK = ((\lambda_\alpha)\kappa; \lambda_\alpha)$. It suffices to prove that $\lambda_\alpha = \omega_\alpha$ for all $\alpha \in Y$ since this forces $(\lambda_\alpha)\kappa = \omega$. Fix $\alpha \in Y$. Recall that

$$a\theta_\alpha b \iff a\varphi_{\alpha,\gamma}\rho_\gamma T b\varphi_{\alpha,\gamma} \qquad \text{for some } \gamma \leq \alpha\,\xi\,\gamma. \tag{7}$$

If $a\rho_\alpha T b$, then by compatibility of T, Lemma 6.4, we have $a\varphi_{\alpha,\beta}\rho_\beta T b\varphi_{\alpha,\beta}$ for any $\beta \leq \alpha$ and, in particular, for $\beta = \gamma$ in (7), so that $a\theta_\alpha b$. It follows that $\rho_\alpha T \subseteq \theta_\alpha$. Now $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$ so let $\rho_\alpha \sim (r, N, \pi)$. By Lemma 6.1, $\rho_\alpha T \sim (r, G_\alpha, \pi)$ which then implies that $\theta_\alpha \sim (r', G_\alpha, \pi')$ for some $r' \supseteq r$ and $\pi' \supseteq \pi$.

Recall that

$$a\lambda_\alpha b \iff a\varphi_{\alpha,\beta}\theta_\alpha K b\varphi_{\alpha,\beta} \qquad \text{for all } \beta \leq \alpha. \tag{8}$$

In view of Lemma 6.1, $\theta_\alpha K \sim (\omega_{I_\alpha}, G_\alpha, \omega_{\Lambda_\alpha})$ so that $\theta_\alpha K = \omega_\alpha$. But then (8) shows that $\lambda_\alpha = \omega_\alpha$, as required. □

Lemma 7.2. *For any $\rho \in \mathcal{C}(S)$, $\rho k T = \rho T k = \mu$.*

Proof. It is easy to see that the above assertion holds in a Rees matrix semigroup. Let $\rho \sim (\xi; \rho_\alpha)$. By Theorems 5.4 and 6.3, we have

$$(\xi; \rho_\alpha)kT = (\varepsilon; \rho_\alpha k)T = (\varepsilon; (\rho_\alpha k T)^\varepsilon) = (\varepsilon; \rho_\alpha k T) = (\varepsilon; \mu_\alpha) \sim \mu.$$

Letting $\theta_\alpha = (\rho_\alpha T)^\xi$, we have $(\xi; \rho_\alpha)Tk = (\xi; \theta_\alpha)k = (\varepsilon; \theta_\alpha k)$. Recall the form of θ_α in (7). Let $a = (i, g, \lambda)$, $b = (j, h, \nu)$, $\theta_\alpha \sim (r, N, \pi)$, $\rho_\gamma \sim (r', N', \pi')$ and $\varphi = \varphi_{\alpha,\gamma}$ be as in (5). Using Lemma 6.1, we get

$$a\varphi_{\alpha,\gamma} = (i\xi, u_i(g\omega)v_\lambda, \lambda\eta)\rho_{(r', G_\gamma, \pi')}(j\xi, u_j(h\omega)v_\nu, \nu\eta) = b\varphi_{\alpha,\gamma}$$

which entails no restriction on g or h implying that $N = G_\alpha$. But then $\theta_\alpha k \sim (\varepsilon_\alpha, G_\alpha, \varepsilon_\alpha)$ so that $\theta_\alpha k = \mu_\alpha$ whence $\rho T k = \mu$. □

Lemma 7.3. *For any $\rho \in \mathcal{C}(S)$, we have $\rho k t = \varepsilon$.*

Proof. It is easy to see that the above equality holds in a Rees matrix semigroup. Letting $\rho \sim (\xi; \rho_\alpha)$, by Theorems 5.4 and 5.6, we obtain

$$(\xi; \rho_\alpha)kt = (\varepsilon; \rho_\alpha k)t = (\varepsilon; (\rho_\alpha k t)^\varepsilon) = (\varepsilon; \rho_\alpha k t) = (\varepsilon; \varepsilon_\alpha).$$

Lemma 7.4. *For any $\rho \in \mathcal{C}(S)$, we have $\rho t K = \rho t K t$.*

Proof. Let

$$\rho \sim (\xi; \rho_\alpha), \qquad \theta_\alpha = (\rho_\alpha t)^\xi, \qquad \lambda_\alpha = (\theta_\alpha K)^\pi,$$
$$\eta = (\theta_\alpha K)\pi\kappa, \qquad \zeta = (\lambda_\alpha t)^\eta$$

so that

$$(\xi; \rho_\alpha)tK = (\xi; \theta_\alpha)K = (\eta; \lambda_\alpha), \qquad (\xi; \rho_\alpha)tKt = (\eta; \zeta_\alpha).$$

Fix $\alpha \in Y$ and let $e, f \in E(S_\alpha)$ be such that $e\rho_\alpha f$. Then $e\rho_\alpha t f$ which evidently implies that $e\theta_\alpha f$. Since $(\xi; \rho_\alpha)t = (\xi; \theta_\alpha)$, we have $(\theta_\alpha) \in \mathcal{LC}(S)$ and hence, for any $\beta \leq \alpha$, we obtain $e\varphi_{\alpha,\beta}\theta_\beta f\varphi_{\alpha,\beta}$. But then $e\varphi_{\alpha,\beta}\theta_\beta K f\varphi_{\alpha,\beta}$ for all $\beta \leq \alpha$ and thus $e\lambda_\alpha f$. We have proved that $\operatorname{tr}\rho_\alpha \subseteq \operatorname{tr}\lambda_\alpha$. It follows from Lemma 6.1 that $\rho_\alpha t \subseteq \lambda_\alpha t$.

Since $(\xi; \rho_\alpha)tK \geq (\xi; \rho_\alpha)tKt$, we have $\lambda_\alpha \supseteq \zeta_\alpha$, and since $\operatorname{tr}(\xi; \rho_\alpha)tK$

$= \operatorname{tr}(\xi; \rho_\alpha)tKt$, we have $\operatorname{tr}\lambda_\alpha = \operatorname{tr}\zeta_\alpha$. Let $a \in \ker\lambda_\alpha$. Then $a\lambda_\alpha a^0$ and thus $a\theta_\alpha K a^0$. But then $a \in \ker(\theta_\alpha K) = \ker\theta_\alpha$ whence $a\theta_\alpha a^0$. Then $a\varphi_{\alpha,\gamma}\rho_\gamma t a^0\varphi_{\alpha,\gamma}$ for some $\gamma \leq \alpha\xi\gamma$. By the first part of the proof, we get $a\varphi_{\alpha,\gamma}\lambda_\gamma t a^0\varphi_{\alpha,\gamma}$. Also $(\xi; \rho_\alpha)t \leq (\xi; \rho_\alpha)tK$ implies that $\xi \subseteq \eta$, whence $\gamma \leq \alpha\eta\gamma$. Therefore $a\zeta_\alpha a^0$ so that $a \in \ker\zeta_\alpha$. Consequently, $\ker\lambda_\alpha \subseteq \ker\zeta_\alpha$ and equality prevails. This, together with $\operatorname{tr}\lambda_\alpha = \operatorname{tr}\zeta_\alpha$ gives $\lambda_\alpha = \zeta_\alpha$. It now follows that $(\xi; \rho_\alpha)tK = (\xi; \rho_\alpha)tKt$, as asserted. $\square$

From this point on, the situation becomes quite analogous to that in Section 4 for the operators G, g, L and l. It becomes almost identical to the situation in ([**7**, Section 3]) for the case of a Clifford semigroup (semilattice of groups). The results of Lemmas 4.2–4.4 are of course valid also for Clifford semigroups. As we have seen in the stated reference, essentially no other relations, except for some trivial ones, are valid for Clifford semigroups. We now summarize briefly the situation in our present case relegating the details and proofs to the cited reference.

Let $\Gamma = \{K, k, T, t\}$. Then operators Γ satisfy the following relations

$$\begin{aligned}
\Sigma = \{\text{(i)}\ \ & K^2 = kK = K, \qquad k^2 = Kk = k,\\
& t^2 = Tt = t, \qquad T^2 = tK = t,\\
\text{(ii)}\ \ & KTK = TKT = TK,\\
& tkt = ktk = kt,\\
\text{(iii)}\ \ & tKt = tK,\\
\text{(iv)}\ \ & kT = Tk\}.
\end{aligned}$$

Let

$$\begin{aligned}
&\varepsilon = kt, \qquad \tau = ktK, \qquad \tau \vee \mu = ktKT, \qquad \mu = kT,\\
&\omega = TK, \qquad \sigma = TKt, \qquad \sigma \wedge \mu = TKtk,\\
&\Delta = \{\varepsilon, \sigma, \mu, \tau, \sigma \wedge \mu, \tau \vee \mu, \omega\}.
\end{aligned}$$

Theorem 7.5. *The set*

$$\Omega = \{K, KT, Kt, KtK, Ktk, KtKT, k, t, tk, tK, tKT, T\} \cup \Delta$$

is a system of representatives for the congruence on Γ^+ generated by the relations Σ.

The $\mathcal{D}$-structure of Ω is given in ([**7**, Proposition 1]).

Lemma 7.6. *For any $\rho \in \mathcal{C}(S)$,*

$$\rho Ktk \subseteq \rho k, \qquad \rho T \subseteq \rho tKT, \qquad \rho KT \subseteq \rho KtKT.$$

Proof. Indeed

$$\begin{gathered}
\ker(\rho Kt) \subseteq \ker(\rho K) = \ker \rho, \qquad \operatorname{tr} \rho = \operatorname{tr}(\rho t) \subseteq \operatorname{tr}(\rho tK),\\
\operatorname{tr}(\rho K) = \operatorname{tr}(\rho Kt) \subseteq \operatorname{tr}(\rho KtK),
\end{gathered}$$

which, by Theorem 5.1, implies the assertions. $\square$

Similarly as in [**7**], we may deduce the following result.

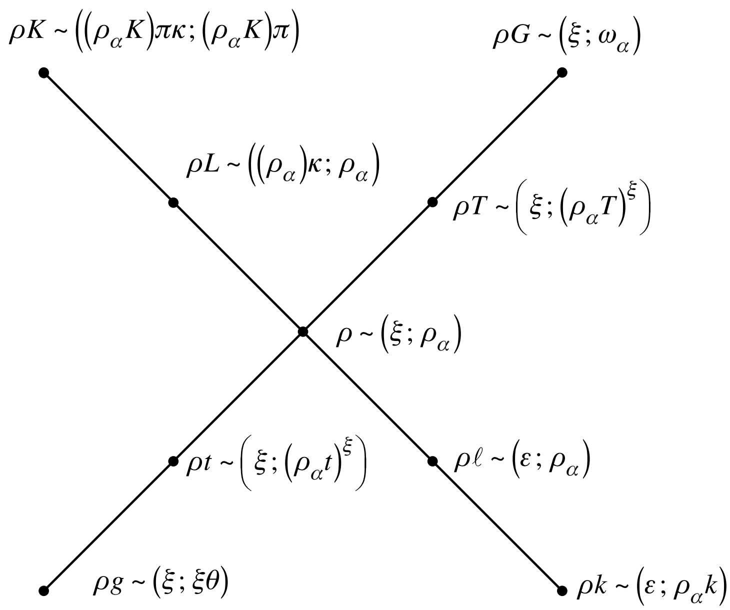

FIGURE 1.

Theorem 7.7. *Let S be a normal cryptogroup. The semigroup $\Omega(S)$ generated by the operators K, k, T and t on $\mathcal{C}(S)$ is a homomorphic image of Ω. For the semigroup S in* ([**7**, *Example 1*]), *we have $\Omega(S) \cong \Omega$.*

Figure 1 illustrates the mutual relationship of the values ρG, ρg, ρL, ρl, ρK, ρk, ρT and ρt for any congruence ρ on a normal cryptogroup.

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