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AUTOMORPHISMS OF THE INTEGRAL GROUP RING OF THE WREATH PRODUCT OF A p-GROUP WITH S_n, THE CASE $n=2$

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1. Introduction. Let $\mathbf{Z}G$ be the integral group ring of the group G which is the wreath product $H\mathrm{wr}S_n$ where H is a finite p-group. It has been proved in [**1**] that if $n \geq 3$, then any normalized automorphism θ of $\mathbf{Z}G$ can be written as $\theta = \tau_u \circ \lambda$ where λ is an automorphism of G and τ_u is the inner automorphism of $\mathbf{Q}G$ induced by a suitable unit u of $\mathbf{Q}G$. We complete this work by proving the same result for $n = 2$. We use the notations of [**1**] and state the

Theorem. *Let G be the wreath product $H\mathrm{wr}S_2$ of a finite p-group H and S_2. Then every normalized automorphism θ of $\mathbf{Z}G$ can be written as $\theta = \tau_u \circ \lambda$ where λ is an automorphism of G and u is a unit of $\mathbf{Q}G$.*

2. Some observations. The group in question is

$$G = (H \times H) \rtimes \langle (12) \rangle = \{(a, b; \sigma) \mid a, b \in H, \sigma = (12) \text{ or } I\},$$
$$H \text{ a finite } p\text{-group}.$$

Identifying $(a, b; I)$ with (a, b) we have $(a, b)^{(12)} = (b, a)$. Denote by C_g the class sum of g and by $\mathcal{C}_g$ the class of g. We note that

$$\mathcal{C}_{(a,b)} = \{(a^x, b^y) \mid x, y \in H\} \cup \{(b^y, a^x) \mid x, y \in H\}.$$

Assume throughout that θ is a given normalized automorphism of $\mathbf{Z}G$. If $p = 2$, then G is a 2-group and the result is true by the Theorem of Weiss [**5**]. Thus we may assume that $p \neq 2$. Therefore, $\theta(\Delta(G, P)) = \Delta(G, P)$ where $P = H \times H$. We recall two crucial lemmas.

Lemma 1. *If $\theta(C_g) = C_x$, $\theta(C_h) = C_y$, then there exist $t, z \in G$ such that $\theta(C_{gh}) = C_{xy^t} = C_{x^z y}$.*

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Proof. [**2**]. □

Lemma 2. *There exists an automorphism λ of P such that, for all $g \in P$, we have $\theta(C_g) = C_{\lambda(g)}$.*

Proof. This is Lemma 3.9 of [**1**]. □

Let us write $\lambda(a,b) = (\lambda_1(a,b), \lambda_2(a,b))$. As the need arises, we will use the same notation for any automorphism of P.

Lemma 3. *For all $a, b \in H$, $\lambda(a,1)$ and $(\lambda(b,1))^{(12)}$ commute.*

Proof. By Lemma 2, two elements, g_1 and g_2, of P are conjugate in G if and only if $\lambda(g_1)$ and $\lambda(g_2)$ are conjugate in G. We have, for $1 \neq a \in H$,

$$\begin{aligned} \mathcal{C}_{(a,1)} &= \{(a^x,1) \mid x \in H\} \cup \{(1,a^x) \mid x \in H\} \\ \mathcal{C}_{\lambda(a,1)} &= \{(\lambda_1(a,1))^x, (\lambda_2(a,1))^y \mid x, y \in H\} \\ &\quad \cup \{(\lambda_2(a,1))^x, (\lambda_1(a,1))^y \mid x, y \in H\}. \end{aligned}$$

Moreover, if $x \in H$,

$$\begin{aligned} \lambda(a^x,1) &= \lambda(x,1)^{-1}\lambda(a,1)\lambda(x,1) \\ &= (\lambda_1(x,1), \lambda_2(x,1))^{-1}(\lambda_1(a,1), \lambda_2(a,1))(\lambda_1(x,1), \lambda_2(x,1)) \\ &= ((\lambda_1(a,1))^{\lambda_1(x,1)}, (\lambda_2(a,1))^{\lambda_2(x,1)}). \end{aligned}$$

This belongs to the first subset of $\mathcal{C}_{\lambda(a,1)}$. Further, if $(e,f) = \lambda^{-1}(x,y)$ for $x, y \in H$, then

$$\begin{aligned} \lambda(a^e,1) &= \lambda((e,f)^{-1}(a,1)(e,f)) \\ &= (x,y)^{-1}(\lambda_1(a,1), \lambda_2(a,1))(x,y) \\ &= ((\lambda_1(a,1))^x, (\lambda_2(a,1))^y). \end{aligned}$$

Thus the image under λ of the first subset of $\mathcal{C}_{(a,1)}$ is the first subset of $\mathcal{C}_{\lambda(a,1)}$. Since the two subsets of $\mathcal{C}_{(a,1)}$ are disjoint, we can find $t \in H$ such that

$$(*) \qquad \lambda(1,a^t) = (\lambda_2(a,1), \lambda_1(a,1)) = (\lambda(a,1))^{(12)}.$$

Since $(a,1)$ and $(1,b)$ commute for all a and b, $\lambda(a,1)$ and $\lambda(1,b)$ must also commute. Hence, $\lambda(a,1)$ commutes with $(\lambda(b,1))^{(12)}$ for all b as claimed. □

We would like to know when an automorphism of P can be extended to an automorphism of G which leaves (12) fixed.

Lemma 4. *An automorphism of P can be extended to an automorphism of G by setting $\mu(12) = (12)$ if and only if $\mu_1(b,a) = \mu_2(a,b)$ for all $a, b \in H$.*

Proof. μ extends as desired if and only if it satisfies $(\mu(g))^{(12)} = \mu(g^{(12)})$ for all $g \in P$. This is equivalent to

$$\begin{aligned}(\mu_1(b,a), \mu_2(b,a)) = \mu(b,a) = \mu((a,b)^{(12)}) = (\mu(a,b))^{(12)} \\ = (\mu_1(a,b), \mu_2(a,b))^{(12)} = (\mu_2(a,b), \mu_1(a,b)).\end{aligned}$$

In other words, $\mu_1(b,a) = \mu_2(a,b)$ for all $a, b \in H$. □

3. Proof of the Theorem. As in [**1**], it is enough to prove that there is an automorphism μ of G such that $(\mu^{-1}\theta)(C_g) = C_g$ for all $g \in G$ (see [**3**, page 117]).

Define a map $\mu : P \to P$ by

$$\mu(a,b) = \lambda(a,1)(\lambda(b,1))^{(12)} \qquad \text{for all } a, b \in H.$$

Note that

$$\begin{aligned}\mu((a,b)(c,d)) &= \mu(ac, bd) \\ &= \lambda(ac,1)(\lambda(bd,1))^{(12)} \\ &= \lambda(a,1)\lambda(c,1)(\lambda(b,1))^{(12)}(\lambda(d,1))^{(12)} \\ &= \lambda(a,1)(\lambda(b,1))^{(12)}\lambda(c,1)(\lambda(d,1))^{(12)} \quad \text{by Lemma 3} \\ &= \mu(a,b)\mu(c,d).\end{aligned}$$

Also, if $\mu(a,b) = 1$, then $\lambda(a,1)(\lambda(b,1))^{(12)} = 1$. But we saw in the proof of Lemma 3 (see $(*)$) that $(\lambda(b,1))^{(12)} = \lambda(1, b^t)$ for some $t \in H$.

Thus, $\lambda(a,1)\lambda(1,b^t)=1$, or $\lambda(a,b^t)=1$ forcing $a=b=1$. We conclude that μ is an automorphism of P. Also

$$\begin{aligned}\mu(b,a)&=\lambda(b,1)(\lambda(a,1))^{(12)}\\&=(\lambda(a,1))^{(12)}\lambda(b,1)\quad\text{by Lemma 3}\\&=(\lambda_2(a,1)\lambda_1(b,1),\lambda_1(a,1)\lambda_2(b,1)).\end{aligned}$$

Thus, $\mu_1(b,a)=\lambda_2(a,1)\lambda_1(b,1)$. But $\mu(a,b)=\lambda(a,1)(\lambda(b,1))^{(12)}$, so $\mu_2(a,b)=\lambda_2(a,1)\lambda_1(b,1)$.

Since $\mu_1(b,a)=\mu_2(a,b)$ for all $a,b\in H$, we conclude from Lemma 4 that μ can be extended to an automorphism of G by setting $\mu(12)=(12)$.

Since $\mu(a,1)=\lambda(a,1)$ and $\mu(1,a)=(\lambda(a,1))^{(12)}$, we conclude that $\theta C_{(a,1)}=C_{\mu(a,1)}$ for all a in H. Replace θ by $\mu^{-1}\theta$, where μ denotes the automorphism of $\mathbf{Z}G$ obtained by extending μ $\mathbf{Z}$-linearly. We then have $\theta C_{(a,1)}=C_{(a,1)}$ for all $a\in H$.

Now consider $C_{(a,b)}$ where $a\neq 1$ and $b\neq 1$. Lemma 1, together with the previous remark and the fact that $C_{(a,1)}=C_{(1,a)}$, tells us that

$$\theta C_{(a,b)}=\theta C_{(a,1)(1,b)}=C_{(a,1)(1,b)^x}\qquad\text{for some } x\in G.$$

So $\theta C_{(a,b)}=C_{(a,b^x)}$ if $x\in P$ and $\theta C_{(a,b)}=C_{(ab^y,1)}$ for some $y\in H$ if $x\notin P$. But the latter case is impossible since all classes $C_{(a,1)}$ are fixed under θ.

Hence, $\theta C_{(a,b)}=C_{(a,b^x)}=C_{(a,b)}$, and all classes $C_{(a,b)}$ are fixed under θ.

Next we claim that $\theta(C_{(12)})=C_{(12)}$; in fact, since all class sums of elements in P are fixed by θ, $\theta(C_{(12)})=C_{(a,b;(12))}$ for some $a,b\in H$. Also, $(12)^2=1$ implies $(a,b;(12))^2=1$ and so, $ab=1$. Hence, $(12)\sim(a,b;(12))$ and the claim is established.

Now we prove that if $g=(a,b;(12))$ for some $a,b\in H$, then $\theta(C_g)=C_g$; in fact, by Lemma 1 and the above we have $\theta(C_g)=\theta(C_{(a,b)(12)})=C_{(a,b)(12)^x}$ for some $x\in G$. Write $(a,b)(12)^x=(a,b)(c,c^{-1};(12))=(ac,bc^{-1};(12))$, for some $c\in H$. Then $C_{(ab,ba)}=\theta(C_{(ab,ba)})=\theta(C_{(a,b;(12))^2})=C((_{ac,bc^{-1};(12))^2})=C((_{acbc^{-1},bc^{-1}ac}))$, and this implies in any case that $ab\sim acbc^{-1}$. Thus $g\sim(ac,bc^{-1};(12))$, and we are done. $\square$

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